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WHEN IS A FOURFOLD MASSEY PRODUCT DEFINED?

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ABSTRACT. We define a new invariant in the homology of a differential graded algebra. This invariant is the obstruction to defining a fourfold Massey product. It can be used to detect differential graded algebras that are not quasi-isomorphic. We also make an explicit calculation in the cohomology of the Steenrod algebra.

1. Introduction

Massey products and Toda brackets are an essential tool for a detailed understanding of the cohomology of the Steenrod algebra and stable homotopy groups of spheres (see, for example, [7], [2], [1]). The standard references on Massey products, such as [4] and [6], typically assume that brackets are strictly defined, i.e., that the subbrackets have no indeterminacy. We have found in our own work on motivic stable homotopy groups [3] that strictly defined brackets are not always general enough.

This note addresses a subtlety with the definition of fourfold Massey products that arises when both of the threefold subbrackets have indeterminacy. We will define a new invariant (see Definition 3.1) of the homology of a differential graded algebra. Our main result (see Theorem 3.4) is that this invariant is the obstruction to defining a fourfold bracket whose threefold subbrackets contain zero.

We work with a differential graded algebra A whose homology is H. We will assume that A has characteristic 2 so that there are no signs to contend with. The reader who is interested in other characteristics can insert appropriate signs. For convenience, we suppress the grading of A. In general, A need not be commutative. We have also avoided the full generality of matric Massey products, but the results carry over to that context as well.

The symbols a_i always represent cycles, and the products $a_i a_{i+1}$ are always assumed to be boundaries. In other words, all threefold brackets are assumed to be defined. For any cycle x in A, we write $\overline{x}$ for the element of H that is represented by x.

2. The problem

Let us first recall how to compute a fourfold Massey product $\langle \overline{a_0}, \overline{a_1}, \overline{a_2}, \overline{a_3} \rangle$. First choose elements a_{01}, a_{12}, and a_{23} such that $d(a_{01}) = a_0 a_1$, $d(a_{12}) = a_1 a_2$, and $d(a_{23}) = a_2 a_3$. Next, choose elements a_{02} and a_{13} such that $d(a_{02}) = a_0 a_{12} + a_{01} a_2$

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and $d(a_{13}) = a_1a_{23} + a_{12}a_3$. The bracket $\langle \overline{a_0}, \overline{a_1}, \overline{a_2}, \overline{a_3} \rangle$ is the subset of H consisting of all elements of the form

$$\overline{a_0a_{13} + a_{01}a_{23} + a_{02}a_3}.$$

There is a subtlety that arises if $\langle \overline{a_0}, \overline{a_1}, \overline{a_2} \rangle$ or $\langle \overline{a_1}, \overline{a_2}, \overline{a_3} \rangle$ has indeterminacy. In this case, one must be careful to choose a_{01}, a_{12}, and a_{23} in such a way that $a_0a_{12} + a_{01}a_2$ and $a_1a_{23} + a_{12}a_3$ are boundaries. The well-known Lemma 2.1 addresses a simple case of this phenomenon.

Lemma 2.1. *Suppose that both $\langle \overline{a_0}, \overline{a_1}, \overline{a_2} \rangle$ and $\langle \overline{a_1}, \overline{a_2}, \overline{a_3} \rangle$ contain zero, and at least one of the brackets is strictly zero. Then $\langle \overline{a_0}, \overline{a_1}, \overline{a_2}, \overline{a_3} \rangle$ is defined.*

Proof. Suppose that $\langle \overline{a_0}, \overline{a_1}, \overline{a_2} \rangle$ is strictly zero. Choose a_{12} and a_{23} such that $a_1a_{23} + a_{12}a_3$ is a boundary. Then any choice of a_{01} makes $a_0a_{12} + a_{01}a_2$ into a boundary, since $\langle \overline{a_0}, \overline{a_1}, \overline{a_2} \rangle$ is strictly zero.

The same argument applies when $\langle \overline{a_1}, \overline{a_2}, \overline{a_3} \rangle$ is strictly zero. □

If both $\langle \overline{a_0}, \overline{a_1}, \overline{a_2} \rangle$ and $\langle \overline{a_1}, \overline{a_2}, \overline{a_3} \rangle$ have indeterminacies, it may be impossible to choose a_{01}, a_{12}, and a_{23} such that both $a_0a_{12} + a_{01}a_2$ and $a_1a_{23} + a_{12}a_3$ are boundaries simultaneously. The problem is that there are two constraints on a_{12}, and it may not be possible to satisfy both constraints.

3. Coindeterminacy

Suppose that $\langle \overline{a_0}, \overline{a_1}, \overline{a_2} \rangle$ and $\langle \overline{a_1}, \overline{a_2}, \overline{a_3} \rangle$ both contain zero, but either possibly has non-zero indeterminacy.

Definition 3.1. The coindeterminacy of the brackets $\langle \overline{a_0}, \overline{a_1}, \overline{a_2} \rangle$ and $\langle \overline{a_1}, \overline{a_2}, \overline{a_3} \rangle$ is the subset of H consisting of all elements of the form $\overline{x + y}$, where x ranges over all elements of A such that $d(x) = a_1a_2$ and $a_0x + za_2$ is a boundary for some z with $d(z) = a_0a_1$; and y ranges over all elements of A such that $d(y) = a_1a_2$ and $a_1w + ya_3$ is a boundary for some w with $d(w) = a_2a_3$.

In other words, x ranges over all possible choices of a_{12} that can be used to construct zero in $\langle \overline{a_0}, \overline{a_1}, \overline{a_2} \rangle$. Similarly, y ranges over all possible choices of a_{12} that can be used to construct zero in $\langle \overline{a_1}, \overline{a_2}, \overline{a_3} \rangle$.

The careful reader can verify that the coindeterminacy is well defined in H, i.e.,

(1) if x and y satisfy the conditions of Definition 3.1, then $x + y$ is a cycle,
(2) if x and y satisfy the conditions of Definition 3.1 and b is a boundary, then $x + b$ and y satisfy the conditions of Definition 3.1,
(3) if a'_i is homologous to a_i for each i, then the coindeterminacy of $\langle \overline{a'_0}, \overline{a'_1}, \overline{a'_2} \rangle$ and $\langle \overline{a'_1}, \overline{a'_2}, \overline{a'_3} \rangle$ is the same as the coindeterminacy of $\langle \overline{a_0}, \overline{a_1}, \overline{a_2} \rangle$ and $\langle \overline{a_1}, \overline{a_2}, \overline{a_3} \rangle$.

Definition 3.2. Let $\overline{a}$ and $\overline{b}$ be elements of H. Then $(\overline{a} \backslash\backslash \overline{b})$ is the additive subgroup of H consisting of all elements $\overline{x}$ such that $\overline{ax} = \overline{zb}$ for some $\overline{z}$ in H, and $(\overline{a} // \overline{b})$ is the additive subgroup of H consisting of all elements $\overline{x}$ such that $\overline{az} = \overline{xb}$ for some $\overline{z}$ in H.

Lemma 3.3. *The coindeterminacy of $\langle \overline{a_0}, \overline{a_1}, \overline{a_2} \rangle$ and $\langle \overline{a_1}, \overline{a_2}, \overline{a_3} \rangle$ is a coset with respect to $(\overline{a_0} \backslash\backslash \overline{a_2}) + (\overline{a_1} // \overline{a_3})$.*

One possible name for $(\overline{a_0} \backslash\backslash \overline{a_2}) + (\overline{a_1} // \overline{a_3})$ is the "indeterminacy of the coindeterminacy".

Proof. Let $x+y$ and $x'+y'$ represent elements of the coindeterminacy. We will consider $(x+y)+(x'+y') = (x+x')+(y+y')$.

The element $x+x'$ is a cycle. There exist elements z and z' such that a_0x+za_2 and $a_0x'+z'a_2$ are boundaries. Therefore, $a_0(x+x')$ is homologous to $(z+z')a_2$. This shows that $\overline{x+x'}$ belongs to $(\overline{a_0}\backslash\backslash\overline{a_2})$. Similarly, $\overline{y+y'}$ belongs to $(\overline{a_1}//\overline{a_3})$.

On the other hand, let $\overline{x+y}$ be an element of the coindeterminacy, and let $\overline{c}$ be an element of $(\overline{a_0}\backslash\backslash\overline{a_2})$. Choose a cycle e such that $\overline{a_0c} = \overline{ea_2}$. There exists z in A such that a_0x+za_2 is a boundary. Then $a_0(x+c)+(z+e)a_2$ is also a boundary. This shows that $\overline{(x+c)+y}$ also belongs to the coindeterminacy. Similarly, if $\overline{c}$ in an element of $(\overline{a_1}//\overline{a_3})$, then $\overline{x+(y+c)}$ also belongs to the coindeterminacy. □

Theorem 3.4. *Suppose that $\langle\overline{a_0},\overline{a_1},\overline{a_2}\rangle$ and $\langle\overline{a_1},\overline{a_2},\overline{a_3}\rangle$ both contain zero but possibly have non-zero indeterminacies. The fourfold bracket $\langle\overline{a_0},\overline{a_1},\overline{a_2},\overline{a_3}\rangle$ is defined if and only if zero is contained in the coindeterminacy of $\langle\overline{a_0},\overline{a_1},\overline{a_2}\rangle$ and $\langle\overline{a_1},\overline{a_2},\overline{a_3}\rangle$.*

In other words, the coindeterminacy is the obstruction to defining a fourfold bracket, given that both of its threefold subbrackets contain zero.

Proof. Suppose that $\langle\overline{a_0},\overline{a_1},\overline{a_2},\overline{a_3}\rangle$ is defined. There are elements a_{01}, a_{12}, and a_{23} such that $a_0a_{12}+a_{01}a_2$ and $a_1a_{23}+a_{12}a_3$ are boundaries. Then $\overline{0} = \overline{a_{12}+a_{12}}$ is an element of the coindeterminacy.

Suppose that zero belongs to the coindeterminacy. In the notation from Definition 3.1, we have $x=y$. Let a_{01}, a_{12}, and a_{23} be z, x, and w respectively. Then $a_0a_{12}+a_{01}a_2$ and $a_1a_{23}+a_{12}a_3$ are boundaries, so there exist choices for a_{02} and a_{13}. This shows that $\langle\overline{a_0},\overline{a_1},\overline{a_2},\overline{a_3}\rangle$ is defined. □

4. Differential graded algebras that are not quasi-isomorphic

A quasi-isomorphism $\phi : A \to A'$ induces isomorphisms of coindeterminacies in the homologies of A and A'. Therefore, coindeterminacies can be used to detect that two differential graded algebras are not quasi-isomorphic. We illustrate this with the following examples.

Example 4.1. Let A be the differential graded algebra whose underlying algebra is a commutative polynomial algebra on the generators listed in the table. The differential on A is the unique derivation that is defined on generators in the following table:

x	$d(x)$
a_0	0
a_1	0
a_2	0
a_3	0
a_{01}	a_0a_1
a_{12}	a_1a_2
a_{23}	a_2a_3
c	0
a_{02}	$a_0a_{12}+a_{01}a_2$
a_{13}	$a_1a_{23}+a_{12}a_3$

Note that the indeterminacy of the subbracket $\langle\overline{a_0},\overline{a_1},\overline{a_2}\rangle$ contains $\overline{a_0c}$, and the indeterminacy of the subbracket $\langle\overline{a_1},\overline{a_2},\overline{a_3}\rangle$ contains $\overline{ca_3}$. Nevertheless, the fourfold bracket $\langle\overline{a_0},\overline{a_1},\overline{a_2},\overline{a_3}\rangle$ is defined because the coindeterminacy contains zero.

Example 4.2. Let A' be the differential graded algebra whose underlying algebra is the same as the underlying algebra of A. The differential on A' is the same as on A, except that $d(a_{13}) = a_1 a_{23} + (a_{12} + c)a_3$.

The homologies of A and of A' are quite similar. They are isomorphic as rings, and they share the same threefold Massey product structure. However, the bracket $\langle \overline{a_0}, \overline{a_1}, \overline{a_2}, \overline{a_3} \rangle$ is not defined in the homology of A' because the coindeterminacy is a non-zero coset of $\overline{c} = \overline{a_{12} + (a_{12} + c)}$.

Therefore, coindeterminacy detects that A and A' are not quasi-isomorphic differential graded algebras.

5. An example

In many cases of interest, it is not easy to compute coindeterminacy directly. We will provide an explicit example in the cohomology of the Steenrod algebra of a coindeterminacy that does not contain zero. However, we will obtain this result indirectly by showing that a fourfold Massey product cannot be defined for other reasons.

Lemma 5.1. *The fourfold bracket* $\langle h_0^2, h_0^2 h_3, h_0^2 h_3, h_0^2 \rangle$ *is not defined.*

Proof. The threefold subbrackets $\langle h_0^2, h_0^2 h_3, h_0^2 h_3 \rangle$ and $\langle h_0^2 h_3, h_0^2 h_3, h_0^2 \rangle$ are both defined and equal to $\{0, h_0^6 h_4\}$.

Note that $P^2 h_1$ equals the bracket $\langle h_0^2 h_3, h_0^2, P h_1 \rangle$ with no indeterminacy, which equals $\langle h_0^2, h_0^2 h_3, \langle h_0^2 h_3, h_0^2, h_1 \rangle \rangle$ also with no indeterminacy. By a standard formal property of Massey products, this last expression would equal $\langle h_0^2, h_0^2 h_3, h_0^2 h_3, h_0^2 \rangle h_1$, if the fourfold bracket were defined. However, $P^2 h_1$ is not divisible by h_1, so the fourfold bracket cannot be defined. □

Remark 5.2. The reader who is familiar with the structure of the cohomology of the Steenrod algebra will recognize that $\langle h_0^2, h_0^2 h_3, h_0^2 h_3, h_0^2 \rangle$ is trying to be $P^2 = v_1^8$. However, P^2 does not exist.

Corollary 5.3. *The coindeterminacy of* $\langle h_0^2, h_0^2 h_3, h_0^2 h_3 \rangle$ *and* $\langle h_0^2 h_3, h_0^2 h_3, h_0^2 \rangle$ *is* $\{h_0^4 h_4, h_0^4 h_4 + h_1 d_0\}$.

Proof. Lemma 3.3 says that the coindeterminacy is a coset of $\{0, h_1 d_0\}$. Theorem 3.4 and Lemma 5.1 imply that the coindeterminacy does not contain zero. There is only one non-zero coset. □

6. Next steps

We leave unanswered a number of interesting and accessible problems.

Problem 6.1. Extend these ideas to higher order Massey products.

Problem 6.2. Extend these results to fourfold (and higher) Toda brackets.

Problem 6.3. Establish a convergence theorem, in the spirit of [8], relating coindeterminacies in the cohomology of the Steenrod algebra to coindeterminacies in stable homotopy groups.

Problem 6.4. In the stable homotopy groups of spheres, find a fourfold bracket that is not defined, even though both of its threefold subbrackets contain zero, i.e., a coindeterminacy that does not contain zero.

Because of Corollary 5.3, we expect that $\langle 4, 4\sigma, 4\sigma\rangle$ and $\langle 4\sigma, 4\sigma, 4\rangle$ are Toda brackets whose coindeterminacy does not contain zero. In fact, the coindeterminacy ought to consist of the elements $k\rho_{15}$ and $k\rho_{15} + \eta\kappa$, where k ranges over the values 2, 10, 18, and 26, and ρ_{15} is the generator of the image of J in the 15-stem.

Problem 6.5. Reinterpret coindeterminacy in terms of the existence or non-existence of certain 5-cell complexes.

Problem 6.6. Reinterpret coindeterminacy in terms of linkages of spheres in Euclidean space, in the spirit of [5].

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References

[1] M. G. Barratt, J. D. S. Jones, and M. E. Mahowald, *Relations amongst Toda brackets and the Kervaire invariant in dimension* 62, J. London Math. Soc. (2) **30** (1984), no. 3, 533–550, DOI 10.1112/jlms/s2-30.3.533. MR810962 (87g:55025)

[2] M. G. Barratt, M. E. Mahowald, and M. C. Tangora, *Some differentials in the Adams spectral sequence. II*, Topology **9** (1970), 309–316. MR0266215 (42 #1122)

[3] D. C. Isaksen, *Motivic stable stems*, preprint, 2014.

[4] David Kraines, *Massey higher products*, Trans. Amer. Math. Soc. **124** (1966), 431–449. MR0202136 (34 #2010)

[5] W. S. Massey, *Higher order linking numbers*, J. Knot Theory Ramifications **7** (1998), no. 3, 393–414, DOI 10.1142/S0218216598000206. MR1625365 (99e:57016)

[6] J. Peter May, *Matric Massey products*, J. Algebra **12** (1969), 533–568. MR0238929 (39 #289)

[7] Mark Mahowald and Martin Tangora, *Some differentials in the Adams spectral sequence*, Topology **6** (1967), 349–369. MR0214072 (35 #4924)

[8] R. Michael F. Moss, *Secondary compositions and the Adams spectral sequence*, Math. Z. **115** (1970), 283–310. MR0266216 (42 #1123)

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