PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 143, Number 7, July 2015, Pages 2959–2968
S 0002-9939(2015)12475-6
Article electronically published on February 13, 2015

A CHARACTERIZATION FOR ELLIPTIC PROBLEMS ON FRACTAL SETS

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(Communicated by Catherine Sulem)

Abstract. In this paper we prove a characterization theorem on the existence of one non-zero strong solution for elliptic equations defined on the Sierpiński gasket. More generally, the validity of our result can be checked studying elliptic equations defined on self-similar fractal domains whose spectral dimension $\nu \in (0,2)$. Our theorem can be viewed as an elliptic version on fractal domains of a recent contribution obtained in a recent work of Ricceri for a two-point boundary value problem.

1. Introduction

In recent years a great deal of effort has gone into investigating PDEs on fractals (see, for instance, [6, 10, 12, 21] and the excellent monograph [7]). A major difficulty is how to define differential operators on non-smooth sets. Analysis on fractal sets has been made possible by the definition of operators that play the role of the Laplacian.

Originally defined as a by-product of the construction of the analog of Brownian motion [1], these Laplace-type operators have been shown by direct limit-of-difference-quotient definitions in the papers by Kigami [13–16], for a class of self-similar fractals that includes the *Sierpiński gasket*. In this way, elliptic equations have been studied by using a suitable energy functional defined on an appropriate Hilbert space (see [2, 3, 8, 11]).

Motivated by this large interest in the current literature, the purpose of this paper is to prove a characterization result on the existence of non-negative and non-zero strong solutions for the following Dirichlet problem:

$$(S^f_{\lambda,\alpha}) \qquad \begin{cases} \Delta u(x) = \lambda\alpha(x)f(u(x)) & x \in V \setminus V_0, \\ u|_{V_0} = 0, \end{cases}$$

where V stands for the Sierpiński gasket in $(\mathbb{R}^{N-1}, |\cdot|)$, $N \geq 2$, V_0 is its intrinsic boundary (consisting of its N corners), Δ denotesthe weak Laplacian on V and λ

Received by the editors December 27, 2013 and, in revised form, February 3, 2014 and February 9, 2014.

2010 *Mathematics Subject Classification.* Primary 35J20; Secondary 28A80, 35J25, 35J60, 47J30, 49J52.

Key words and phrases. Sierpiński gasket, non-linear elliptic equation, Dirichlet form, weak Laplacian.

is a positive real parameter. We assume that

(h_f) $f : [0, +\infty) \to [0, +\infty)$ is a continuous function with $f(0) = 0$ and such that, for some $a > 0$, the map $h_F : (0, +\infty) \to [0, +\infty)$ defined by

$$h_F(\xi) := \frac{F(\xi)}{\xi^2}$$

is non-increasing in the real interval $(0, a]$, where

$$F(\xi) := \int_0^\xi f(t)dt$$

for each $\xi \in [0, +\infty)$.

We assume that the variable potential $\alpha : V \to \mathbb{R}$ satisfies the following hypothesis:

(h_α) $\alpha \in C(V)$ with $\alpha(x) < 0$, for every $x \in V$.

The main result of this paper is the following.

Theorem 1.1. *Assume that hypotheses* (h_f) *and* (h_α) *are fulfilled. Then, the following properties are equivalent*:

(i_1) h_F *is not constant in* $(0, b]$ *for each* $b > 0$;

(i_2) *for each* $r > 0$ *there exists an open interval* $I \subseteq (0, +\infty)$ *such that, for every* $\lambda \in I$, *problem* $(S^f_{\lambda,\alpha})$ *has a strong non-negative and non-zero solution, whose norm in* $H^1_0(V)$ *is less than* r.

The above characterization can be regarded as an elliptic version, for some classes of fractal sets, of a very recent result obtained by Ricceri for a two-point boundary value problem (see [19, Theorem 1]).

The extension of the cited result to $(S^f_{\lambda,\alpha})$ is not trivial and is required to overcome some difficulties which arise in this new geometrical context. In particular, some analytical properties on the Hilbert space $H^1_0(V)$ and the distribution of the spectrum of the corresponding linear problem defined on fractal sets, need special care. More precisely, in our setting, a key ingredient is the validity of the following Morrey-type inequality:

$$\sup_{x,y \in V_*} \frac{|u(x) - u(y)|}{|x - y|^\sigma} \leq (2N + 3)\sqrt{W(u)}, \tag{1}$$

where

$$\sigma := \frac{\log((N + 2)/N)}{2 \log 2},$$

and V_* is defined as in the next section (see [8, Lemma 2.4] for details). Inequality (1) and the Arzela-Ascoli theorem yield that the embedding

$$H^1_0(V) \hookrightarrow C_0(V) \tag{2}$$

is compact (see [9]). This fact will be crucial in our approach.

Taking into account the results contained in [6], our method adopted here can be useful for studying the existence of weak solutions for elliptic equations defined on self-similar sets, whose spectral dimension $\nu \in (0, 2)$. In such a case, the Laplacian may be defined via a suitable Dirichlet form, following the variational fractal approach developed by Mosco in [17]. An open and more delicate problem is to attack the case $\nu \geq 2$ in which the compact embedding (2) is false.

We emphasize that, as suggested by Ricceri, a possible extension of his result to the elliptic case requires a more sophisticated and delicate analysis also for equations involving the classical Laplacian and defined on bounded Euclidean domains.

This paper is organized as follows. In Section 2 we recall the geometrical construction of the Sierpiński gasket and our variational framework. Successively, Section 3 is devoted to the proof of the main theorem.

We refer to the recent book by Ciarlet [4] as a general reference for the basic notions used in the present paper.

2. Preliminaries and abstract result

Let $N \geq 2$ be a natural number and let $p_1, \ldots, p_N \in \mathbb{R}^{N-1}$ be so that $|p_i - p_j| = 1$ for $i \neq j$. Define, for every $i \in \{1, \ldots, N\}$, the map $S_i\colon \mathbb{R}^{N-1} \to \mathbb{R}^{N-1}$ by

$$S_i(x) = \frac{1}{2}\, x + \frac{1}{2}\, p_i\,.$$

Let $\mathcal{S} := \{S_1, \ldots, S_N\}$ and denote by $L\colon \mathcal{P}(\mathbb{R}^{N-1}) \to \mathcal{P}(\mathbb{R}^{N-1})$ the map assigning to a subset A of $\mathbb{R}^{N-1}$ the set

$$L(A) = \bigcup_{i=1}^{N} S_i(A).$$

It is known that there is a unique non-empty compact subset V of $\mathbb{R}^{N-1}$, called the *attractor* of the family $\mathcal{S}$, such that $L(V) = V$; see, Theorem 9.1 in Falconer [7]. The set V is called the *Sierpiński gasket* in $\mathbb{R}^{N-1}$ of *intrinsic boundary* $V_0 := \{p_1, \ldots, p_N\}$.

Consider H to be the convex hull of the set V_0 and observe that S satisfies the *open set condition* (see [7, p. 129]) taking $\operatorname{int}(H)$ the interior of H, which is a non-empty bounded open set such that

$$\bigcup_{i=1}^{N} S_i(\operatorname{int}(H)) \subset \operatorname{int}(H).$$

Since the above condition holds and V is the attractor of $\mathcal{S}$, applying [7, Theorem 9.3], we deduce that V has Hausdorff and box dimensions equal to the value of d satisfying

$$\sum_{i=1}^{N} \frac{1}{2^d} = 1, \tag{3}$$

and we also have $\mathcal{H}^d(V) \in (0, +\infty)$, where $\mathcal{H}^d$ is the d-dimensional Hausdorff measure on $\mathbb{R}^{N-1}$. By relation (3), we immediately get that $d = \log N / \log 2$. Let μ be the normalized restriction of the Hausdorff measure $\mathcal{H}^d$ on $\mathbb{R}^{N-1}$ to the subsets of V, so $\mu(V) = 1$.

We also recall, for completeness, that if $0 \leq d < \infty$, $0 < \delta < \infty$ and $A \subset \mathbb{R}^k$, then

$$\mathcal{H}^d(A) = \lim_{\delta \to 0+} \mathcal{H}^d_\delta(A) = \sup_{\delta > 0} \mathcal{H}^d_\delta(A),$$

where

$$\mathcal{H}^d_\delta(A) = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2} + 1\right)} \inf \left\{ \sum_{j=1}^{\infty} \left(\frac{\operatorname{diam} C_j}{2} \right)^d ;\; A \subset \bigcup_{j=1}^{\infty} C_j,\; \operatorname{diam} C_j \leq \delta \right\}$$

and

$$\Gamma(s) = \int_0^\infty e^{-x}x^{s-1}dx$$

is Euler's Gamma function (see Evans and Gariepy [5, p. 60] for details).

Further, the following property of μ will be useful in the sequel:

$$\mu(B) > 0, \text{ for every non-empty open subset } B \text{ of } V. \tag{4}$$

In other words, the support of μ coincides with V; see, for instance, Breckner, Rădulescu and Varga [2] for more details.

Denote by $C(V)$ the space of real-valued continuous functions on V and by

$$C_0(V) := \{u \in C(V) \mid u|_{V_0} = 0\}.$$

The spaces $C(V)$ and $C_0(V)$ are endowed with the usual supremum norm $\|\cdot\|_\infty$. For a function $u\colon V \to \mathbb{R}$ and for $m \in \mathbb{N}$ let

$$W_m(u) = \left(\frac{N+2}{N}\right)^m \sum_{\substack{x,y\in V_m\\ |x-y|=2^{-m}}} (u(x)-u(y))^2, \tag{5}$$

where $V_m := L(V_{m-1})$, for $m \geq 1$ and $V_* := \bigcup_{m\geq 0} V_m$.

We have $W_m(u) \leq W_{m+1}(u)$ for very natural m, so we can put

$$W(u) = \lim_{m\to\infty} W_m(u). \tag{6}$$

Now define

$$H_0^1(V) := \{u \in C_0(V) \mid W(u) < \infty\}.$$

It turns out that $H_0^1(V)$ is a dense linear subset of $L^2(V,\mu)$ equipped with the $\|\cdot\|_2$ norm. We now endow $H_0^1(V)$ with the norm

$$\|u\| = \sqrt{W(u)}.$$

In fact, there is an inner product defining this norm: for $u, v \in H_0^1(V)$ and $m \in \mathbb{N}$ let

$$\mathcal{W}_m(u,v) = \left(\frac{N+2}{N}\right)^m \sum_{\substack{x,y\in V_m\\ |x-y|=2^{-m}}} (u(x)-u(y))(v(x)-v(y)).$$

Set

$$\mathcal{W}(u,v) = \lim_{m\to\infty} \mathcal{W}_m(u,v).$$

Then $\mathcal{W}(u,v) \in \mathbb{R}$ and the space $H_0^1(V)$, equipped with the inner product $\mathcal{W}$, which induces the norm $\|\cdot\|$, becomes a real Hilbert space.

We now state a useful property of the space $H_0^1(V)$ which shows, together with the facts that $(H_0^1(V), \|\cdot\|)$ is a Hilbert space and $H_0^1(V)$ is dense in $L^2(V,\mu)$, that $\mathcal{W}$ is a Dirichlet form on $L^2(V,\mu)$.

Lemma 2.1. *Let $h\colon \mathbb{R} \to \mathbb{R}$ be a Lipschitz mapping with Lipschitz constant $L \geq 0$ and such that $h(0) = 0$. Then, for every $u \in H_0^1(V)$, we have $h \circ u \in H_0^1(V)$ and $\|h \circ u\| \leq L\|u\|$.*

Following Falconer and Hu [8] we can define in a standard way a linear self-adjoint operator $\Delta\colon Z \to L^2(V,\mu)$, where Z is a linear subset of $H_0^1(V)$ which is dense in $L^2(V,\mu)$ (and dense also in $(H_0^1(V), \|\cdot\|)$), such that

$$-\mathcal{W}(u,v) = \int_V \Delta u \cdot v d\mu, \text{ for every } (u,v) \in Z \times H_0^1(V).$$

The operator Δ is called the (*weak*) *Laplacian* on V.

Precisely, let $H^{-1}(V)$ be the closure of $L^2(V,\mu)$ with respect to the pre-norm

$$\|u\|_{-1} = \sup_{\substack{h\in H_0^1(V)\\ \|h\|=1}} |\langle u,h\rangle|,$$

where

$$\langle v,h\rangle = \int_V v(x)h(x)d\mu, \qquad v\in L^2(V,\mu),\ h\in H_0^1(V).$$

Then $H^{-1}(V)$ is a Hilbert space. Then, the relation

$$-\mathcal{W}(u,v) = \langle \Delta u, v\rangle, \quad \forall v\in H_0^1(V),$$

uniquely defines a function $\Delta u\in H^{-1}(V)$ for every $u\in H_0^1(V)$.

Finally, fix $\lambda>0$. Let $\alpha\colon V\to\mathbb{R}$ and $f:\mathbb{R}\to\mathbb{R}$ be as in the Introduction. We say that a function $u\in H_0^1(V)$ is a *weak solution* of problem $(S^f_{\lambda,\alpha})$ if

$$\mathcal{W}(u,v) = -\lambda\int_V \alpha(x)f(u(x))v(x)d\mu,$$

for every $v\in H_0^1(V)$.

While we mainly work with the weak Laplacian, there is also a directly defined version. We say that Δ_s is the *standard Laplacian* of u if $\Delta_s u: V\to\mathbb{R}$ is continuous and

$$\lim_{m\to\infty}\sup_{x\in V\setminus V_0} |(N+2)^m(H_m u)(x) - \Delta_s u(x)| = 0,$$

where

$$(H_m u)(x) := \sum_{\substack{y\in V_m\\ |x-y|=2^{-m}}} (u(y)-u(x)),$$

for $x\in V_m$. We say that $u\in C_0(V)$ is a *strong solution* of problem $(S^f_{\lambda,\alpha})$ if $\Delta_s u$ exists and is continuous for all $x\in V\setminus V_0$, and

$$\Delta_s u(x) = \lambda\alpha(x)f(u(x)), \quad \forall\, x\in V\setminus V_0.$$

The existence of the standard Laplacian of a function $u\in H_0^1(V)$ implies the existence of the weak Laplacian Δ; see, for completeness, Falconer and Hu [8].

Remark 2.1. Since $f\colon\mathbb{R}\to\mathbb{R}$ is continuous and $\alpha\in C(V)$, then using Lemma 2.16 of Falconer and Hu [8], it follows that every weak solution of the problem $(S^f_{\lambda,\alpha})$ is also a strong solution.

Now, let $(X,\langle\cdot,\cdot\rangle)$ be a real Hilbert space and, for each $\gamma>0$, put

$$B_\gamma := \{u\in X : \|u\|^2\le\gamma\}.$$

Further, denote by $\operatorname{int}(B_\gamma)$ the interior of B_γ.

The proof of our main result is obtained by exploiting the following abstract theorem due to Ricceri [19] whose proof is entirely based on the results contained in [18].

Theorem 2.1. *Let $J: X\to\mathbb{R}$ be a sequentially weakly upper semicontinuous and Gâteaux differentiable functional, with $J(0_X)=0$. Assume that, for some $\gamma>0$, there exists a global maximum $\widehat{u}$ of $J|_{B_\gamma}$, such that*

$$\langle J'(\widehat{u}),\widehat{u}\rangle < 2J(\widehat{u}).$$

Then, there exists an open interval $I \subseteq (0, +\infty)$ such that, for every $\lambda \in I$, the equation

$$u = \lambda J'(u)$$

has a non-zero solution lying in $\mathrm{int}\,(B_\gamma)$.

3. Proof of the main theorem

Let us put $X := H_0^1(V)$ endowed by the inner product $\mathcal{W}$ and define

$$\widetilde{f}(t) := \begin{cases} f(t) & \text{if } t \geq 0, \\ 0 & \text{if } t < 0. \end{cases}$$

Since $f(0) = 0$ it follows that $\widetilde{f}$ is a continuous function.

We consider the truncated problem

$$(S^{\widetilde{f}}_{\lambda,\alpha}) \qquad \begin{cases} \Delta u(x) = \lambda\alpha(x)\widetilde{f}(u(x)) & x \in V \setminus V_0, \\ u|_{V_0} = 0, \end{cases}$$

and set

$$J(u) := -\int_V \alpha(x)\widetilde{F}(u(x))d\mu,$$

for every $u \in X$, where

$$\widetilde{F}(\xi) := \int_0^\xi \widetilde{f}(t)dt,$$

for every $\xi \in \mathbb{R}$.

By [3, Proposition 4.5] it follows that J is a Gâteaux differentiable and sequentially weakly continuous functional with $J(0_X) = 0$. Moreover, fixing $\lambda > 0$, the weak solutions of $(S^{\widetilde{f}}_{\lambda,\alpha})$ are exactly the solutions $u \in X$ of the following equation:

$$u = \lambda J'(u)\,;$$

see Proposition 2.19 in [8].

Further, Remark 2.1 ensures that every weak solution of problem $(S^{\widetilde{f}}_{\lambda,\alpha})$ is a strong one. Hence, exploiting the maximum principle proved by Strichartz in [20, Theorem 2.1], every solution $u \in X$ of $(S^{\widetilde{f}}_{\lambda,\alpha})$ is non-negative, so that u also solves the original problem $(S^{f}_{\lambda,\alpha})$.

$(\mathrm{i}_1) \Rightarrow (\mathrm{i}_2)$

By hypothesis (h_f), taking into account that h_F is non-increasing in $(0, a]$ and since

$$h'_F(\xi) = \frac{f(\xi)\xi - 2F(\xi)}{\xi^3}, \quad \forall\,\xi \in (0, a],$$

we obtain

$$f(\xi)\xi \leq 2F(\xi), \tag{7}$$

for every $\xi \in (0, a]$.

On the other hand, Fukushima and Shima proved in [9] that the embedding

$$(X, \|\cdot\|) \hookrightarrow (C_0(V), \|\cdot\|_\infty)$$

is compact and

$$\|u\|_\infty \leq (2N + 3)\|u\|, \tag{8}$$

for every $u \in X$.

Thus, let us fix

$$r \in \left(0, \frac{a^2}{(2N+3)^2}\right],$$

and denote $\gamma := r^2$.

By relations (7) and (8) it follows that

$$\widetilde{f}(u(x))u(x) \leq 2\widetilde{F}(u(x)), \tag{9}$$

for every $u \in B_\gamma$ and $x \in V$.

Let $u_0 \in X$ be a non-negative function in V with $\|u_0\| = y > 0$. Take $\varepsilon > y/\gamma$ and consider the function

$$v_\varepsilon(x) := \frac{u_0(x)}{\varepsilon}, \quad \forall\, x \in V.$$

Clearly $v_\varepsilon \in B_\gamma$ and

$$J(v_\varepsilon) = -\int_{V_a^\varepsilon} \alpha(x)F(v_\varepsilon(x))d\mu > 0, \tag{10}$$

where

$$V_a^\varepsilon := \{x \in V : 0 < v_\varepsilon(x) \leq a\},$$

with $\mu(V_a^\varepsilon) > 0$.

Now, let $\widehat{u} \in B_\gamma$ be a global maximum of J in B_γ. Then, condition (10) ensures that $J(\widehat{u}) > 0$ and consequently

$$\max_{x \in V} \widehat{u}(x) > 0.$$

At this point, note that

$$S_f := \{x \in V : f(\widehat{u}(x))\widehat{u}(x) < 2F(\widehat{u}(x))\} \neq \emptyset. \tag{11}$$

Indeed, arguing by contradiction, if $S_f = \emptyset$, bearing in mind relation (9), we would have

$$f(\widehat{u}(x))\widehat{u}(x) = 2F(\widehat{u}(x)),$$

for every $x \in V$.

Then, since $h'_F(\xi) = 0$ for every

$$\xi \in A := \left(0, \max_{x \in V} \widehat{u}(x)\right],$$

the function h_F would be constant in A against (i_1).

Finally, since α is negative in S_f, relations (4) and (11) yield

$$-\int_{S_f} \alpha(x)f(\widehat{u}(x))\widehat{u}(x)d\mu < -2\int_{S_f} \alpha(x)F(\widehat{u}(x))d\mu. \tag{12}$$

Moreover,

$$\int_{V \setminus S_f} \alpha(x)(f(\widehat{u}(x))\widehat{u}(x) - 2F(\widehat{u}(x)))d\mu = 0. \tag{13}$$

Thus, by (12) and (13), we write

$$-\int_V \alpha(x)f(\widehat{u}(x))\widehat{u}(x)d\mu < -2\int_V \alpha(x)F(\widehat{u}(x))d\mu.$$

Bearing in mind that

$$\langle J'(\widehat{u}), \widehat{u}\rangle = -\int_V \alpha(x) f(\widehat{u}(x))\widehat{u}(x)d\mu,$$

the above inequality can be rewritten as

$$\langle J'(\widehat{u}), \widehat{u}\rangle < 2J(\widehat{u}). \tag{14}$$

On the other hand, if

$$r \in \left(\frac{a}{(2N+3)}, +\infty\right),$$

by choosing $\gamma := a^2/(2N+3)^2$, and arguing as before, there exists a global maximum $\widehat{u}$ of J in B_γ such that condition (14) holds.

Hence, Theorem 2.1 ensures that there exists an open interval $I \subseteq (0, +\infty)$ such that, for every $\lambda \in I$, problem $(S^f_{\lambda,\alpha})$ has a strong non-negative and non-zero solution, whose norm in X is less than $\sqrt{\gamma}$. The conclusion follows.

$(\mathrm{i}_1) \Leftarrow (\mathrm{i}_2)$

Let us start recalling a preliminary fact on the spectrum of linear elliptic problems on the Sierpiński gasket. More precisely, let $a : V \to \mathbb{R}$ be such that

(a_1) $a(x) \geq 0$ in V and $0 < \int_V a(x)d\mu < +\infty$,

and consider the following elliptic eigenvalue problem:

$$(S_{\lambda,a}) \qquad \begin{cases} \Delta u(x) + \lambda a(x)u(x) = 0 & x \in V \setminus V_0, \\ u|_{V_0} = 0. \end{cases}$$

By [11], under the structural hypothesis (a_1), it follows that problem $(S_{\lambda,a})$ possesses a sequence $\{\lambda_n\}$ of eigenvalues fulfilling

$$0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots, \quad \lambda_n \to \infty \text{ as } n \to \infty; \tag{15}$$

see also [8, pp. 563–564].

From now on, we argue by contradiction. Hence, assume that there are two positive real constants b and c such that

$$\frac{F(\xi)}{\xi^2} = c, \quad \forall\, \xi \in (0, b].$$

Fixing

$$r \in \left(0, \frac{b}{(2N+3)}\right],$$

by (i_2), there exists an open interval I such that, for every $\lambda \in I$, problem $(S^f_{\lambda,\alpha})$ admits a (strong) non-negative solution $u \in C_0(V) \setminus \{0_X\}$ such that

$$\|u\| < r\,. \tag{16}$$

In view of (8) and (16), we also obtain

$$\|u\|_\infty < b,$$

and it follows that

$$f(u(x)) = 2cu(x), \quad \forall\, x \in V.$$

Then, for every $\lambda \in I$, the linear problem

$$(S_{\lambda,c\alpha}) \qquad \begin{cases} \Delta u(x) - 2\lambda c\alpha(x)u(x) = 0 & x \in V \setminus V_0, \\ u|_{V_0} = 0, \end{cases}$$

admits a strong non-zero solution. This fact contradicts (15) and the proof is complete. □

Acknowledgements

The authors are grateful to the referee for the careful analysis of this paper and for constructive remarks. This paper was written when the first author was visiting the University of Ljubljana in 2013. He expresses his gratitude to the host institution for the warm hospitality. The manuscript was realized within the auspices of the GNAMPA Project 2013 titled *Problemi non-locali di tipo Laplaciano frazionario*. The second author acknowledges the support of grant CNCS-PCE-47/2011.

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