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LANGLANDS PARAMETERS ASSOCIATED TO SPECIAL MAXIMAL PARAHORIC SPHERICAL REPRESENTATIONS

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ABSTRACT. We describe the image, under the local Langlands correspondence for tori, of the characters of a torus which are trivial on its Iwahori subgroup. Let k be a non-archimedian local field. Let $\boldsymbol{G}$ be a connected reductive group defined over k, which is quasi-split and split over a tamely ramified extension. Let K be a special maximal parahoric subgroup of $\boldsymbol{G}(k)$. To the class of representations of $\boldsymbol{G}(k)$ having a non-zero vector fixed under K, we establish a bijection, in a natural way, with the twisted semisimple conjugacy classes of the inertia fixed subgroup of the dual group $\hat{\boldsymbol{G}}$. These results generalize the well known classical results to the ramified case.

INTRODUCTION

Let k be a non-archimedian local field. Let $\boldsymbol{G}$ be an unramified connected reductive group defined over k and let π be a smooth, irreducible, admissible representation of $\boldsymbol{G}$ which is unramified; i.e., there is a hyperspecial subgroup K of $\boldsymbol{G}(k)$ such that the K-invariant subspace π^K of the space realizing π is non-zero. One can associate a Langlands parameter to the representation π via the following recipe.

The representation π corresponds, in a natural way, to a character of the Hecke algebra $\mathscr{H}(\boldsymbol{G}(k), K)$. Then via the Satake isomorphism, this character corresponds to an unramified character χ of $\boldsymbol{T}(k)$ for certain maximal torus $\boldsymbol{T}$. This character χ is unique up to the relative Weyl group conjugacy. The character χ, under the local Langlands correspondence for tori, corresponds to a Langlands parameter $\varphi_\chi \in \mathrm{H}^1(W_k, \hat{\boldsymbol{T}})$, where $\hat{\boldsymbol{T}}$ is the dual torus and W_k is the Weil group. This parameter is induced from a cocycle in $\mathrm{H}^1(W_k/I_k, \hat{\boldsymbol{T}})$, where I_k is the inertia subgroup of W_k. Using this, one can then associate to the parameter φ_χ a semisimple conjugacy class in $\hat{\boldsymbol{G}} \rtimes (W_k/I_k)$, where $\hat{\boldsymbol{G}}$ is the complex dual of $\boldsymbol{G}$. This semisimple conjugacy class describes the Langlands parameter associated to π.

All these classic results are well known and can be found in [2]. We wish to find analogous statements when $\boldsymbol{G}$ is not necessarily unramified. Let $\boldsymbol{G}$ be quasi-split and tamely ramified. Let K be a special maximal parahoric subgroup of $\boldsymbol{G}$ and let π be a smooth, irreducible, admissible representation of $\boldsymbol{G}$ which is K-spherical, i.e., $\pi^K \neq 0$. We associate to π a character χ of $\boldsymbol{T}(k)$ of a certain maximal torus $\boldsymbol{T}$ in a similar way as above, using the description of special maximal parahoric Hecke algebras given in [4]. We show that χ is trivial on the Iwahori subgroup

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$\boldsymbol{T}(k)_0$ of $\boldsymbol{T}(k)$. In Theorem 1, we calculate the image, under the local Langlands correspondence for tori, of all such characters which are trivial on $\boldsymbol{T}(k)_0$ and show that it is precisely inflation of the cocycles in $\mathrm{H}^1(W_k/I_k, \hat{\boldsymbol{T}}^{I_k})$. In Theorem 2, we show that the orbits in $\mathrm{H}^1(W_k/I_k, \hat{\boldsymbol{T}}^{I_k})$ of the relative Weyl group are in bijection with the semisimple conjugacy classes in $\hat{\boldsymbol{G}}^I \rtimes (W_k/I_k)$.

In [10, Chapter 11], a character of $\boldsymbol{T}(k)$ is called elementary if under the Langlands reciprocity map, it corresponds to a cocycle in $\mathrm{H}^1(W_k, \hat{\boldsymbol{T}})$ which is the inflation of a cocyle in $\mathrm{H}^1(W_k/I_k, \hat{\boldsymbol{T}}^{I_k})$. The question of characterizing the elementary characters is a natural one and some partial results are presented in [10, Chapter 11]. Our first theorem answers this question by showing that a character is elementary if and only if it is trivial on the Iwahori subgroup $\boldsymbol{T}(k)_0$.

The referee has pointed out that Lemma 5 for arbitrary tori was independently observed in a recent work by Haines [3, Sec. 3.3.1]. In that article, Haines mentions that the result is implied by a more general result of Kaletha [5, Prop. 4.5.2]. A characteristic zero assumption is made in that result of Kaletha.

1. Notation

Let k be a non-archimedian local field. Let $\boldsymbol{G}$ be a connected reductive group defined by k which is quasi-split and split over a tamely ramified extension. We denote $\boldsymbol{G}(k)$ by G and likewise for other algebraic groups. Let K be a special maximal parahoric subgroup of G corresponding to a special vertex ν in the Bruhat-Tits building $\mathcal{B}(G_{\mathrm{ad}})$. Let $\boldsymbol{A}$ denote a maximal split k-torus whose corresponding apartment in $\mathcal{B}(G_{\mathrm{ad}})$ contains ν. Let $\boldsymbol{T} = Z_{\boldsymbol{G}}\boldsymbol{A}$, the centralizer of $\boldsymbol{A}$ in $\boldsymbol{T}$. Then $\boldsymbol{T}$ is a maximal torus in $\boldsymbol{G}$ since $\boldsymbol{G}$ is quasi-split. Let W denote the relative Weyl group of $\boldsymbol{G}$. Let $\mathscr{H}(G, K)$ be the Hecke algebra of K-bi-invariant compactly supported complex-valued functions on G. Let T_c and T_0 denote respectively the maximal compact subgroup and the Iwahori subgroup of T. Let $\hat{\boldsymbol{G}}$ denote the complex dual of $\boldsymbol{G}$ and $\hat{\boldsymbol{G}}_{\mathrm{ss}}$ the set of semisimple elements in $\hat{\boldsymbol{G}}$. Let $\mathrm{Inn}(\hat{\boldsymbol{G}})$ be the group of inner automorphisms of $\hat{\boldsymbol{G}}$. Let $\sigma = \sigma_k$ denote the Frobenius element in W_k/I_k, where W_k is the Weil group of k and $I = I_k$ is its inertia subgroup. We denote the identity component of an algebraic group $\mathscr{G}$ by $\mathscr{G}^\circ$.

2. Statement of the theorems

Theorem 1. *A character is elementary if and only if it is trivial on the Iwahori subgroup. In other words, we have a commutative diagram:*

$$\begin{tikzcd} \mathrm{Hom}(T, \mathbb{C}^\times) \arrow[r, "LLC", "\cong"'] & \mathrm{H}^1(W_k, \hat{\boldsymbol{T}}) \\ \mathrm{Hom}(T/T_0, \mathbb{C}^\times) \arrow[u, hook] \arrow[r, "\cong"] & \mathrm{H}^1(W_k/I, \hat{\boldsymbol{T}}^I) \arrow[u, hook, "\mathrm{infl}"'] \end{tikzcd}$$

where LLC is the local Langlands correspondence for tori and infl *is the inflation homomorphism.*

Let $\mathrm{Rep}(G)$ denote the set of equivalent classes of smooth, irreducible admissible representations of G.

Theorem 2. *The K-spherical representations are in a natural bijection with the semisimple conjugacy classes in $\hat{\boldsymbol{G}}^I \rtimes \sigma$ via the local Langlands correspondence for tori. More precisely,*

$$\begin{aligned} \{\pi \mid \pi \in Rep(G),\ \pi^K \neq 0\} &\leftrightarrow \mathrm{Hom}_{\mathbb{C}}(\mathscr{H}(G,K),\mathbb{C}) \\ &\cong \mathrm{Hom}(T/T_0,\mathbb{C}^\times)/W \\ &\cong (\hat{\boldsymbol{T}}^I)_\sigma/W \\ &\cong (\hat{\boldsymbol{G}}^I \rtimes \sigma)_{\mathrm{ss}}/\operatorname{Inn}(\hat{\boldsymbol{G}}^I). \end{aligned}$$

3. Langlands correspondence for tori

The following treatment of local Langlands correspondence for tori can be found in [11].

3.1. **The special case of an induced torus.** Let $\boldsymbol{T} = R_{k'/k}\mathbb{G}_m$ be an induced torus, when k' is a finite separable extension of k. Then $T = k'^\times$ and the group of characters $X^*(\boldsymbol{T})$ is canonically a free $\mathbb{Z}$-module with basis $W_k/W_{k'}$. From this, it follows that $\hat{\boldsymbol{T}}$ is canonically isomorphic to $\mathrm{ind}_{W_{k'}}^{W_k}\mathbb{C}^\times$. We get

$$\begin{aligned} \mathrm{Hom}(T,\mathbb{C}^\times) &\cong \mathrm{Hom}(k'^\times,\mathbb{C}^\times) \\ &\cong \mathrm{Hom}(W_{k'},\mathbb{C}^\times) && (3.1) \\ &\cong \mathrm{H}^1(W_{k'},\mathbb{C}^\times) \\ &\cong \mathrm{H}^1(W_k,\mathrm{ind}_{W_{k'}}^{W_k}\mathbb{C}^\times) && (3.2) \\ &\cong \mathrm{H}^1(W_k,\hat{\boldsymbol{T}}). \end{aligned}$$

The isomorphism (3.1) follows by class field theory and the isomorphism (3.2) by Shapiro's lemma.

3.2. **The LLC for tori in general.**

Theorem ([11, 7.5 Theorem])**.** *There is a unique family of homomorphisms*

$$\varphi_{\boldsymbol{T}} : \mathrm{Hom}(T,\mathbb{C}^\times) \to \mathrm{H}^1(W_k,\hat{\boldsymbol{T}})$$

with the following properties:

(1) *$\varphi_{\boldsymbol{T}}$ is additive functorial in $\boldsymbol{T}$; i.e., it is a morphism between two additive functors from the category of tori over k to the category of abelian groups.*
(2) *For $\boldsymbol{T} = R_{k'/k}\mathbb{G}_m$, where k'/k is a finite separable extension, $\varphi_{\boldsymbol{T}}$ is the isomorphism described in Section* 3.1*.*

4. Proof of Theorem 1

Lemma 3. *Let $\boldsymbol{T}$ be a torus defined over k. Then there exists an isomorphism*

$$\kappa_{\boldsymbol{T}} : \mathrm{Hom}(T/T_0,\mathbb{C}^\times) \to \mathrm{H}^1(W_k/I,\hat{\boldsymbol{T}}^I).$$

Moreover, the isomorphism $k_{\boldsymbol{T}}$ is additive functorial in $\boldsymbol{T}$.

Proof. We have by the Kottwitz isomorphism [7, Sec. 7] (see also [4, Prop. 1.0.2])

$$\begin{aligned} T/T_0 &\cong ((X^*(\hat{\boldsymbol{T}}))_I)^\sigma \\ &\cong X^*((\hat{\boldsymbol{T}}^I)_\sigma). && (4.1) \end{aligned}$$

Therefore,

$$\begin{aligned}
\operatorname{Hom}(T/T_0,\mathbb{C}^\times) &\cong \operatorname{Hom}(X^*((\hat{\boldsymbol{T}}^I)_\sigma),\mathbb{C}^\times)\\
&\cong (\hat{\boldsymbol{T}}^I)_\sigma \qquad (4.2)\\
&\cong \mathrm{H}^1(W_k/I,\hat{\boldsymbol{T}}^I).
\end{aligned}$$

The isomorphism in equation (4.2) is by Cartier duality. The functoriality of $\kappa_{\boldsymbol{T}}$ follows from the functoriality of the Kottwitz isomorphism. □

Remark 4. From the relation

$$T/T_c \cong (X^*(\hat{\boldsymbol{T}})_I)^\sigma/\text{torsion},$$

one similarly obtains the isomorphism

$$\operatorname{Hom}(T/T_c,\mathbb{C}^\times) \cong \mathrm{H}^1(W_k/I,(\hat{\boldsymbol{T}}^I)^\circ). \tag{4.3}$$

Lemma 5. *Let k'/k be a finite separable extension and put $\boldsymbol{T}=R_{k'/k}(\mathbb{G}_m)$. Then the isomorphism $\operatorname{Hom}(T/T_0,\mathbb{C}^\times)\cong \mathrm{H}^1(W_k/I,\hat{\boldsymbol{T}}^I)$ obtained from the Kottwitz isomorphism in Lemma 3 is the same as the one induced from the local Langlands correspondence for tori.*

Proof. Since $\boldsymbol{T}$ is an induced torus, $X^*(\boldsymbol{T})$ is canonically a free $\mathbb{Z}$-module with basis $W_k/W_{k'}$. Consequently, $\hat{\boldsymbol{T}}$ is simply $\operatorname{ind}_{W_{k'}}^{W_k}\mathbb{C}^\times$. Let $\mathscr{O}$ and $\mathscr{O}'$ be the ring of integers in k and k' respectively. We have

$$\begin{aligned}
\mathrm{H}^1(W_k/I_k,\hat{\boldsymbol{T}}^{I_k}) &\cong (\hat{\boldsymbol{T}}^{I_k})_{\sigma_k}\\
&\cong ((\operatorname{ind}_{W_{k'}}^{W_k}\mathbb{C}^\times)^{I_k})_{\sigma_k}\\
&\cong \mathrm{H}^1(W_k/I_k,(\operatorname{ind}_{W_{k'}}^{W_k}\mathbb{C}^\times)^{I_k})\\
&\cong \mathrm{H}^1(W_{k'}/I_{k'},\mathbb{C}^\times) \qquad (4.4)\\
&\cong \operatorname{Hom}(k'^\times/\mathscr{O}'^\times,\mathbb{C}^\times) \qquad (4.5)\\
&\cong \operatorname{Hom}(T/T_0,\mathbb{C}^\times).
\end{aligned}$$

Here, the isomorphism in (4.4) is induced by the isomorphism in Shapiro's lemma:

$$\begin{tikzcd}
\mathrm{H}^1(W_k,\operatorname{ind}_{W_{k'}}^{W_k}\mathbb{C}^\times) \arrow[r, "\sim"] & \mathrm{H}^1(W_{k'},\mathbb{C}^\times)\\
\mathrm{H}^1(W_k/I_k,(\operatorname{ind}_{W_{k'}}^{W_k}\mathbb{C}^\times)^{I_k}) \arrow[r, "\sim"] \arrow[u, hook, "\mathrm{infl}"] & \mathrm{H}^1(W_{k'}/I_{k'},\mathbb{C}^\times) \arrow[u, hook, "\mathrm{infl}"']
\end{tikzcd}$$

Thus the local Langlands correspondence also induces an isomorphism:

$$\varphi_{\boldsymbol{T}}:\operatorname{Hom}(T/T_0,\mathbb{C}^\times)\cong \mathrm{H}^1(W_k/I,\hat{\boldsymbol{T}}^I).$$

The Kottwitz isomorphism for induced tori is constructed as follows (see [7, Sec. 7.2]): the homomorphism

$$v'_T : T \to \operatorname{Hom}(X^*(\boldsymbol{T}),\mathbb{Z}),$$

sending $t\in T$ to the homomorphism

$$\lambda \mapsto \operatorname{ord}(\lambda(t)),$$

induces an isomorphism

$$v_T : T/T_0 \cong \mathrm{Hom}(X^*(\boldsymbol{T})^I, \mathbb{Z})^\sigma. \tag{4.6}$$

Also, the homomorphism

$$q'_T : X_*(\boldsymbol{T}) \to \mathrm{Hom}(X^*(\boldsymbol{T}), \mathbb{Z}),$$

which sends $\mu \in X_*(T)$ to the homomorphism

$$\lambda \mapsto \langle \lambda, \mu \rangle,$$

induces an isomorphism

$$q_T : (X_*(\boldsymbol{T})_I)^\sigma \cong \mathrm{Hom}(X^*(\boldsymbol{T})^I, \mathbb{Z})^\sigma. \tag{4.7}$$

From the equations (4.6) and (4.7), we get the Kottwitz isomorphism for the induced torus $\boldsymbol{T}$:

$$w_T : T/T_0 \cong X^*((\hat{\boldsymbol{T}}^I)_\sigma),$$

where we used the identifications $(X_*(\boldsymbol{T})_I)^\sigma = (X^*(\hat{\boldsymbol{T}})_I)^\sigma \cong X^*((\hat{\boldsymbol{T}}^I)_\sigma)$. This map w_T induces the map $\kappa_{\boldsymbol{T}} : \mathrm{Hom}(T/T_0, \mathbb{C}^\times) \cong \mathrm{H}^1(W_k/I, \hat{\boldsymbol{T}}^I)$ in Lemma 3.

We identify T/T_0 with $\mathbb{Z}$ by the isomorphisms

$$T/T_0 \cong k'^\times/\mathscr{O}'^\times \cong \varpi'^{\mathbb{Z}} \cong \mathbb{Z}. \tag{4.8}$$

Here ϖ' is the uniformizer in k'. Let J be a W_k-stable basis of $X^*(\boldsymbol{T})$. Choose any $\lambda \in J$ and let $J_\lambda \subset J$ be the orbit of λ in J under the action of I_k. Let $\chi = \sum_{\mu \in J_\lambda} \mu$. Then $\chi \in X^*(\hat{\boldsymbol{T}})^{I_k}$.

Let $f \in \mathrm{Hom}(T/T_0, \mathbb{C}^\times)$ and let $c = f(1)$ (under the identification in equation (4.8)). Then both $\kappa_{\boldsymbol{T}}$ and $\varphi_{\boldsymbol{T}}$ map f to the cocycle $\phi_f \in \mathrm{H}^1(W_k/I, \hat{\boldsymbol{T}}^I)$ defined by $\sigma \mapsto \chi \otimes c$. Thus $\kappa_{\boldsymbol{T}} = \varphi_{\boldsymbol{T}}$. This completes the proof of the lemma. □

Proposition 6. *There is a unique family of homomorphisms,*

$$\varphi_{\boldsymbol{T}} : \mathrm{Hom}(T/T_0, \mathbb{C}^\times) \to \mathrm{H}^1(W_k, \hat{\boldsymbol{T}}),$$

with the following properties:

(1) *$\varphi_{\boldsymbol{T}}$ is additive functorial in $\boldsymbol{T}$; i.e., it is a morphism between two additive functors from the category of tori over k to the category of abelian groups.*
(2) *For $\boldsymbol{T} = R_{k'/k}\mathbb{G}_m$, where k'/k is a finite Galois extension, $\varphi_{\boldsymbol{T}}$ is the homomorphism induced from the local Langlands correspondence for tori.*

Proof. Since the isomorphism $\kappa_{\boldsymbol{T}}$ in Lemma 3 is additive functorial in $\boldsymbol{T}$, we thus get an additive functorial family of homomorphisms

$$\varphi_{\boldsymbol{T}} : \mathrm{Hom}(T/T_0, \mathbb{C}^\times) \longrightarrow \mathrm{H}^1(W_k/I, \hat{\boldsymbol{T}}^I) \xhookrightarrow{\text{infl}} \mathrm{H}^1(W_k, \hat{\boldsymbol{T}}).$$

This shows existence. To show uniqueness, let $\boldsymbol{T}$ be a given torus defined over k and let k'/k be a finite Galois extension such that $\boldsymbol{T}$ is split over k'. Let $\boldsymbol{T}' = R_{k'/k}(\boldsymbol{T} \otimes_k k')$. Then $\boldsymbol{T}'$ is isomorphic to a direct sum of $d = \dim(\boldsymbol{T})$ tori of the form $R_{k'/k}(\mathbb{G}_m)$ and there is a natural embedding $\boldsymbol{T} \hookrightarrow \boldsymbol{T}'$. This gives an embedding $T/T_0 \hookrightarrow T'/T_0'$. By (1), there is a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}(T/T_0, \mathbb{C}^\times) \arrow[r, "\varphi_{\boldsymbol{T}}"] & \mathrm{H}^1(W_k, \hat{\boldsymbol{T}}) \\ \mathrm{Hom}(T'/T_0', \mathbb{C}^\times) \arrow[r, "\varphi_{\boldsymbol{T}'}"] \arrow[u, two heads] & \mathrm{H}^1(W_k, \hat{\boldsymbol{T}}'). \arrow[u] \end{tikzcd}$$

Notice that $\varphi_{\boldsymbol{T}'}$ is completely determined by (2). Now given $x \in \mathrm{Hom}(T/T_0, \mathbb{C}^\times)$, we can lift it to $x' \in \mathrm{Hom}(T'/T_0', \mathbb{C}^\times)$. It follows that $\varphi_{\boldsymbol{T}}(x)$ is the image of $\varphi_{\boldsymbol{T}'}(x')$ under the vertical arrow on the right and is hence determined by (1) and (2). □

Theorem 7. *Let $\boldsymbol{T}$ be a torus defined over k. Then the local Langlands correspondence for tori induces the isomorphism*

$$\mathrm{Hom}(T/T_0, \mathbb{C}^\times) \cong \mathrm{H}^1(W_k/I, \hat{\boldsymbol{T}}^I).$$

Proof. By the above proposition, it follows that the homomorphism $\mathrm{Hom}(T/T_0, \mathbb{C}^\times) \to \mathrm{H}^1(W_k, \hat{\boldsymbol{T}})$, determined by the Kottwitz map, must be the same as the one determined by LLC. Therefore, the image of $\mathrm{Hom}(T/T_0, \mathbb{C}^\times)$ under LLC must be the same as the one determined by $\kappa_{\boldsymbol{T}}$, which is $\mathrm{H}^1(W_k/I, \hat{\boldsymbol{T}}^I)$. This proves the theorem. □

5. Some results about inertia fixed groups

Let $\boldsymbol{B}$ be a Borel subgroup of $\boldsymbol{G}$ containing the maximal torus $\boldsymbol{T}$ of $\boldsymbol{G}$. The triple $(\boldsymbol{G}, \boldsymbol{B}, \boldsymbol{T})$ determines a based root datum. Let $(\hat{\boldsymbol{G}}, \hat{\boldsymbol{B}}, \hat{\boldsymbol{T}})$ be the triple defined over $\mathbb{C}$ which corresponds to the dual based root datum. Since $\boldsymbol{G}$ is tamely ramified, the inertia group acts on $(\hat{\boldsymbol{G}}, \hat{\boldsymbol{B}}, \hat{\boldsymbol{T}})$ via a cyclic group (τ). Let $\boldsymbol{W}$ be the Weyl group of $\hat{\boldsymbol{G}}$. Let $\hat{\boldsymbol{T}}_{\mathrm{ad}}$ be the image of $\hat{\boldsymbol{T}}$ in $\hat{\boldsymbol{G}}_{\mathrm{ad}}$, the adjoint group of $\hat{\boldsymbol{G}}$. Let $\hat{\boldsymbol{Z}}$ be the center of $\hat{\boldsymbol{G}}$.

The following results are given in [8, Sec. 1.1 (Theorem 1.1.A and the paragraphs following it)].

Theorem 8 (Kottwitz-Shelstad)**.**

(1) $(\hat{\boldsymbol{G}}^\tau)^\circ$ *is a connected reductive group.*
(2) $(\hat{\boldsymbol{B}}^\tau)^\circ$ *is a Borel subgroup of* $(\hat{\boldsymbol{G}}^\tau)^\circ$ *containing its maximal torus* $(\hat{\boldsymbol{T}}^\tau)^\circ$.
(3) $\boldsymbol{W}^\tau$ *is the Weyl group of* $(\hat{\boldsymbol{G}}^\tau)^\circ$.
(4) $\hat{\boldsymbol{G}}^\tau = \hat{\boldsymbol{Z}}^\tau(\hat{\boldsymbol{G}}^\tau)^\circ$.
(5) $(\hat{\boldsymbol{T}}_{\mathrm{ad}})^\tau$ *is connected.*

From (5)*, we immediately obtain*

(5′) $\hat{\boldsymbol{T}}^\tau = \hat{\boldsymbol{Z}}^\tau(\hat{\boldsymbol{T}}^\tau)^\circ$.

Remark 9. As pointed out by the referee, these facts can be proved as follows. The fact that $(\hat{\boldsymbol{T}}_{\mathrm{ad}})^\tau$ is connected follows from [9, remark at the end of Cor. 9.12]. Using this and [12, Lemma 4.6], it follows that $(\hat{\boldsymbol{G}}_{\mathrm{ad}})^\tau$ is connected. From this, (4) follows. The fact that $(\hat{\boldsymbol{G}}^\tau)^\circ$ is reductive follows from a general lemma of Kottwitz [6, 10.1.2]. The fact that $\boldsymbol{W}^\tau$ is the Weyl group of $(\hat{\boldsymbol{G}}^\tau)^\circ$ can be proved by adapting the proof of Theorem 8.2 of [9]. Since τ fixes a splitting, this allows one to take $t = 1$ in that proof.

6. Proof of Theorem 2

Lemma 10. $\mathrm{Hom}_{\mathbb{C}}(\mathscr{H}(G,K),\mathbb{C}) \cong \mathrm{Hom}(T/T_0,\mathbb{C}^\times)/W$.

Proof. $\mathscr{H}(G,K) \cong \mathbb{C}[T/T_0]^W$ by [4, Theorem 1.0.1]. Therefore

$$\begin{aligned}\mathrm{Hom}_{\mathbb{C}}(\mathscr{H}(G,K),\mathbb{C}) &\cong \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[T/T_0]^W,\mathbb{C})\\ &\cong \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[T/T_0],\mathbb{C})/W\\ &\cong \mathrm{Hom}(T/T_0,\mathbb{C}^\times)/W.\end{aligned}$$

□

Proposition 11.

$$(\hat{\boldsymbol{T}}^I)_\sigma/W \cong Z(\hat{\boldsymbol{G}})^I(\hat{\boldsymbol{G}}^I)^\circ_{\mathrm{ss}} \rtimes \sigma/\,\mathrm{Inn}(Z(\hat{\boldsymbol{G}})^I(\hat{\boldsymbol{G}}^I)^\circ).$$

Proof. Let $\boldsymbol{W}$ be, as before, the Weyl group of $\hat{\boldsymbol{G}}$. By Theorem 8(3), $\boldsymbol{W}^\tau$ is the Weyl group of $(\hat{\boldsymbol{G}}^\tau)^\circ$. Since $(\hat{\boldsymbol{G}}^\tau)^\circ$ is reductive (by Theorem 8(1)), it follows from the proof of [1, Lemma 6.5] that we have a surjection

$$(\hat{\boldsymbol{T}}^\tau)^\circ \twoheadrightarrow ((\hat{\boldsymbol{G}}^\tau)^\circ \rtimes \sigma)_{\mathrm{ss}}/\,\mathrm{Inn}((\hat{\boldsymbol{G}}^\tau)^\circ). \tag{6.1}$$

By Theorem 8(5′), this implies

$$\hat{\boldsymbol{T}}^\tau \twoheadrightarrow (\hat{\boldsymbol{Z}}^\tau(\hat{\boldsymbol{G}}^\tau)^\circ \rtimes \sigma)_{\mathrm{ss}}/\,\mathrm{Inn}(\hat{\boldsymbol{Z}}^\tau(\hat{\boldsymbol{G}}^\tau)^\circ). \tag{6.2}$$

Denote the σ-action on an element $g \in \hat{\boldsymbol{G}}$ by g^σ. Let $s,t \in \hat{\boldsymbol{T}}^\tau$ be such that there exists $g \in \hat{\boldsymbol{Z}}^\tau(\hat{\boldsymbol{G}}^\tau)^\circ$ satisfying $g^{-1}sg^\sigma = t$. As defined in Section 5, let $\hat{\boldsymbol{B}}$ be the Borel subgroup of $\hat{\boldsymbol{G}}$ containing the maximal torus $\hat{\boldsymbol{T}}$. Write $g = unv$ using Bruhat decomposition, where u, v are in the unipotent radical of $\hat{\boldsymbol{B}}$ and n is in the normalizer $\hat{\boldsymbol{N}}$ of $\hat{\boldsymbol{T}}$. Also, $g = zg_0$ for some $z \in \hat{\boldsymbol{Z}}^\tau$ and $g_0 \in (\hat{\boldsymbol{G}}^\tau)^\circ$. Let $g_0 = u_0n_0v_0$ be the Bruhat decomposition of g_0 in $(\hat{\boldsymbol{G}}^\tau)^\circ$ with respect to the Borel $(\hat{\boldsymbol{B}}^\tau)^\circ$ and maximal torus $(\hat{\boldsymbol{T}}^\tau)^\circ$. Then $u = u_0$, $v = v_0$ and $n = zn_0$. Thus,

$$\begin{aligned} g^{-1}sg^\sigma = t &\implies su^\sigma n^\sigma v^\sigma = unvt\\ &\implies su^\sigma s^{-1}sn^\sigma v^\sigma = untt^{-1}vt\\ &\implies sn^\sigma = nt\\ &\implies sz^\sigma n_0^\sigma = zn_0t. \end{aligned} \tag{6.3}$$

Let $\bar{n}_0$ denote the image of n_0 in $\boldsymbol{W}$. Then $n_0^\tau = n_0 \implies \bar{n}_0 \in \boldsymbol{W}^\tau$. Also (6.3) implies $\bar{n}_0^\sigma = \bar{n}_0$. Thus $\bar{n}_0 \in \boldsymbol{W}^{\tau,\sigma}$. Using again the fact that $(\hat{\boldsymbol{G}}^\tau)^\circ$ is reductive, it follows from the proof of Lemma 6.2 in [1] that there exists $p \in N_{(\hat{\boldsymbol{G}}^\tau)^\circ}(\hat{\boldsymbol{T}}^\tau)^\circ$ such that $p^\sigma = p$ and $n_0 \in p(\hat{\boldsymbol{T}}^\tau)^\circ$. This fact is also shown in the proof of [12, Lemma 4.7]. Let $n_0 = pq$ for some $q \in (\hat{\boldsymbol{T}}^\tau)^\circ$. Then,

$$z^{-1}z^\sigma q^{-1}p^{-1}sp^\sigma q^\sigma = t.$$

Let $r = zq \in \hat{\boldsymbol{T}}^\tau$. Then $r^{-1}p^{-1}spr^\sigma = t$. Thus $\bar{s} = \bar{t}$ where $\bar{s}$ and $\bar{t}$ represent the class of s and t in $\hat{\boldsymbol{T}}^\tau_\sigma/\boldsymbol{W}^{\tau,\sigma} = (\hat{\boldsymbol{T}}^I)_\sigma/W$. □

Using Theorem 8(4), we obtain

Corollary 12. $(\hat{\boldsymbol{T}}^I)_\sigma/W \cong (\hat{\boldsymbol{G}}^I \rtimes \sigma)_{\mathrm{ss}}/\,\mathrm{Inn}(\hat{\boldsymbol{G}}^I)$.

Write $\boldsymbol{B} = \boldsymbol{TU}$, where $\boldsymbol{B}$ is as before, the Borel subgroup of $\boldsymbol{G}$ containing $\boldsymbol{T}$, and where $\boldsymbol{U}$ is the unipotent radical of $\boldsymbol{B}$. Using the Iwasawa decomposition [4, Corr. 9.1.2], we can define the spherical functions

$$\Phi_{K,\chi}(tuk) = \chi(m)\delta^{1/2}(m),$$

as in [2], where $t \in T$, $u \in U$, $k \in K$ and δ is the modulus function. The proof of the bijection

$$\{\pi | \pi \in \mathrm{Rep}(G),\ \pi^K \neq 0\} \leftrightarrow \mathrm{Hom}_{\mathbb{C}}(\mathscr{H}(\boldsymbol{G}, K), \mathbb{C})$$

is then identical to the case when $\boldsymbol{G}$ is unramified and K is hyperspecial, which is given in [2]. This completes the proof of Theorem 2.

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