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MINIMAL SURFACES IN THE COMPLEX HYPERQUADRIC Q_2 II

JUN WANG AND XIAOWEI XU

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Abstract. In this paper, minimal surfaces with parallel second fundamental form in Q_2 are classified, which are uniquely determined up to a rigidity motion. It is also proved that minimal surfaces in Q_2 with constant Gauss curvature and constant normal curvature are totally geodesic.

1. Introduction

There is a long history of studying submanifolds with particular geometry properties in symmetric spaces. Minimal surfaces with parallel mean curvature vectors (pmcv) in the complex space forms have been well studied in the past two decades. T. Ogata [13] studied the non-minimal surfaces with pmcv in CP^2 by using the Kähler function; K. Kenmotsu and D. T .Zhou [8] completely classified surfaces with pmcv in two-dimensional complex space forms, and such surfaces are not of constant Kähler angles; S. Hirakawa [6] classified constant Gaussian curvature surfaces with non-zero pmcv vector in two-dimensional complex space forms; F. Torralbo and F. Urbano [14] considered surfaces with pmcv vector in $S^2 \times S^2$ and $H^2 \times H^2$; D. Fetcu [5] obtained some characterization results concerning surfaces with pmcv vector in complex space forms by using holomorphic quadratic forms defined on these surfaces. The results about minimal surfaces with constant curvature in space forms are beautiful. E. Calabi [4] showed that minimal two-spheres with constant curvature in $S^n(1)$ are the Boruvka spheres; J. Bolton, G. R. Jensen, M. Rigoli and L. M. Woodward [1], and S. Bando, and Y. Ohnita [2] proved that minimal two-spheres with constant curvature in the complex projective space CP^{n+1} are the Veronese surfaces; Kenmotsu and Masuda [7] determined all the minimal surfaces of constant curvature in CP^2.

To study minimal surfaces in the complex hyperquadric Q_n, the first author and X. X. Jiao [9] introduced two analytic type functions τ_X and τ_Y (see also Section 2), which are closely related to the Gauss curvature K and Kähler angle θ of minimal surfaces in Q_n. In their previous works [10] and [11], they completely classified minimal surfaces with parallel second fundamental form or with constant Gauss curvature in Q_2 for the cases $\tau_X \equiv 0$ or $\tau_Y \equiv 0$. The purpose of this paper is to study the minimal surfaces in Q_2 with parallel second fundamental form or constant

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curvatures under the condition that both τ_X and τ_Y are not identically zero. Our main results are given as follows:

Theorem A. *Let $f : M \to Q_2$ be a minimal immersion from a connected surface M into Q_2 with parallel second fundamental form, and K, θ be the Gauss curvature and Kähler angle respectively. Then f is totally geodesic and the pair (K, θ) belongs to $\{(0, \pi/2), (2, 0), (2, \pi), (2, \pi/2), (4, 0), (4, \pi)\}$. Moreover, in terms of a holomorphic coordinate $z = x + \mathtt{i}y$ on M, f is congruent to one of the following:*

(I) $f(z) = [(\cos x, \sin x, \mathtt{i}\cos y, \mathtt{i}\sin y)^T]$ *if* $(K, \theta) = (0, \pi/2)$*;*
(II) $f(z) = [(2z, 0, 1 - z^2, \mathtt{i}(1 + z^2))^T]$ *if* $(K, \theta) = (2, 0)$*;*
(III) $f(z) = [(2\bar{z}, 0, 1 - \bar{z}^2, -\mathtt{i}(1 + \bar{z}^2))^T]$ *if* $(K, \theta) = (2, \pi)$*;*
(IV.a) $f(z) = [(1, \mathtt{i}, z, \mathtt{i}z)^T]$ *if* $(K, \theta) = (4, 0)$*;*
(IV.b) $f(z) = [(1, -\mathtt{i}, \bar{z}, -\mathtt{i}\bar{z})^T]$ *if* $(K, \theta) = (4, \pi)$*;*
(IV.c) $f(z) = [(z\bar{z} - 1, \mathtt{i}(z\bar{z} + 1), z + \bar{z}, -\mathtt{i}(z - \bar{z}))^T]$ *if* $(K, \theta) = (2, \pi/2)$.

Remark. Part of this theorem has been obtained in the first author's work [11].

Theorem B. *Let $f : M \to Q_2$ be a minimal immersion from a connected surface M into Q_2 with both τ_X and τ_Y not identically zero. If its Gauss curvature K and normal curvature $K^\perp$ are constants, then M is part of a totally real totally geodesic torus with $K = K^\perp = 0$.*

Remark. Does it also hold if we use the condition "K (or $K^\perp$) is a constant" instead of "K and $K^\perp$ are constants" in this theorem?

Our paper is organized as follows. In section 2, we recall some known facts and formulas about minimal surfaces in Q_2, and we introduce two functions of analytic type τ_X and τ_Y, which play an important role in our studies. In section 3, we recall some notation in [11], and we prove Theorem A by choosing the best frame field. In section 4, we prove Theorem B by calculating the Laplacian of various geometric invariants.

Throughout this paper, we will agree on the following ranges of indices:

$$1 \le A, B, C, \cdots \le 4, \quad 1 \le i, j, k, \cdots \le 2, \quad 3 \le \alpha, \beta, \gamma \le 4,$$

and we also take the convention

$$\mathtt{i}^2 = -1.$$

2. Preliminaries

Denote the Maurer-Cartan forms of $SO(4)$ by $\Omega = e^{-1}de = (\Omega_{AB})$. Then the Maurer-Cartan equation of $SO(4)$ is given by

$$d\Omega_{AB} = -\sum_C \Omega_{AC} \wedge \Omega_{CB}, ; \Omega_{AB} + \Omega_{BA} = 0. \tag{2.1}$$

The standard metric on $Q_2 = SO(4)/SO(2) \times SO(2)$ induced from the Fubini-Study metric of CP^3 can be written as

$$ds^2_{Q_2} = \Omega_3\bar{\Omega}_3 + \Omega_4\bar{\Omega}_4, \tag{2.2}$$

where $\Omega_\alpha = (\Omega_{\alpha 1} + \mathrm{i}\Omega_{\alpha 2})/\sqrt{2}$ are local one-forms of (1,0)-type on Q_2.

Let $f : M \longrightarrow Q_2$ be a minimal immersion from a Riemann surface M into Q_2 and U be a domain of M. Locally, let $e = (e_A) : U \longrightarrow SO(4)$ be a moving

frame along f, i.e., $f(p) = [e_1(p) + \mathtt{i}e_2(p)]$ for $p \in U$. Set $\omega_{AB} = e^*\Omega_{AB}$ and $\omega_\alpha = (\omega_{\alpha 1} + \mathrm{i}\omega_{\alpha 2})/\sqrt{2}$. The induced metric on M is given by

$$ds^2 = \sum_\alpha \omega_\alpha \bar{\omega}_\alpha = \varphi\bar{\varphi}, \tag{2.3}$$

where φ are local one-forms of (1,0)-type, which is determined up to a factor of absolute value 1. Splitting φ into real and imaginary parts, i.e., $\varphi = \varphi_1 + \mathtt{i}\varphi_2$, the first structure equation of ds^2 is given by

$$d\varphi_1 = -\rho \wedge \varphi_2, \quad d\varphi_2 = \rho \wedge \varphi_1, \tag{2.4}$$

or equivalently

$$d\varphi = -\mathtt{i}\rho \wedge \varphi. \tag{2.5}$$

The Gauss curvature K satisfies the equation

$$d\rho = \frac{\mathtt{i}}{2} K\varphi \wedge \bar{\varphi}. \tag{2.6}$$

Define local complex-valued functions X_α and Y_α by

$$\omega_\alpha = X_\alpha \varphi + Y_\alpha \bar{\varphi}. \tag{2.7}$$

Set $X = (X_3, X_4)^T$ and $Y = (Y_3, Y_4)^T$. By the conformal condition (2.3) together with (2.7), we have

$$|X|^2 + |Y|^2 = 1 \tag{2.8}$$

and

$$X^T \bar{Y} = 0. \tag{2.9}$$

The Kähler angle [3] $\theta : M \longrightarrow [0, \pi]$ is given by

$$\cos\theta = |X|^2 - |Y|^2, \tag{2.10}$$

which gives a measure of the failure of f to be a holomorphic one. Explicitly, f is holomorphic (resp. totally real, anti-holomorphic) if $\theta = 0$ (resp. $\theta = \pi/2$, $\theta = \pi$).

Taking the exterior derivatives on both sides of (2.7) and using structure equation (2.1) and (2.5), we obtain

$$DX_\alpha \wedge \varphi + DY_\alpha \wedge \bar{\varphi} = 0,$$

where the covariant differentials DX_α, DY_α are defined by

$$\begin{aligned} DX_\alpha &= dX_\alpha - \mathtt{i}X_\alpha(\omega_{12} - \rho) + \sum_\beta \omega_{\alpha\beta} X_\beta, \\ DY_\alpha &= dY_\alpha - \mathtt{i}Y_\alpha(\omega_{12} + \rho) + \sum_\beta \omega_{\alpha\beta} Y_\beta. \end{aligned} \tag{2.11}$$

It is known that f is minimal if and only if

$$DX_\alpha = a_\alpha \varphi, \quad \text{or} \quad DY_\alpha = c_\alpha \bar{\varphi}, \tag{2.12}$$

where a_α and c_α are locally smooth complex-valued functions. By using (2.11) and (2.12), we also have

$$\begin{aligned} d(X^TX) &= 2\mathtt{i}(X^TX)(\omega_{12}-\rho)+2\sum_\alpha X_\alpha a_\alpha\varphi, \\ d(Y^TY) &= 2\mathtt{i}(Y^TY)(\omega_{12}+\rho)+2\sum_\alpha Y_\alpha c_\alpha\overline{\varphi}, \end{aligned} \tag{2.13}$$

which imply that X^TX and Y^TY are functions of analytic type (see the definition on page 56 in [16]). It is easy to check that $\tau_X = |X^TX|$ and $\tau_Y = |Y^TY|$ are globally defined. So, the zeroes set $\mathcal{Z}_{\tau_X}$ (resp. $\mathcal{Z}_{\tau_Y}$) of τ_X (resp. τ_Y) is either M or an isolated set. Hence, according to functions τ_X and τ_Y, minimal surfaces in Q_2 are divided into the following cases:

(I) both $\mathcal{Z}_{\tau_X}$ and $\mathcal{Z}_{\tau_Y}$ are isolated;
(II) $\mathcal{Z}_{\tau_X}$ is isolated and $\mathcal{Z}_{\tau_Y} = M$;
(III) $\mathcal{Z}_{\tau_X} = M$ and $\mathcal{Z}_{\tau_Y}$ is isolated;
(IV.a) f is holomorphic with $\mathcal{Z}_{\tau_X} = M$;
(IV.b) f is anti-holomorphic with $\mathcal{Z}_{\tau_Y} = M$;
(IV.c) f is neither holomorphic nor anti-holomorphic with $\mathcal{Z}_{\tau_X} = \mathcal{Z}_{\tau_Y} = M$.

3. Proof of Theorem A

In this section, we will adopt the notation in [11], in which they proved that minimal surfaces of parallel second fundamental form in Q_2 are totally geodesic for cases (II)–(IV.c). In the following, we will study minimal surfaces with parallel second fundamental form in Q_2 for the only remaining case (I).

First, we recall some results in [11]. Locally, we can choose a moving frame such that

$$\omega_3 = s\varphi + \mathtt{i}\frac{t\mu}{s}\overline{\varphi},\ \omega_4 = \mathtt{i}t\varphi + \mu\overline{\varphi},$$

where s and t are local real-valued functions satisfying $s > |t|$, and μ is a local complex-valued function (see Lemma 3.1 in [11]).

By (2.8) and (2.10), we can write $|X|^2 = \cos^2\frac{\theta}{2}$ and $|Y|^2 = \sin^2\frac{\theta}{2}$. Since $\tau_X = |X_3^2 + X_4^2| \le |X_3|^2 + |X_4|^2 = \cos^2\frac{\theta}{2}$, we define smooth function $\zeta : M \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ such that

$$\tau_X = \cos^2\frac{\theta}{2}\cos\zeta. \tag{3.1}$$

In fact, combining with the above frame, we have

$$s^2 + t^2 = \cos^2\frac{\theta}{2},\ s^2 - t^2 = \tau_X.$$

It's easy to get that

$$s = \cos\frac{\theta}{2}\cos\frac{\zeta}{2},\ t = \cos\frac{\theta}{2}\sin\frac{\zeta}{2},\ |\mu| = \sin\frac{\theta}{2}\cos\frac{\zeta}{2}.$$

Therefore, we have

$$\tau_Y = \sin^2\frac{\theta}{2}\cos\zeta.$$

Observe that f is neither holomorphic nor anti-holomorphic for case (I). Together with (2.9), we can write

$$\omega_3 = \cos\frac{\theta}{2}\cos\frac{\zeta}{2}\varphi + \mathtt{i}\sin\frac{\theta}{2}\sin\frac{\zeta}{2}e^{\mathtt{i}r}\bar{\varphi}, \quad \omega_4 = \mathtt{i}\cos\frac{\theta}{2}\sin\frac{\zeta}{2}\varphi + \sin\frac{\theta}{2}\cos\frac{\zeta}{2}e^{\mathtt{i}r}\bar{\varphi},$$

where r is a local real-valued smooth function. Without loss of generality, we suppose $r = 0$ for brevity. So, locally, we choose a moving frame along f such that

$$(\omega_3, \omega_4) = (\varphi, \bar{\varphi}) \begin{pmatrix} \cos\frac{\theta}{2}\cos\frac{\zeta}{2} & \mathtt{i}\cos\frac{\theta}{2}\sin\frac{\zeta}{2} \\ \mathtt{i}\sin\frac{\theta}{2}\sin\frac{\zeta}{2} & \sin\frac{\theta}{2}\cos\frac{\zeta}{2} \end{pmatrix}. \tag{3.2}$$

Let $\{\theta_A\}$ be another frame field on Q_2 which satisfies

$$\begin{aligned} \theta_1 + \mathtt{i}\theta_2 &= \cos\tfrac{\theta}{2}(\cos\tfrac{\zeta}{2}\Omega_3 - \mathtt{i}\sin\tfrac{\zeta}{2}\Omega_4) + \sin\tfrac{\theta}{2}(\mathtt{i}\sin\tfrac{\zeta}{2}\bar{\Omega}_3 + \cos\tfrac{\zeta}{2}\bar{\Omega}_4), \\ \theta_3 + \mathtt{i}\theta_4 &= \sin\tfrac{\theta}{2}(\cos\tfrac{\zeta}{2}\Omega_3 - \mathtt{i}\sin\tfrac{\zeta}{2}\Omega_4) - \cos\tfrac{\theta}{2}(\mathtt{i}\sin\tfrac{\zeta}{2}\bar{\Omega}_3 + \cos\tfrac{\zeta}{2}\bar{\Omega}_4). \end{aligned} \tag{3.3}$$

It follows that

$$(\theta_1, \theta_2, \theta_3, \theta_4) = \frac{1}{\sqrt{2}}(\Omega_{31}, \Omega_{32}, \Omega_{41}, \Omega_{42})A, \tag{3.4}$$

where

$$A = \begin{pmatrix} \cos\frac{\theta}{2}\cos\frac{\zeta}{2} & \sin\frac{\theta}{2}\sin\frac{\zeta}{2} & \sin\frac{\theta}{2}\cos\frac{\zeta}{2} & -\cos\frac{\theta}{2}\sin\frac{\zeta}{2} \\ \sin\frac{\theta}{2}\sin\frac{\zeta}{2} & \cos\frac{\theta}{2}\cos\frac{\zeta}{2} & -\cos\frac{\theta}{2}\sin\frac{\zeta}{2} & \sin\frac{\theta}{2}\cos\frac{\zeta}{2} \\ \sin\frac{\theta}{2}\cos\frac{\zeta}{2} & -\cos\frac{\theta}{2}\sin\frac{\zeta}{2} & -\cos\frac{\theta}{2}\cos\frac{\zeta}{2} & -\sin\frac{\theta}{2}\sin\frac{\zeta}{2} \\ \cos\frac{\theta}{2}\sin\frac{\zeta}{2} & -\sin\frac{\theta}{2}\cos\frac{\zeta}{2} & \sin\frac{\theta}{2}\sin\frac{\zeta}{2} & \cos\frac{\theta}{2}\cos\frac{\zeta}{2} \end{pmatrix} \in SO(4). \tag{3.5}$$

By using (3.2), we have

$$f^*(\theta_1 + \mathtt{i}\theta_2) = \varphi_1 + \mathtt{i}\varphi_2, \ f^*(\theta_3 + \mathtt{i}\theta_4) = 0. \tag{3.6}$$

This means $\{\theta_A\}$ is a Darboux frame associated to f.

We now calculate the connection one-forms θ_{AB} with respect to the frame fields $\{\theta_A\}$, which are determined by

$$d\theta_A = -\sum_B \theta_{AB} \wedge \theta_B, \ \theta_{AB} + \theta_{BA} = 0. \tag{3.7}$$

First, by using (2.1), we have

$$d\Omega_3 = \mathtt{i}\Omega_{12} \wedge \Omega_3 - \Omega_{34} \wedge \Omega_4, \quad d\Omega_4 = \mathtt{i}\Omega_{12} \wedge \Omega_4 + \Omega_{34} \wedge \Omega_3. \tag{3.8}$$

It is easy to get that

$$\begin{aligned} d(\cos\tfrac{\zeta}{2}\Omega_3 - \mathtt{i}\sin\tfrac{\zeta}{2}\Omega_4) &= \mathtt{i}\Omega_{12} \wedge (\cos\tfrac{\zeta}{2}\Omega_3 - \mathtt{i}\sin\tfrac{\zeta}{2}\Omega_4) \\ &\quad -\Omega_{34} \wedge (\mathtt{i}\sin\tfrac{\zeta}{2}\Omega_3 + \cos\tfrac{\zeta}{2}\Omega_4) \\ &\quad -\tfrac{d\zeta}{2} \wedge (\sin\tfrac{\zeta}{2}\Omega_3 + \mathtt{i}\cos\tfrac{\zeta}{2}\Omega_4), \\ d(\mathtt{i}\sin\tfrac{\zeta}{2}\bar{\Omega}_3 + \cos\tfrac{\zeta}{2}\bar{\Omega}_4) &= -\mathtt{i}\Omega_{12} \wedge (\mathtt{i}\sin\tfrac{\zeta}{2}\bar{\Omega}_3 + \cos\tfrac{\zeta}{2}\bar{\Omega}_4) \\ &\quad +\Omega_{34} \wedge (\cos\tfrac{\zeta}{2}\bar{\Omega}_3 - \mathtt{i}\sin\tfrac{\zeta}{2}\bar{\Omega}_4) \\ &\quad +\tfrac{d\zeta}{2} \wedge (\mathtt{i}\cos\tfrac{\zeta}{2}\bar{\Omega}_3 - \sin\tfrac{\zeta}{2}\bar{\Omega}_4). \end{aligned} \tag{3.9}$$

On the other hand, from (3.3), it follows that

$$\begin{aligned} \Omega_3 &= (\cos\tfrac{\theta}{2}\cos\tfrac{\zeta}{2} + \mathtt{i}\sin\tfrac{\theta}{2}\sin\tfrac{\zeta}{2})\theta_1 + (\mathtt{i}\cos\tfrac{\theta}{2}\cos\tfrac{\zeta}{2} + \sin\tfrac{\theta}{2}\sin\tfrac{\zeta}{2})\theta_2 \\ &\quad +(\sin\tfrac{\theta}{2}\cos\tfrac{\zeta}{2} - \mathtt{i}\cos\tfrac{\theta}{2}\sin\tfrac{\zeta}{2})\theta_3 + (-\cos\tfrac{\theta}{2}\sin\tfrac{\zeta}{2} + \mathtt{i}\sin\tfrac{\theta}{2}\cos\tfrac{\zeta}{2})\theta_4, \\ \Omega_4 &= (\mathtt{i}\cos\tfrac{\theta}{2}\sin\tfrac{\zeta}{2} + \sin\tfrac{\theta}{2}\cos\tfrac{\zeta}{2})\theta_1 + (-\cos\tfrac{\theta}{2}\sin\tfrac{\zeta}{2} - \mathtt{i}\sin\tfrac{\theta}{2}\cos\tfrac{\zeta}{2})\theta_2 \\ &\quad +(\mathtt{i}\sin\tfrac{\theta}{2}\sin\tfrac{\zeta}{2} - \cos\tfrac{\theta}{2}\cos\tfrac{\zeta}{2})\theta_3 + (-\sin\tfrac{\theta}{2}\sin\tfrac{\zeta}{2} + \mathtt{i}\cos\tfrac{\theta}{2}\cos\tfrac{\zeta}{2})\theta_4. \end{aligned} \tag{3.10}$$

Thus, we obtain the following:

$$\cos\frac{\zeta}{2}\Omega_3 - \mathtt{i}\sin\frac{\zeta}{2}\Omega_4 = \cos\frac{\theta}{2}(\theta_1+\mathtt{i}\theta_2) + \sin\frac{\theta}{2}(\theta_3+\mathtt{i}\theta_4),$$
$$\sin\frac{\zeta}{2}\Omega_3 + \mathtt{i}\cos\frac{\zeta}{2}\Omega_4 = \sin\frac{\theta}{2}(\mathtt{i}\theta_1+\theta_2) - \cos\frac{\theta}{2}(\mathtt{i}\theta_3+\theta_4),$$
$$\mathtt{i}\sin\frac{\zeta}{2}\bar{\Omega}_3 + \cos\frac{\zeta}{2}\bar{\Omega}_4 = \sin\frac{\theta}{2}(\theta_1+\mathtt{i}\theta_2) - \cos\frac{\theta}{2}(\theta_3+\mathtt{i}\theta_4),$$
$$\mathtt{i}\cos\frac{\zeta}{2}\bar{\Omega}_3 - \sin\frac{\zeta}{2}\bar{\Omega}_4 = \cos\frac{\theta}{2}(\mathtt{i}\theta_1+\theta_2) + \sin\frac{\theta}{2}(\mathtt{i}\theta_3+\theta_4),$$

$$\begin{aligned}\mathtt{i}\sin\tfrac{\zeta}{2}\Omega_3 + \cos\tfrac{\zeta}{2}\Omega_4 &= (\mathtt{i}\cos\tfrac{\theta}{2}\sin\zeta + \sin\tfrac{\theta}{2}\cos\zeta)\theta_1 - (\cos\tfrac{\theta}{2}\sin\zeta + \mathtt{i}\sin\tfrac{\theta}{2}\cos\zeta)\theta_2\\ &\quad +(\mathtt{i}\sin\tfrac{\theta}{2}\sin\zeta - \cos\tfrac{\theta}{2}\cos\zeta)\theta_3 + (\mathtt{i}\cos\tfrac{\theta}{2}\cos\zeta - \sin\tfrac{\theta}{2}\sin\zeta)\theta_4,\\ \cos\tfrac{\zeta}{2}\bar{\Omega}_3 - \mathtt{i}\sin\tfrac{\zeta}{2}\bar{\Omega}_4 &= (\cos\tfrac{\theta}{2}\cos\zeta - \mathtt{i}\sin\tfrac{\theta}{2}\sin\zeta)\theta_1 - (\mathtt{i}\cos\tfrac{\theta}{2}\cos\zeta - \sin\tfrac{\theta}{2}\sin\zeta)\theta_2\\ &\quad +(\sin\tfrac{\theta}{2}\cos\zeta + \mathtt{i}\cos\tfrac{\theta}{2}\sin\zeta)\theta_3 - (\cos\tfrac{\theta}{2}\sin\zeta + \mathtt{i}\sin\tfrac{\theta}{2}\cos\zeta)\theta_4.\end{aligned}$$

Taking the exterior derivatives on both sides of the second equation in (3.3) and using (3.9) and (3.10), we obtain

$$\begin{aligned}d(\theta_1+\mathtt{i}\theta_2) &= \mathtt{i}(\cos\theta\Omega_{12} - \sin\zeta\Omega_{34})\wedge(\theta_1+\mathtt{i}\theta_2)\\ &\quad +(\mathtt{i}\tfrac{d\zeta}{2} + \cos\zeta\Omega_{34})\wedge(\theta_3-\mathtt{i}\theta_4)\\ &\quad -(\tfrac{d\theta}{2} - \mathtt{i}\sin\theta\Omega_{12})\wedge(\theta_3+\mathtt{i}\theta_4)\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}d(\theta_3+\mathtt{i}\theta_4) &= -\mathtt{i}(\cos\theta\Omega_{12} + \sin\zeta\Omega_{34})\wedge(\theta_3+\mathtt{i}\theta_4)\\ &\quad -(\mathtt{i}\tfrac{d\zeta}{2} + \cos\zeta\Omega_{34})\wedge(\theta_1-\mathtt{i}\theta_2)\\ &\quad +(\tfrac{d\theta}{2} + \mathtt{i}\sin\theta\Omega_{12})\wedge(\theta_1+\mathtt{i}\theta_2).\end{aligned} \tag{3.12}$$

Comparing with (3.7), the connection one-forms θ_{AB} are given by

$$\begin{aligned}\theta_{13} &= \frac{d\theta}{2} - \cos\zeta\Omega_{34}, & \theta_{23} &= -\frac{d\zeta}{2} - \sin\theta\Omega_{12},\\ \theta_{14} &= -\frac{d\zeta}{2} + \sin\theta\Omega_{12}, & \theta_{24} &= \frac{d\theta}{2} + \cos\zeta\Omega_{34},\\ \theta_{12} &= \cos\theta\Omega_{12} - \sin\zeta\Omega_{34},\\ \theta_{34} &= -\cos\theta\Omega_{12} - \sin\zeta\Omega_{34}.\end{aligned} \tag{3.13}$$

Note that along f, equation (3.12) becomes

$$(\frac{d\theta}{2} + \mathtt{i}\sin\theta\omega_{12})\wedge\varphi - (\mathtt{i}\frac{d\zeta}{2} + \cos\zeta\omega_{34})\wedge\bar{\varphi} = 0. \tag{3.14}$$

By using the minimality of f, we can write

$$d\theta + \mathtt{i}2\sin\theta\omega_{12} = 2a\varphi, \tag{3.15}$$

$$d\zeta + \mathtt{i}2\cos\zeta\omega_{34} = 2c\varphi, \tag{3.16}$$

where a and c are local complex-valued smooth functions. The square of the length of the second fundamental form B is given by

$$\|B\|^2 = 4(|a|^2 + |c|^2). \tag{3.17}$$

By comparing the real part of (3.15) and (3.16) respectively, we obtain

$$|d\theta|^2 = 4|a|^2, \quad |d\zeta|^2 = 4|c|^2. \tag{3.18}$$

The curvature forms Ψ_{AB} on Q_2 are determined by

$$\Psi_{AB} = d\theta_{AB} + \theta_{AC}\wedge\theta_{CB}, \tag{3.19}$$

where

$$\Psi_{AB} = \frac{1}{2}\sum_{C,D} K_{ABCD}\theta_C \wedge \theta_D, \quad K_{ABCD} = -K_{ABDC}.$$

From (3.13) we have

$$f^*\theta_{12} = \cos\theta\omega_{12} - \sin\zeta\omega_{34} \tag{3.20}$$

and

$$f^*\theta_{34} = -\cos\theta\omega_{12} - \sin\zeta\omega_{34}. \tag{3.21}$$

Taking the exterior derivative on both sides of equation (3.20), together with (2.1), (3.15) and (3.18), we get the Gauss equation

$$K = 2\cos^2\theta + 2\sin^2\zeta - 2(|a|^2 + |c|^2). \tag{3.22}$$

Similarly, from (3.21) we obtain the Ricci equation

$$K^\perp = 2\sin^2\zeta - 2\cos^2\theta + 2(|a|^2 - |c|^2), \tag{3.23}$$

where $K^\perp$ is the normal curvature. From (3.22) and (3.23), it follows that

$$K - K^\perp = 4(\cos^2\theta - |a|^2), \tag{3.24}$$

$$K + K^\perp = 4(\sin^2\zeta - |c|^2). \tag{3.25}$$

Similarly, by routine calculations, we also have

$$K_{3112} = -\sin 2\zeta,\ K_{3212} = \sin 2\theta,\ K_{4112} = -\sin 2\theta,\ K_{4212} = \sin 2\zeta. \tag{3.26}$$

Proof of Theorem A. We only prove the Case (I) here, for the Cases (II)–(IV.c), one can refer to [10] and [11].

Set $f^*\theta_{\alpha i} = h^\alpha_{ij}\varphi_i$. The Codazzi equations are given by

$$h^\alpha_{ijk} - h^\alpha_{ikj} = -K^\alpha_{ijk}, \tag{3.27}$$

where

$$Dh^\alpha_{ij} = \sum_k h^\alpha_{ijk}\varphi_k = dh^\alpha_{ij} - h^\alpha_{kj}f^*\theta_{ki} - h^\alpha_{ik}f^*\theta_{kj} + h^\beta_{ij}f^*\theta_{\alpha\beta}. \tag{3.28}$$

The parallel second fundamental form implies that $h^\alpha_{ijk} = 0$. So, by (3.26) and (3.27) we obtain

$$\sin 2\theta = \sin 2\zeta = 0, \tag{3.29}$$

which implies both θ and ζ are constants. From (3.1), $\theta = \pi/2$ and $\zeta = 0$. By using (3.15) and (3.16), we obtain that f is totally geodesic with $K = K^\perp = 0$, and $\omega_{12} = \omega_{34} = 0$. Therefore, the pull back of the Maurer-Cartan forms ω_{AB} are given by

$$\begin{pmatrix} 0 & 0 & -\varphi_1 & -\varphi_1 \\ 0 & 0 & -\varphi_2 & \varphi_2 \\ \varphi_1 & \varphi_2 & 0 & 0 \\ \varphi_1 & -\varphi_2 & 0 & 0 \end{pmatrix}. \tag{3.30}$$

Comparing with Example 4.4 in [15], M is part of a totally real flat torus. More explicitly, M is a part of the orbit of the Lie group

$$T^2 = \left\{ \begin{pmatrix} \cos x & -\sin x & 0 & 0 \\ \sin x & \cos x & 0 & 0 \\ 0 & 0 & \cos y & -\sin y \\ 0 & 0 & \sin y & \cos y \end{pmatrix} \;\middle|\; x, y \in \mathbb{R} \right\}$$

through the base point $[(1, 0, \mathtt{i}, 0)^T]$. This completes the proof. □

4. Proof of Theorem B

In this section, we study the minimal surfaces with constant Gauss curvature and constant normal curvature in Q_2, by using the Laplacian of the globally defined functions θ, ζ, $|a|^2$ and $|c|^2$.

First, we use the Laplacian operator $\Delta = *d * d$, where $*$ is the Hodge-star operator with respect to the metric ds^2. So, from (3.15), we obtain

$$*d\theta = 2\sin\theta\omega_{12}.$$

From (2.1), (2.7), (2.10) and (3.15), we have

$$d * d\theta = (4|a|^2 \cot\theta + 2\sin 2\theta)\varphi_1 \wedge \varphi_2.$$

It follows that

$$\Delta\theta = 4|a|^2 \cot\theta + 2\sin 2\theta. \tag{4.1}$$

Similarly, from (3.16) we have

$$\Delta\zeta = -4|c|^2 \tan\zeta - 2\sin 2\zeta. \tag{4.2}$$

On the other hand, to calculate $\Delta|a|^2$ and $\Delta|c|^2$, we will work in the holomorphic coordinate of M. Let $\{z\}$ be a holomorphic coordinate of M. So, φ can be written as $\varphi = \lambda dz$ for a local positive function λ. Then, we have

$$\Delta = \frac{4}{\lambda^2}\frac{\partial^2}{\partial z \partial \bar{z}} \tag{4.3}$$

and the Gauss curvature

$$K = -\Delta \log\lambda. \tag{4.4}$$

In the following, we assume that the Gauss curvature K and the normal curvature $K^\perp$ are constants. By using (3.15), we obtain

$$\frac{\partial\theta}{\partial z} = \lambda a. \tag{4.5}$$

Taking partial derivative $\frac{\partial}{\partial\bar{z}}$ on both sides of (4.5), we have

$$\frac{\partial^2\theta}{\partial z\partial\bar{z}} = a\frac{\partial\lambda}{\partial\bar{z}} + \lambda\frac{\partial a}{\partial\bar{z}}. \tag{4.6}$$

Together with (4.1) and (4.3), we obtain

$$\frac{\partial a}{\partial\bar{z}} = \lambda\left(|a|^2\cot\theta + \frac{\sin 2\theta}{2}\right) - a\frac{\partial\log\lambda}{\partial\bar{z}}. \tag{4.7}$$

Taking the partial derivative $\frac{\partial}{\partial z}$ on both sides of (3.24) and using (4.5) and (4.7), we obtain

$$\bar{a}\frac{\partial a}{\partial z} = |a|^2\frac{\partial \log\lambda}{\partial z} - \lambda a\left(|a|^2\cot\theta + \frac{3\sin 2\theta}{2}\right). \tag{4.8}$$

Similarly, by taking the partial derivative $\frac{\partial}{\partial \bar{z}}$ on both side of (4.7) and using (4.4) and (4.8), we obtain

$$\begin{aligned}\bar{a}\tfrac{\partial^2 a}{\partial z\partial\bar{z}} &= \lambda^2|a|^2\left(\tfrac{K}{4} - 1 - \tfrac{|a|^2}{\sin^2\theta}\right) - |a|^2\left|\tfrac{\partial\log\lambda}{\partial z}\right|^2 \\ &\quad +\bar{a}\tfrac{\partial\lambda}{\partial z}\left(|a|^2\cot\theta + \tfrac{\sin 2\theta}{2}\right) + a\tfrac{\partial\bar{\lambda}}{\partial\bar{z}}\left(|a|^2\cot\theta + \tfrac{3\sin 2\theta}{2}\right),\end{aligned}$$

which implies

$$\begin{aligned}\bar{a}\tfrac{\partial^2 a}{\partial z\partial\bar{z}} + a\tfrac{\partial^2\bar{a}}{\partial z\partial\bar{z}} &= \lambda^2|a|^2\left(\tfrac{K}{2} - 2 - 2\tfrac{|a|^2}{\sin^2\theta}\right) - 2|a|^2\left|\tfrac{\partial\log\lambda}{\partial z}\right|^2 \\ &\quad +2\left(\bar{a}\tfrac{\partial\lambda}{\partial z} + a\tfrac{\partial\bar{\lambda}}{\partial\bar{z}}\right)\left(|a|^2\cot\theta + \sin 2\theta\right).\end{aligned} \tag{4.9}$$

From (4.7) and (4.8), we have

$$\begin{aligned}|\tfrac{\partial a}{\partial\bar{z}}|^2 &= \lambda^2\left(|a|^2\cot\theta + \tfrac{\sin 2\theta}{2}\right)^2 + |a|^2|\tfrac{\partial\log\lambda}{\partial\bar{z}}|^2 \\ &\quad -\left(|a|^2\cot\theta + \tfrac{\sin 2\theta}{2}\right)(\bar{a}\tfrac{\partial\lambda}{\partial z} + a\tfrac{\partial\bar{\lambda}}{\partial\bar{z}})\end{aligned} \tag{4.10}$$

and

$$\begin{aligned}|\tfrac{\partial a}{\partial z}|^2 &= \lambda^2\left(|a|^2\cot\theta + \tfrac{3\sin 2\theta}{2}\right)^2 + |a|^2|\tfrac{\partial\log\lambda}{\partial z}|^2 \\ &\quad -\left(|a|^2\cot\theta + \tfrac{3\sin 2\theta}{2}\right)(\bar{a}\tfrac{\partial\lambda}{\partial z} + a\tfrac{\partial\bar{\lambda}}{\partial\bar{z}}).\end{aligned} \tag{4.11}$$

By (4.3), we have

$$\Delta|a|^2 = \frac{4}{\lambda^2}\left(\bar{a}\frac{\partial^2 a}{\partial z\partial\bar{z}} + a\frac{\partial^2\bar{a}}{\partial z\partial\bar{z}} + |\frac{\partial a}{\partial\bar{z}}|^2 + |\frac{\partial a}{\partial z}|^2\right).$$

So, by using (4.9)–(4.11), we obtain

$$\Delta|a|^2 = 2|a|^2(K - 4 - 4|a|^2 + 16\cos^2\theta) + 10\sin^2 2\theta. \tag{4.12}$$

Notice that $\Delta|a|^2 = \Delta\cos^2\theta$ by (3.24). So, by using (3.18), (4.1) gives

$$\Delta|a|^2 = 8|a|^2(1 - 3\cos^2\theta) - 2\sin^2 2\theta. \tag{4.13}$$

By comparing (4.12) and (4.13) and using (3.24), we get

$$(20K^\perp - 16K + 64)\cos^2\theta + (K^\perp - K)(2K - K^\perp - 8) = 0. \tag{4.14}$$

Similarly, from (3.16), it follows that

$$\frac{\partial\zeta}{\partial z} = \lambda c. \tag{4.15}$$

Taking the partial derivative $\frac{\partial}{\partial\bar{z}}$ on both side of (4.15), we have

$$\frac{\partial^2\zeta}{\partial z\partial\bar{z}} = c\frac{\partial\lambda}{\partial\bar{z}} + \lambda\frac{\partial c}{\partial\bar{z}}. \tag{4.16}$$

Together with (4.2) and (4.3), we obtain

$$\frac{\partial c}{\partial\bar{z}} = -\lambda\left(|c|^2\tan\zeta + \frac{\sin 2\zeta}{2}\right) - c\frac{\partial\log\lambda}{\partial\bar{z}}. \tag{4.17}$$

Taking the partial derivative $\frac{\partial}{\partial z}$ on both side of (3.25), with (4.15) and (4.17), we obtain

$$\bar{c}\frac{\partial c}{\partial z} = |c|^2\frac{\partial \log\lambda}{\partial z} + \lambda c\left(|c|^2\tan\zeta + \frac{3\sin 2\zeta}{2}\right). \tag{4.18}$$

Taking the partial derivative $\frac{\partial}{\partial z}$ on both sides of (4.17), together with (4.4) and (4.18), we obtain

$$\begin{aligned}\bar{c}\tfrac{\partial^2 c}{\partial z\partial\bar{z}} &= \lambda^2|c|^2\left(\tfrac{K}{4} - 1 - \tfrac{|c|^2}{\cos^2\zeta}\right) - |c|^2\left|\tfrac{\partial\log\lambda}{\partial z}\right|^2 \\ &\quad -\bar{c}\tfrac{\partial\lambda}{\partial z}\left(|c|^2\tan\zeta + \tfrac{\sin 2\zeta}{2}\right) - c\tfrac{\partial\lambda}{\partial\bar{z}}\left(|c|^2\tan\zeta + \tfrac{3\sin 2\zeta}{2}\right).\end{aligned}$$

As before, we obtain

$$\begin{aligned}\bar{c}\tfrac{\partial^2 c}{\partial z\partial\bar{z}} + c\tfrac{\partial^2\bar{c}}{\partial z\partial\bar{z}} &= \lambda^2|c|^2\left(\tfrac{K}{2} - 2 - \tfrac{2|c|^2}{\cos^2\zeta}\right) - 2|c|^2\left|\tfrac{\partial\log\lambda}{\partial z}\right|^2 \\ &\quad -2\left(\bar{c}\tfrac{\partial\lambda}{\partial z} + c\tfrac{\partial\lambda}{\partial\bar{z}}\right)\left(|c|^2\tan\zeta + \sin 2\zeta\right).\end{aligned} \tag{4.19}$$

By using (4.18) and (4.18), we obtain

$$\begin{aligned}|\tfrac{\partial c}{\partial\bar{z}}|^2 &= \lambda^2\left(|c|^2\tan\zeta + \tfrac{\sin 2\zeta}{2}\right)^2 + |c|^2|\tfrac{\partial\log\lambda}{\partial\bar{z}}|^2 \\ &\quad + \left(|c|^2\tan\zeta + \tfrac{\sin 2\zeta}{2}\right)(\bar{c}\tfrac{\partial\lambda}{\partial z} + c\tfrac{\partial\lambda}{\partial\bar{z}}),\end{aligned} \tag{4.20}$$

$$\begin{aligned}|\tfrac{\partial c}{\partial z}|^2 &= \lambda^2\left(|c|^2\tan\zeta + \tfrac{3\sin 2\zeta}{2}\right)^2 + |c|^2|\tfrac{\partial\log\lambda}{\partial\bar{z}}|^2 \\ &\quad + \left(|c|^2\tan\zeta + \tfrac{3\sin 2\zeta}{2}\right)(\bar{c}\tfrac{\partial\lambda}{\partial z} + c\tfrac{\partial\lambda}{\partial\bar{z}}).\end{aligned} \tag{4.21}$$

Similarly, we have

$$\Delta|c|^2 = 2|c|^2(K - 4 - 4|c|^2 + 16\sin^2\zeta) + 10\sin^2 2\zeta. \tag{4.22}$$

Notice that $\Delta|c|^2 = \Delta\sin^2\zeta$ by (3.25). So, by using (3.18) and (4.2), we obtain

$$\Delta|c|^2 = 8|c|^2(1 - 3\sin^2\zeta) - 2\sin^2 2\zeta. \tag{4.23}$$

By comparing (4.22) and (4.23) and using (3.25), we have

$$(20K^\perp + 16K - 64)\sin^2\zeta + (K + K^\perp)(2K + K^\perp - 8) = 0. \tag{4.24}$$

Proof of Theorem B. If K and $K^\perp$ are constants, identities (4.14) and (4.24) imply θ and ζ are constants. It is easy to check that f is totally geodesic with $\theta = \frac{\pi}{2}$ and $\zeta = 0$. So, by Theorem A, we know that f is congruent to part of the flat torus. This completes the proof. □

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School of Mathematics Sciences and Institute of Mathematics, Nanjing Normal University, Nanjing 210023, People's Republic of China

E-mail address: wangjun706@mails.ucas.ac.cn

School of Mathematical Sciences, University of Science and Technology of China, Hefei, 230026, Anhui province, People's Republic of China; and Wu Wen-Tsun Key Laboratory of Mathematics, USTC, Chinese Academy of Sciences, Hefei, 230026, Anhui, People's Republic of China

E-mail address: xwxu09@ustc.edu.cn