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SO(2)-CONGRUENT PROJECTIONS OF CONVEX BODIES WITH ROTATION ABOUT THE ORIGIN

BENJAMIN MACKEY

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Abstract. We prove that if two convex bodies $K, L \subset \mathbb{R}^3$ satisfy the property that the orthogonal projections of K and L onto every plane containing the origin are rotations of each other, then either K and L coincide or L is the image of K under a reflection about the origin.

1. Introduction

In this paper, we will prove the following theorem:

Theorem 1. *Let $K, L \subset \mathbb{R}^3$ be convex bodies containing the origin as an interior point such that for every $\xi \in S^2$, the projection $K_{|\xi^\perp}$ can be rotated about the origin into $L_{|\xi^\perp}$. Then either $K = L$ or K can be obtained by reflecting L about the origin.*

Several related results have been proven under various assumptions about the type of congruence the projections satisfy. Wilhelm Süss proved that if each projection $K_{|\xi^\perp}$ is some parallel translation of $L_{|\xi^\perp}$, then K and L are parallel ([2], page 8). Vladimir Golubyatnikov allowed for both shifts and rotations, and proved two different theorems with variations on the symmetry of the projections ([2], page 13) and smoothness of the bodies ([2], page 22). This new result differs in that no symmetry assumptions are made about the bodies, but the freedom to translate projections is lost.

2. Notation and definitions

Throughout this paper, $\mathbb{R}^n$ will refer to the n-dimensional Euclidean space, and $S^{n-1} = \{\xi \in \mathbb{R}^n : |x| = 1\}$ will denote the unit sphere. The Euclidean inner product of two vectors $x, y \in \mathbb{R}^n$ will be denoted by $x \cdot y$. For a unit vector $\xi \in S^{n-1}$, the hyperplane orthogonal to ξ is denoted by $\xi^\perp = \{x \in \mathbb{R}^n : x \perp \xi\}$. The set $\xi^\perp \cap S^2$ is the great circle of unit vectors orthogonal to ξ. If $K \subset \mathbb{R}^3$ is a convex body containing the origin and $\xi \in S^2$, the section of K orthogonal to ξ is the set $K \cap \xi^\perp$. Given $E \subset S^2$ endowed with the spherical metric, the interior of E will be denoted by $\mathrm{int}(E)$ and the closure of E will be denoted by $\overline{E}$.

Let $\xi \in S^2$, r be some nonnegative number, and let $x \in \mathbb{R}^3$. Then the image of x rotated by an angle of $r\pi$ about the linear subspace spanned by ξ will be denoted

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by $R_{\xi,r}(x)$. Given $\epsilon > 0$, the spherical disk of radius $\epsilon\pi$ centered at ξ in S^2 will be called $S(\xi, \epsilon)$. We recall some standard concepts in the study of convexity:

Definition 1. Let $\xi \in S^{n-1}$ and $K \subset \mathbb{R}^n$ be a convex body. The **orthogonal projection** of K in the direction ξ is the set $K_{|\xi^\perp} = \{y \in \xi^\perp : \exists\lambda \in \mathbb{R}, y+\lambda\xi \in K\}$.

Definition 2. Let $K \subset \mathbb{R}^n$ be a convex body. The **support function** of K is the map $h_K : S^{n-1} \mapsto \mathbb{R}$ defined by $h_K(\xi) = \max\{u \cdot \xi : u \in K\}$. The **width function** of K is defined by $width_K(\xi) = (h_K(\xi) + h_K(-\xi))/2$. A body K has **constant width** if $width_K$ is a constant function on S^{n-1}.

Definition 3. If $K \subset \mathbb{R}^n$ is a convex body, the **polar dual** of K is the set $K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1, \forall y \in K\}$.

Definition 4. Let $K \subset \mathbb{R}^n$ be a convex body containing the origin. The **radial function** $\rho_K : S^{n-1} \mapsto \mathbb{R}$ is defined by $\rho_K(\xi) = \max\{\lambda \in \mathbb{R} : \lambda\xi \in K\}$.

3. Auxillary results

For any $r \in \mathbb{R}$, define $F_r \subset S^2$ by $F_r = \{\xi \in S^2 : K_{|\xi^\perp}$ rotated by $r\pi$ is $L_{|\xi^\perp}\}$. Observe that if a rotation of magnitude between π and 2π in the clockwise direction is necessary for $K_{|\xi^\perp}$ to coincide with $L_{|\xi^\perp}$, then a rotation of less than π in the counterclockwise direction makes the projections coincide. Thus, only $r \in [0, 1]$ need be considered. It follows that $\xi \in F_0$ if and only if $K_{|\xi^\perp} = L_{|\xi^\perp}$, and $\xi \in F_1$ if and only if the image of $K_{|\xi^\perp}$ under a reflection about the origin is $L_{|\xi^\perp}$. The conclusion of Theorem 1 can be rewritten as "either $S^2 = F_0$ or $S^2 = F_1$".

Lemma 1. *For all $r \in [0, 1]$, the set F_r is closed.*

Proof. If F_r is empty, then it is trivially closed. If F_r is nonempty, let ξ_n be a sequence in F_r, and suppose ξ_n converges to $\xi \in S^2$. Let $\theta \in \xi^\perp$ be arbitrary. For each n, pick some $\theta_n \in \xi_n^\perp$ so that θ_n converges to θ. Since each $\xi_n \in F_r$, we have $h_L(R_{\xi_n,r}(\theta_n)) = h_K(\theta_n)$ for each n.

By Rodrigues' rotation formula ([3], page 147),

$$R_{\xi_n,r}(\theta_n) = \theta_n \cos(r\pi) + (\xi_n \times \theta_n) \sin(r\pi) + \xi_n(\xi_n \cdot \theta_n)(1 - \cos(r\pi)).$$

Taking the limit as n approaches infinity, we see that $R_{\xi_n,r}(\theta_n)$ converges to

$$\theta \cos(r\pi) + (\xi \times \theta) \sin(r\pi) + \xi(\xi \cdot \theta)(1 - \cos(r\pi)) = R_{\xi,r}(\theta).$$

By the continuity of h_K and h_L, the function $h_L(R_{\xi_n,r}(\theta_n))$ converges to $h_L(R_{\xi,r}(\theta))$ and $h_K(\theta_n)$ converges to $h_K(\theta)$. It follows that $h_L(R_{\xi,r}(\theta)) = h_K(\theta)$ for every $\theta \in \xi^\perp$. This means that $K_{|\xi^\perp}$ rotated by $r\pi$ coincides with $L_{|\xi^\perp}$, and so $\xi \in F_r$. □

Two-dimensional bodies of constant width play an important role in our analysis. Define the set $\Sigma \subset S^2$ by $\Sigma = \{\xi \in S^2 : K_{|\xi^\perp}$ has constant width$\}$. If $\xi_1, \xi_2 \in \Sigma$, then $\xi_1^\perp \cap S^2$ and $\xi_2^\perp \cap S^2$ must intersect, which implies that $K_{|\xi_1^\perp}$ and $K_{|\xi_2^\perp}$ must have the same width, which will be denoted M.

Lemma 2. *Σ is closed.*

Proof. Let $\{\xi_n\}_{n=1}^\infty \subset \Sigma$ be a sequence such that ξ_n converges to $\xi \in S^2$, and let $\theta \in \xi^\perp \cap S^2$ be arbitrary. For each n, there is some $\theta_n \in \xi_n^\perp$ so that θ_n converges to θ. Since the width function is continuous, $width_K(\theta_n)$ converges to $width_K(\theta)$, but $width_K(\theta_n) = M$ for all M. Therefore, $width_K(\theta) = M$ for all $\theta \in \xi^\perp \cap S^2$. □

The next lemma is used during the proof of the central lemma in the next section.

Lemma 3. *Let $\theta \in S^2$ and $\epsilon > 0$, and suppose there exists a countable collection of closed subsets $\{F_n\}$ of S^2 with $\overline{S(\theta,\epsilon)} = \bigcup_{n=1}^{\infty} F_n$. Then there exists some $n \in \mathbb{N}$ where* $\mathrm{int}(F_n)$ *is nonempty.*

Proof. The set $\overline{S(\theta,\epsilon)}$ is compact in S^2 and therefore is a complete metric space. The Baire category theorem (see for example [4] page 98) implies that if $\overline{S(\theta,\epsilon)} = \bigcup_{n=1}^{\infty} F_n$, there is some $n \in \mathbb{N}$ with $\mathrm{int}(\overline{F_n}) = \mathrm{int}(F_n) \neq \emptyset$. $\square$

4. Main results

Our strategy in this section will be to combine information about the original bodies and the corresponding dual bodies to reduce the problem to a statement about a system of equations. Solving this system will then reduce further to solving a quartic polynomial equation in one variable. Using this, we will see that any projection which can be rotated by an amount between 0 and π into the other projection must be a disk. What this shows is that there actually are no rotations except for the zero rotation and a reflection about the origin.

The following lemma is due to Golubyatnikov ([2], page 17) and is essential to the proof of the main theorem of this paper.

Lemma 4. *If the projections of two convex bodies $K, L \subset \mathbb{R}^3$ are all SO(2) congruent, then $S^2 = F_0 \cup F_1 \cup \Sigma$.*

We will prove a lemma analogous to Lemma 4 which is concerned with the dual bodies K^* and L^* of K and L. The key is that duality takes projections into sections: $(K_{|\xi^\perp})^* = K^* \cap \xi^\perp$ for any $\xi \in S^2$ ([1], page 22). This is used in the proof of the following lemma which shows that rotational congruence is inherited by sections of the dual bodies.

Lemma 5. *For all $r \in [0,1]$, $F_r = \{\xi \in S^2 : K^* \cap \xi^\perp$ rotated by $r\pi$ is $L^* \cap \xi^\perp\}$.*

Proof. If $\xi \in F_r$, let $\Phi_r : \xi^\perp \mapsto \xi^\perp$ be the rotation of $\xi^\perp$ about the origin by $r\pi$. Since Φ_r is a 1-1 linear transformation such that $\Phi_r(K_{|\xi^\perp}) = L_{|\xi^\perp}$, we have $(\Phi_r(K_{|\xi^\perp}))^* = (L_{|\xi^\perp})^* = L^* \cap \xi^\perp$. Also, $(\Phi_r(K_{|\xi^\perp}))^* = (\Phi_r^{-1})^t (K_{|\xi^\perp})^*$ ([1], page 21) and $(\Phi_r^{-1})^t = \Phi_r$, which implies that $\Phi_r(K^* \cap \xi^\perp) = L^* \cap \xi^\perp$. $\square$

Define the function $\tau_{K^*} : S^2 \mapsto \mathbb{R}$ by $\tau_{K^*}(\xi) = (\rho^2_{K^*}(\xi) + \rho^2_{K^*}(-\xi))/2$, and define τ_{L^*} similarly. Lemma 4 is a statement about projections and the width function of K, whereas the function τ_{K^*} contains information about the sections of K^*. We can define the set $\Lambda \subset S^2$ analogously to Σ by $\Lambda = \{\xi \in S^2 : \tau_{K^*}$ restricted to $\xi^\perp \cap S^2$ is constant$\}$. Observe that if $\xi \in F_r$, since the radial function measures distance from the origin, it follows from Lemma 5 that $\tau_{K^*}(\theta) = \tau_{L^*}(R_{\xi,r}(\theta))$ for every $\theta \in \xi^\perp \cap S^2$.

Lemma 6. *$\tau_{K^*}(\xi) = \tau_{L^*}(\xi)$ for every $\xi \in S^2$.*

Proof. Since all sections of K^* are congruent to corresponding sections of L^*, $\mathrm{area}(K^* \cap \xi^\perp) = \mathrm{area}(L^* \cap \xi^\perp)$ for every unit vector ξ. Since the area of the section can be expressed as $\frac{1}{2}\int_0^{2\pi} \rho^2_{K^* \cap \xi^\perp}(\theta)d\theta$, we can conclude that

$$\int_{\xi^\perp \cap S^2} \frac{\rho^2_{K^*}(\theta) + \rho^2_{K^*}(-\theta)}{2} d\theta = \int_{\xi^\perp \cap S^2} \frac{\rho^2_{L^*}(\theta) + \rho^2_{L^*}(-\theta)}{2} d\theta$$

for every $\xi \in S^2$. This can be rewritten as

$$\int_{\xi^\perp \cap S^2} \tau_{K^*}(\theta) - \tau_{L^*}(\theta) d\theta = 0$$

for all unit vectors ξ. Thus, the spherical Radon transform of the even function $\tau_{K^*} - \tau_{L^*}$ is identically zero on S^2, and so ([1], page 430) implies that τ_{K^*} and τ_{L^*} coincide everywhere. □

Lemma 7. *Let $K, L \subset \mathbb{R}^3$ be convex bodies containing the origin as an interior point so that for all $\xi \in S^2$, $K_{|\xi^\perp}$ can be rotated about the origin into $L_{|\xi^\perp}$. Then $S^2 = F_0 \cup F_1 \cup \Lambda$.*

The proof of this lemma closely resembles Golubyatnikov's proof starting on page 17 of [2], and will be postponed until the end of the paper. If we assume that this lemma has been proven, we can complete the proof of Theorem 1. From Lemma 7 and Lemma 4, we know that if $\xi \in S^2$ is not in $F_0 \cup F_1$, then it must be that $\xi \in \Sigma \cap \Lambda$. The main idea is to use duality to show that if $\xi \in \Sigma \cap \Lambda$, then $\xi \in F_0 \cup F_1$, and therefore $S^2 = F_0 \cup F_1$. Golubyatnikov has proven that this then implies that $S^2 = F_0$ or $S^2 = F_1$ ([2], page 22), which completes the proof of Theorem 1. All that remains is to prove the following corollary.

Corollary 1. $S^2 = F_0 \cup F_1$.

Proof. By Lemma 4 and Lemma 7, $S^2 \backslash (F_0 \cup F_1)$ is contained in $\Sigma \cap \Lambda$, so it suffices to prove that $\Sigma \cap \Lambda$ is a subset of $F_0 \cup F_1$. If $\xi \in \Sigma \cap \Lambda$, Lemma 4 implies that there exists a constant $a \in \mathbb{R}$ so that for all $\theta \in \xi^\perp$,

$$h_K(\theta) + h_K(-\theta) = a.$$

By Lemma 7, there is a constant $b \in \mathbb{R}$ so that for all $\theta \in \xi^\perp$,

$$\rho_{K^*}^2(\theta) + \rho_{K^*}^2(-\theta) = b.$$

Since $h_K(\theta) = 1/\rho_{K^*}(\theta)$ for any unit vector θ ([1], page 20), the second equation can be rewritten as

$$h_K(\theta)^{-2} + h_K(-\theta)^{-2} = b$$

for all $\theta \in \xi^\perp$.

Consider the system of equations $x + y = a$ and $x^{-2} + y^{-2} = b$. Since the origin is an interior point of K, we can assume that neither x nor $y = a - x$ is equal to zero, and thus this system can be expressed as the quartic equation

$$(x-a)^2 + x^2 = bx^2(a-x)^2,$$

which has at most four real valued solutions in x. For each $\theta \in \xi^\perp$, $h_K(\theta)$ is a solution to this equation. Since h_K is a continuous function of θ, the intermediate value theorem implies that h_K is constant on $\xi^\perp$. This implies that both $K_{|\xi^\perp}$ and $L_{|\xi^\perp}$ are disks, and thus $K_{|\xi^\perp} = L_{|\xi^\perp}$. It follows that $\xi \in F_0$, and therefore $\Sigma \cap \Lambda \subset F_0 \cup F_1$. □

We conclude this paper by returning to the proof of Lemma 4.

Proof of Lemma 7. To prove the lemma, we will show that the set

$$F = S^2 \backslash (F_0 \cup F_1 \cup \Lambda)$$

is empty. An argument similar to the proof of Lemma 2 (just replace $width_K$ with τ_{K^*}) shows that Λ is closed, so it follows from Lemma 1 and Lemma 5 that F is an open set. Suppose there is a unit vector $\xi \in F$ such that $\xi \in F_r$ for some irrational $r \in (0,1)$. By Lemma 5 and Lemma 6, we can conclude that for every $\theta \in \xi^\perp \cap S^2$,

$$\tau_{K^*}(\theta) = \tau_{L^*}(R_{\xi,r}(\theta)) = \tau_{K^*}(R_{\xi,r}(\theta)).$$

Fixing some $\theta_0 \in \xi^\perp \cap S^2$, an inductive argument shows that

$$\tau_{K^*}(R_{\xi,nr}(\theta_0)) = \tau_{K^*}(\theta_0) \forall n \in \mathbb{N}.$$

Since r is irrational, the set $\{R_{\xi,nr}(\theta_0) : n \in \mathbb{N}\}$ is dense in the circle $\xi^\perp \cap S^2$. The function τ_{K^*} is continuous on S^2, and it takes on a single value on a dense subset of $\xi^\perp \cap S^2$, so it follows that τ_{K^*} takes a single value on $\xi^\perp \cap S^2$. Therefore $K^* \cap \xi^\perp$ is a disk, and thus so is $L^* \cap \xi^\perp$, which contradicts the assumption that $\xi \notin \Lambda$.

If F is nonempty and none of the sections orthogonal to elements of F coincide after some irrational angle, then F can be rewritten as $F = \bigcup_r (F \cap F_r)$ for some subset of the rational numbers contained in $(0,1)$. We claim that there exists some rational $r_0 \in (0,1)$ with $\mathrm{int}(F \cap F_{r_0}) \neq \emptyset$. To see this, since F is open and assumed nonempty, there exists $\theta \in F$ and $\epsilon > 0$ with $\overline{S(\theta,\epsilon)} \subset F$. Then $\overline{S(\theta,\epsilon)} = \cup_r \overline{S(\theta,\epsilon)} \cap F_r$, and Lemma 3 implies there exists some rational $r_0 \in (0,1)$ with $\emptyset \neq \mathrm{int}(\overline{S(\theta,\epsilon)} \cap F_{r_0}) \subset \mathrm{int}(F \cap F_{r_0})$.

Fix some ξ and $\epsilon > 0$ with the spherical disk $S(\xi,\epsilon)$ contained in $F \cap F_{r_0}$. Then the continuous function τ_{K^*} is not constant along $\xi^\perp \cap S^2$. Therefore, infinitely many values c exist with corresponding unit vectors $w_c \in \xi^\perp \cap S^2$ so that $\tau_{K^*}(w_c) = c$. The rest of the proof will be spent using these values to construct marks on the sphere which are geometrically impossible.

For each value $c = \tau_{K^*}(w_c)$ for some $w_c \in \xi^\perp \cap S^2$, denote by w'_c the vector obtained by rotating w_c by $r_0\pi$ along $\xi^\perp \cap S^2$. We claim there is an open arc $l_c^1 \subset S(w_c, r_0)$ containing w'_c with τ_{K^*} identically equal to c on l_c^1. We will then construct another arc l_c^3 on S^2 which intersects l_c^1 at w'_c on which τ_{K^*} is also constant, and we will define $X_c = l_c^1 \cup l_c^3$ (see Figure 1).

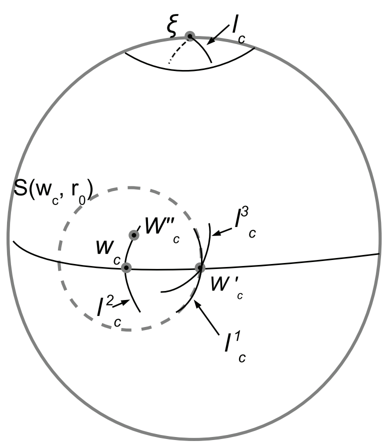

FIGURE 1. The construction of $l_c^1 \cup l_c^3$ for a clockwise rotation by an angle between 0 and 2π

Since the spherical disk $S(\xi,\epsilon)$ is contained in F_{r_0} and $\xi \in w_c^\perp \cap S^2$, there exists an open arc $l_c \subset w_c^\perp \cap S^2$ centered at ξ and contained in $S(\xi,\epsilon) \subset F_{r_0}$. For any $v \in l_c$, v is orthogonal to w_c, and so $w_c \in v^\perp$, which implies that the great circle

$v^{\perp} \cap S^2$ intersects $S(w_c, r_0)$ at the point $R_{v,r_0}(w_c)$. Call $l_c^1 = \{R_{v,r_0}(w_c) : v \in l_c\}$. If $\theta \in l_c^1$ with $\theta = R_{v,r_0}(w_c)$, where $v \in l_c$, the fact that $l_c \subset F_{r_0}$ and Lemma 6 imply that

$$c = \tau_{K^*}(w_c) = \tau_{L^*}(R_{v,r_0}(\theta)) = \tau_{L^*}(\theta) = \tau_{K^*}(\theta),$$

which proves the claim.

We now construct the second arc l_c^3. If we start at w_c' and rotate the projections of L^* in the reverse direction, by a similar argument applied to L^* we can create $S(w_c', r_0)$ and an open arc $l_c^2 \subset S(w_c', r_0)$ so that τ_{K^*} takes only the value c on l_c^2. Next, we can pick a third unit vector $w_c'' \in l_c^2$ distinct from w_c, and consider the circle $S(w_c'', r_0)$. Using a similar argument (create an arc in the spherical disk $S(\xi, \epsilon)$ centered at the preimage of w_c'' and consider the image of this arc under the rotation), we can create an arc $l_c^3 \subset S(w_c'', r_0)$ on which τ_{K^*} takes only the value c. If we define $X_c = l_c^1 \cup l_c^3$ to be the cross mark formed by the two arcs, we see that the function τ_{K^*} takes only the value c on X_c.

This construction can be done identically for every value taken by τ_{K^*} on $\xi^{\perp} \cap S^2$, so we have constructed an infinite family of congruent "X" figures on the unit sphere. For distinct values c_1, c_2 taken by τ_{K^*} on $\xi^{\perp} \cap S^2$, it follows from the construction that X_{c_1} and X_{c_2} are disjoint. Since it is impossible to construct infinitely many congruent mutually disjoint "X" figures on the sphere, this completes the proof. $\square$

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Department of Mathematics, Michigan State University, East Lansing, Michigan 48824

E-mail address: mackeybe@msu.edu