

CATEGORIFIED INVARIANTS AND THE BRAID GROUP

JOHN A. BALDWIN AND J. ELISENDA GRIGSBY

(Communicated by Daniel Ruberman)

ABSTRACT. We investigate two “categorified” braid conjugacy class invariants, one coming from Khovanov homology and the other from Heegaard Floer homology. We prove that each yields a solution to the word problem but not the conjugacy problem in the braid group. In particular, our proof in the Khovanov case is completely combinatorial.

1. INTRODUCTION

Recall that the n -strand braid group B_n admits the presentation

$$B_n = \left\langle \sigma_1, \dots, \sigma_{n-1} \left| \begin{array}{ll} \sigma_i \sigma_j = \sigma_j \sigma_i & \text{if } |i - j| \geq 2, \\ \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j & \text{if } |i - j| = 1 \end{array} \right. \right\rangle,$$

where σ_i corresponds to a positive half twist between the i th and $(i + 1)$ st strands. Given a word w in the generators $\sigma_1, \dots, \sigma_{n-1}$ and their inverses, we will denote by $\sigma(w)$ the corresponding braid in B_n . Also, we will write $\sigma \sim \sigma'$ if σ and σ' are conjugate elements of B_n . As with any group described in terms of generators and relations, it is natural to look for combinatorial solutions to the *word* and *conjugacy problems* for the braid group:

- (1) Word problem: Given words w, w' as above, is $\sigma(w) = \sigma(w')$?
- (2) Conjugacy problem: Given words w, w' as above, is $\sigma(w) \sim \sigma(w')$?

The fastest known algorithms for solving Problems (1) and (2) exploit the Garside structure(s) of the braid group (cf. [7] for a survey and [10] for an implementation). In addition, any faithful representation of B_n for which the images of the generators and the product rule can be described combinatorially—for example, the Lawrence-Krammer representation [6, 23, 25]—provides a solution to Problem (1).

The present work is an attempt to understand what two popular combinatorial link homology theories coming from representation theory and symplectic geometry—namely, Khovanov homology and link Floer homology—can tell us about Problems (1) and (2). Both theories are powerful enough to detect the unknot [24, 36]. Can they detect the trivial braid? Can they distinguish braid conjugacy classes?

Received by the editors July 26, 2013 and, in revised form, February 3, 2014 and February 11, 2014.

2010 *Mathematics Subject Classification*. Primary 20F36, 57M27, 57R58, 81R50.

The first author was partially supported by NSF grant number DMS-1104688.

The second author was partially supported by NSF grant number DMS-0905848 and NSF CAREER award DMS-1151671.

In this short note, we explain how to extract braid conjugacy class invariants from both theories and we prove that each of these invariants provides a solution to Problem (1) but not (naively) to Problem (2). Although the approaches to Problems (1) and (2) described in Theorems 3.1 and 4.3 are *a priori* more computationally involved than the solutions mentioned above, preliminary calculations suggest that our solution to Problem (1) described in Remark 3.12 may in fact compare favorably to previously existing solutions.

Below, we provide a brief description of the “categorified” braid conjugacy class invariants studied in this note. Both are invariants of isotopy classes of oriented links in the solid torus complement of a neighborhood of an oriented unknot $B \subset S^3$. We will think of B as the compactification of the oriented z -axis in \mathbb{R}^3 ,

$$B = \{(r, \theta, z) \mid r = 0\} \cup \{\infty\} \subset \mathbb{R}^3 \cup \{\infty\} = S^3,$$

and the complement $S^3 - N(B)$ as the product

$$A \times I = \{(r, \theta, z) \mid r \in [1, 2], \theta \in [0, 2\pi), z \in [0, 1]\}$$

of an annulus A with the interval $I = [0, 1]$.

Given a braid $\sigma \in B_n$ (read and oriented, by convention, from left to right, with strand 1 at top), we will imagine its oriented closure $\widehat{\sigma}$ as living in $A \times I$ such that $\widehat{\sigma}$ intersects every disk

$$D_t = \{(r, \theta, z) \mid \theta = t\} \cup \{\infty\}$$

positively in n points, where D_t is oriented so that $B = \partial D_t$. Note that $\widehat{\sigma} \subset A \times I$ is well-defined up to isotopy and, moreover, that conjugate braids give rise to isotopic closures. Thus, any invariant of isotopy classes of links in $A \times I$ gives rise to a braid conjugacy class invariant.

The central objects of study in this paper are the sutured annular Khovanov homology $\text{SKh}(\widehat{\sigma} \subset A \times I)$ (cf. Section 2.1) and the link Floer homology $\widehat{\text{HFL}}(\widehat{\sigma} \cup B)$ (cf. Section 2.2). Both are invariants of the conjugacy class of σ per the observation above. As mentioned earlier, we will prove that both detect the trivial conjugacy class—hence, can be used to distinguish unequal braids (cf. Theorem 3.1)—but that neither gives a complete conjugacy class invariant (cf. Theorem 4.3 and Corollary 4.4). These invariants share other structural features, including a relationship with the Burau representation

$$\Psi : B_n \rightarrow GL_n(\mathbb{Z}[T^{\pm 1}])$$

(cf. Remarks 2.2 and 2.7). In particular, the graded Euler characteristic of $\widehat{\text{HFL}}(\widehat{\sigma} \cup B)$ is (more or less) the characteristic polynomial of $\Psi(\sigma)$. We will show, however, that the homology contains more braid information than does the polynomial in general (cf. Proposition 2.8).

It bears mentioning that the standard (i.e. non-sutured) versions of Khovanov homology and knot Floer homology can also be used to detect the trivial braid, thanks to a combination of recent work on the Khovanov/Floer side with a classical braid theory result. Specifically, Batson-Seed [4] have shown, building on work of Hedden-Ni [19], which, in turn, builds on work of Kronheimer-Mrowka [24], that if L_n is a link with n -components, then $\text{Kh}(L_n) \cong \text{Kh}(U_n)$ iff $L_n = U_n$, where U_n is the n -component unlink. Analogously, Ni [35] has shown, building on work of Ozsváth-Szabó [36], that $\widehat{\text{HFK}}(L_n) \cong \widehat{\text{HFK}}(U_n)$ iff $L_n = U_n$. To determine whether $\sigma = \mathbb{1}$ in the n -strand braid group, it then suffices to compute $\text{Kh}(\widehat{\sigma})$

(resp. $\widehat{\text{HFK}}(\widehat{\sigma})$). If one obtains the same answer as for U_n , then the Batson-Seed (resp. Ni) result implies that $L_n = U_n$. A classical result – a proof of which appears in [11] – that an n -strand braid whose closure is U_n is conjugate to the trivial braid – then tells us that $\sigma = \mathbb{1}$.

There are two primary reasons for focusing on the invariants $\widehat{\text{HFL}}(\widehat{\sigma} \cup B)$ and $\text{SKh}(\widehat{\sigma})$ in this paper. First, they’re more intrinsically braid invariants and carry more conjugacy class information than $\widehat{\text{HFK}}(\widehat{\sigma})$ and $\text{Kh}(\widehat{\sigma})$, which depend only on the isotopy class of $\widehat{\sigma}$ as a link in S^3 , and not at all on its embedding in $S^3 - N(B)$. Second, and perhaps most significantly, our proof that $\text{SKh}(\widehat{\sigma})$ detects the trivial braid is *entirely combinatorial*. It does not rely on Floer homology or gauge theory at all, unlike the proof that $\text{Kh}(\widehat{\sigma})$ detects the trivial braid, which relies on deep results in both Heegaard and instanton Floer homology.

Finally, we note that Proposition 3.10 and Corollary 3.11 should be of independent interest, as they demonstrate that Plamenevskaya’s invariant—while not known to be an effective invariant of transverse links—does detect interesting geometric features of braids.

2. CATEGORIFIED BRAID CONJUGACY CLASS INVARIANTS

In this section, we briefly recall the construction of sutured annular Khovanov homology and some basic features of link Floer homology. We will assume the reader is familiar with ordinary Khovanov homology and knot Floer homology. All chain complexes and homology theories considered in this paper are with coefficients in $\mathbb{F} := \mathbb{Z}/2\mathbb{Z}$.

2.1. Sutured annular Khovanov homology. Sutured annular Khovanov homology was originally defined in [1] as a categorification of the Kauffman bracket skein module of $A \times I$. It was studied further in [17, 41], where a connection with sutured Floer homology was discovered (hence, the name). The theory associates to an oriented link $\mathbb{L} \subset A \times I$ a triply-graded vector space

$$\text{SKh}(\mathbb{L}) = \bigoplus_{i,j,k} \text{SKh}^i(\mathbb{L}; j, k),$$

which is an invariant of the oriented isotopy class of $\mathbb{L} \subset A \times I$.

Its construction starts with a projection of \mathbb{L} onto the annulus $A \times \{1/2\}$. This projection may be viewed as a planar diagram D in $S^2 - \{X, O\}$, where X and O are basepoints in S^2 corresponding to the inner and outer boundary circles of $A \times \{1/2\}$. Forgetting the data of the X basepoint temporarily, we may think of D as a planar diagram in $\mathbb{R}^2 = S^2 - \{O\}$ and form the ordinary bigraded Khovanov complex

$$\text{CKh}(D) = \bigoplus_{i,j} \text{CKh}^i(D; j)$$

from a cube of resolutions of D in the usual way. Here, i and j are the homological and quantum gradings, respectively.¹ The basepoint X gives rise to a filtration on $\text{CKh}(D)$, and $\text{SKh}(\mathbb{L})$ is defined to be the (co)homology of the associated graded object.

¹The grading, i , is really a *cohomological* grading, as the Khovanov differential *increases* it by 1.

To define this filtration, we choose an oriented arc from X to O missing all crossings of the diagram D . As described in [18, Sec. 4.2], the generators of $\text{CKh}(D)$ are in one-to-one correspondence with *enhanced* (i.e., *oriented*) resolutions. We define the “ k ” grading of an oriented resolution to be the algebraic intersection number of this resolution with our oriented arc, up to some overall shift. Roberts proves ([41, Lem. 1]) that the Khovanov differential does not increase this extra grading. One therefore obtains a filtration,

$$0 \subseteq \dots \subseteq \mathcal{F}_{n-1}(D) \subseteq \mathcal{F}_n(D) \subseteq \mathcal{F}_{n+1}(D) \subseteq \dots \subseteq \text{CKh}(D),$$

where $\mathcal{F}_n(D)$ is the subcomplex of $\text{CKh}(D)$ generated by oriented resolutions with k grading at most n . Let

$$\mathcal{F}_n(D; j) = \mathcal{F}_n(D) \cap \bigoplus_i \text{CKh}^i(D; j).$$

The sutured annular Khovanov homology groups of \mathbb{L} are defined to be

$$\text{SKh}^i(\mathbb{L}; j, k) := H^i \left(\frac{\mathcal{F}_k(D; j)}{\mathcal{F}_{k-1}(D; j)} \right).$$

The lemma below follows directly.

Lemma 2.1. *There is a spectral sequence whose E_1 term is $\text{SKh}(\mathbb{L} \subset A \times I)$ and whose E_∞ term (ignoring the k grading) is isomorphic to $\text{Kh}(\mathbb{L} \subset S^3)$. Moreover, the d_n differential shifts the (i, j, k) multi-grading by $(1, 0, -n)$.*

Given a braid σ , we will be interested in $\text{SKh}(\widehat{\sigma})$, where $\widehat{\sigma} \subset A \times I$ is the oriented closure of σ as described in the Introduction.

Remark 2.2. It is shown in [2] that the summand of $\text{SKh}(\widehat{\sigma})$ in the next-to-top k grading is equal to the Hochschild homology of the braid bimodule constructed by Khovanov and Seidel in [22]. This bimodule detects the trivial braid [22, Cor. 1.2] (and its action categorifies the Burau representation [22, Prop. 2.8]). These facts together with Theorem 4.3 imply that a categorical group representation may be faithful even while its Hochschild homology is not faithful on conjugacy classes. Similar results have been obtained in the Heegaard Floer setting by Hedden and Watson [20] in combination with Lipshitz, Ozsváth and Thurston [26, Thm. 14], [27].

On the other hand, Theorem 3.1 suggests the following question.

Question 2.3. Suppose one has a faithful weak action, in the sense of [22, Def. 2.6], of a group G on a (derived) category of modules over an (A_∞) algebra A , where the functor associated to an element $g \in G$ is given by taking a (derived) tensor product with a (derived equivalence class of) bimodule \mathcal{M}_g . Let $HH(A, \mathcal{M}_g)$ denote the Hochschild homology of A with coefficients in \mathcal{M}_g . Does

$$HH(A, \mathcal{M}_g) = HH(A, \mathcal{M}_\mathbb{1})$$

necessarily imply that $g = \mathbb{1}$?

2.2. Link Floer homology. Link Floer homology was defined in [37] as a generalization of knot Floer homology and a categorification of the multi-variable Alexander polynomial. One version of the theory associates to an oriented link $\mathbb{L} \subset S^3$ expressed as a union of k sublinks, $\mathbb{L} = L_1 \cup \dots \cup L_k$, a graded vector space

$$\widehat{\text{HFL}}(\mathbb{L}) = \bigoplus_{d, A_{L_1}, \dots, A_{L_k}} \widehat{\text{HFL}}_d(\mathbb{L}; A_{L_1}, \dots, A_{L_k})$$

which is an invariant of the oriented isotopy class of $\mathbb{L} \subset S^3$. Here, d is the Maslov grading and the A_{L_i} are the Alexander gradings associated to the sublinks L_i .² When $k = 1$, $\widehat{\text{HFL}}(\mathbb{L})$ as described here is isomorphic to the knot Floer homology $\widehat{\text{HFK}}(\mathbb{L})$ after an overall shift of the Maslov grading [38, Thm. 1.1].

Link Floer homology enjoys many symmetries. To begin with, $\widehat{\text{HFL}}(\mathbb{L})$ is supported in Alexander multi-gradings that are symmetric about the origin in \mathbb{R}^k ,

$$(1) \quad \widehat{\text{HFL}}_d(\mathbb{L}; A_{L_1}, \dots, A_{L_k}) \cong \widehat{\text{HFL}}_{d-2S}(\mathbb{L}; -A_{L_1}, \dots, -A_{L_k}),$$

where $S = \sum_i A_{L_i}$. Moreover, if \mathbb{L}' is the link obtained by replacing L_i with $-L_i$ (i.e., by reversing the orientation of L_i) then

$$(2) \quad \widehat{\text{HFL}}_d(\mathbb{L}; A_{L_1}, \dots, A_{L_i}, \dots, A_{L_k}) \cong \widehat{\text{HFL}}_{d-2A_{L_i}+l_i}(\mathbb{L}'; A_{L_1}, \dots, -A_{L_i}, \dots, A_{L_k}),$$

where $l_i = \text{lk}(L_i, \mathbb{L} - L_i)$. Similarly, if $m(\mathbb{L})$ denote the mirror of \mathbb{L} , then

$$(3) \quad \widehat{\text{HFL}}_d(\mathbb{L}; A_{L_1}, \dots, A_{L_k}) \cong \widehat{\text{HFL}}_{2S-d+1-|\mathbb{L}|}(m(\mathbb{L}); A_{L_1}, \dots, A_{L_k}).$$

See [31, Sec. 5] and [38, Sec. 8] for discussions of these symmetries.

The link Floer homology of \mathbb{L} is the homology of a chain complex defined in terms of a multi-pointed Heegaard diagram for \mathbb{L} . In an abuse of notation, we will denote this complex by $\widehat{\text{CFL}}(\mathbb{L})$. For each sublink L_i , $\widehat{\text{CFL}}(\mathbb{L})$ can be realized as the associated graded object of a filtration on some complex $C(\mathbb{L} - L_i)$, where

$$H_*(C(\mathbb{L} - L_i)) \cong \widehat{\text{HFL}}(\mathbb{L} - L_i) \otimes V^{\otimes |L_i|}$$

up to a shift of the Alexander multi-grading, for $k \geq 2$. Here, V is the triply-graded vector space $\mathbb{F} \oplus \mathbb{F}$ whose summands are supported in Maslov gradings 0 and -1 and Alexander multi-grading $(0, \dots, 0) \subset \mathbb{R}^{k-1}$. This leads to the following lemma (the claim about grading shifts follows immediately from the discussions in Subsections 3.7 and 8.1 of [38]).

Lemma 2.4. *For $k \geq 2$, there is a spectral sequence whose E_1 term is $\widehat{\text{HFL}}(\mathbb{L})$ and whose E_∞ term (ignoring the A_{L_i} grading) is isomorphic to the vector space obtained from $\widehat{\text{HFL}}(\mathbb{L} - L_i) \otimes V^{\otimes |L_i|}$ by shifting each Alexander grading A_{L_j} by $\frac{1}{2} \text{lk}(L_j, L_i)$.*

The following describes a similar spectral sequence in the case that $k = 1$ (cf. [38]).

Lemma 2.5. *For $k = 1$, there is a spectral sequence whose E_1 term is $\widehat{\text{HFL}}(\mathbb{L})$ and whose E_∞ term is a rank $2^{|\mathbb{L}|-1}$ vector space.*

We now restrict our attention to the case $\mathbb{L} = \hat{\sigma} \cup B$. Note that $\widehat{\text{HFL}}(\hat{\sigma} \cup B)$ is an invariant of the oriented isotopy class of $\hat{\sigma} \subset A \times I$.

Remark 2.6. That $\widehat{\text{HFL}}(\hat{\sigma} \cup B)$ is the homology of the associated graded object of a filtration on a complex which computes $\widehat{\text{HFL}}(\hat{\sigma}) \otimes V$ (up to an Alexander grading shift) closely parallels the relationship between $\text{SKh}(\hat{\sigma} \subset A \times I)$ and $\text{Kh}(\hat{\sigma} \subset S^3)$ described in the previous subsection.

²Typically, one associates an Alexander grading to each *component* of \mathbb{L} . The theory we describe here is obtained by “flattening” the multi-grading associated to each sublink L_i into a single grading by summing.

Remark 2.7. The fact that link Floer homology categorifies the multi-variable Alexander polynomial, combined with an older result of Morton [33], implies that $\widehat{HFL}(\widehat{\sigma} \cup B)$ categorifies the characteristic polynomial $\det(\lambda - \Psi(\sigma))$, where

$$\Psi : B_n \rightarrow GL_n(\mathbb{Z}[T^{\pm 1}])$$

is the Burau representation of B_n . More precisely,

$$\sum_{d, A_{\widehat{\sigma}}, A_B} (-1)^d \cdot \text{rk} \widehat{HFL}_d(\widehat{\sigma} \cup B; A_{\widehat{\sigma}}, A_B) \cdot T^{A_{\widehat{\sigma}}} \cdot \lambda^{A_B} = \det(\lambda - \Psi(\sigma)) \cdot (T - 1)^{|\widehat{\sigma}|},$$

up to an overall factor of $\pm T^{m_1} \cdot \lambda^{m_2}$ for some $m_1, m_2 \in \frac{1}{2}\mathbb{Z}$.

On the other hand, since the Burau representation is not faithful for $n \geq 5$ (cf. [5, 29, 32]), Theorem 3.1.(b) implies that $\widehat{HFL}(\widehat{\sigma} \cup B)$ contains strictly more information about σ than does $\det(\lambda - \Psi(\sigma))$:

Proposition 2.8. *For each $n \geq 5$ there exists a braid $\sigma \in B_n$ for which*

$$\det(\lambda - \Psi(\sigma)) = \det(\lambda - \Psi(\mathbb{1})) \quad \text{but} \quad \widehat{HFL}(\widehat{\sigma} \cup B) \not\cong \widehat{HFL}(\widehat{\mathbb{1}} \cup B).$$

3. TRIVIAL BRAID DETECTION: SOLUTION TO WORD PROBLEM

The goal of this section is to prove that $SKh(\widehat{\sigma})$ and $\widehat{HFL}(\widehat{\sigma} \cup B)$ detect the trivial braid and therefore provide solutions to Problem (1), per the theorem below.

Theorem 3.1. *Suppose $\sigma \in B_n$.*

- (a) *If $SKh(\widehat{\sigma}) \cong SKh(\widehat{\mathbb{1}})$, then $\sigma = \mathbb{1}$.*
- (b) *If $\widehat{HFL}(\widehat{\sigma} \cup B) \cong \widehat{HFL}(\widehat{\mathbb{1}} \cup B)$ and σ is a pure braid, then $\sigma = \mathbb{1}$.³*

In particular, both $SKh(\widehat{\sigma})$ and $\widehat{HFL}(\widehat{\sigma} \cup B)$ can be used to give solutions to the word problem in the braid group.

The proof of Theorem 3.1.(a) relies on certain properties of Plamenevskaya’s invariant [40] of transverse links. We pause to discuss these first, delaying the proof of Theorem 3.1 to Subsection 3.2.

3.1. Plamenevskaya’s invariant and trivial braid detection. Let D_n denote the standard unit disk in the complex plane with n distinct marked points p_1, \dots, p_n positioned along the real axis.

Definition 3.2. We say an arc $\gamma : [0, 1] \rightarrow D_n$ is *admissible* if it satisfies

- (1) γ is a smooth imbedding transverse to ∂D_n ,
- (2) $\gamma(0) = -1 \in \mathbb{C}$ and $\gamma(1) \in \{p_1, \dots, p_n\}$,
- (3) $\gamma(t) \in D_n \setminus (\partial D_n \cup \{p_1, \dots, p_n\})$ for all $t \in (0, 1)$, and
- (4) $\frac{d\gamma}{dt} \neq 0$ for all $t \in [0, 1]$.

We will often abuse notation and use γ to refer to the image of γ in D_n .

Remark 3.3. Note that the admissible arcs (Definition 3.2) used in this section differ from those in [3, Sec. 4], where admissible arcs originate and terminate on ∂D_n . We use Definition 3.2 and its corresponding notion of “right-veering braid” (Defn. 3.7) here, as it is more directly related to the results we reference from [14] and [12] in the proof of Proposition 3.10. It is unknown (to us) whether the *a priori* different notions of “right-veering braid” described in Definition 3.7 and [3, Sec. 4] are, in fact, equivalent.

³If σ is not a pure braid, which is easy to check by hand, then $\sigma \neq \mathbb{1}$.

Definition 3.4. Two admissible arcs γ, γ' are said to be *pulled tight* if they satisfy one of:

- $\gamma = \gamma'$, or
- γ and γ' intersect transversely, and if $t_1, t_2, t'_1, t'_2 \in [0, 1]$ satisfy the property that $\gamma([t_1, t_2]) \cup \gamma'([t'_1, t'_2])$ bounds an imbedded disk $A \subset D_n$, then $A \cap \{p_1, \dots, p_n\} \neq \emptyset$ (i.e., γ and γ' are transverse and form no empty bigons).

Note that if γ, γ' are admissible arcs, there exist admissible arcs δ, δ' isotopic to γ, γ' , resp., such that δ, δ' are pulled tight (cf. [12, Chp. 10]).

Definition 3.5. Let γ, γ' be two admissible arcs in D_n . We say γ is *right* of γ' if, when pulled tight via isotopy, the orientation induced by the tangent vectors $\frac{d\gamma}{dt}|_{t=0}, \frac{d\gamma'}{dt}|_{t=0}$ agrees with the standard orientation on $D \subset \mathbb{C}$.

There is a well-known isomorphism $B_n \cong \mathcal{MCG}(D_n)$ obtained by identifying the elementary Artin generator σ_i (resp., σ_i^{-1}) with a homeomorphism of D_n that acts as the identity outside of a small disk enclosing p_i, p_{i+1} and acts as a 180° counterclockwise (resp., clockwise) rotation on a (slightly smaller) disk enclosing p_i, p_{i+1} . Note that when a braid is viewed as a mapping class, it acts on D_n *on the right* (since Artin braids are read from left to right).

Via the above isomorphism, B_n acts on the set of isotopy classes of admissible arcs. Let $(\gamma)\sigma$ denote the image of (the isotopy class of) γ under $\sigma \in B_n$.

Remark 3.6. Note that in [12] (cf. Prop. 3.3), positive (resp., negative) Artin generators are identified with clockwise (resp., counterclockwise) rotations so that braids may act on D_n *on the left*. We chose our convention to match those in [21] and [3].

Definition 3.7. Let $\sigma \in B_n$. We say σ is *right-veering* if, for all admissible arcs γ , $(\gamma)\sigma$ is right of γ when pulled tight.

Remark 3.8. We may analogously define the notion of *left-veering*. A braid $\sigma \in B_n$ is left-veering (resp. right-veering) iff its mirror $m(\sigma) \in B_n$ is right-veering (resp. left-veering).

The following lemma is well-known:

Lemma 3.9. *If $\sigma \in B_n$ is both right-veering and left-veering, then $\sigma = \mathbb{1}$.*

Proof. If $\sigma \in B_n$ is both right- and left-veering, it must send each admissible arc to an isotopic admissible arc. Consider the collection $\{Q_1, \dots, Q_n\}$ of admissible arcs pictured in Figure 1. A straightforward inductive argument then shows that σ is isotopic to a map which fixes each Q_i in this collection. The Alexander lemma (cf. [13, Lem. 2.1]) then implies that $\sigma = \mathbb{1}$, as desired. □

Let $\psi(\widehat{\sigma}) \in \text{Kh}(\widehat{\sigma})$ denote Plamenevskaya’s invariant [40] of the transverse link in the tight contact structure on S^3 represented by the braid $\widehat{\sigma}$.

Proposition 3.10. *If σ is not right-veering, then $\psi(\widehat{\sigma}) = 0$.*

Proof. Suppose σ is not right-veering. Then there exists some admissible arc γ which is sent to the left by σ . Let $\tau \in B_n$ satisfy $\gamma = (Q_1)\tau$. Then $(Q_1)\tau\sigma\tau^{-1}$ is left of Q_1 . Via the algorithm described in the proof of [14, Prop. 4.3], one can explicitly construct a σ_1 -negative word (cf. [12, Defn. 1.12]) representing $\tau\sigma\tau^{-1}$. Such a word contains at least one letter σ_1^{-1} and no letters σ_1 . It then follows from [40, Prop. 3] that $\psi(\widehat{\tau\sigma\tau^{-1}}) = \psi(\widehat{\sigma}) = 0$. □

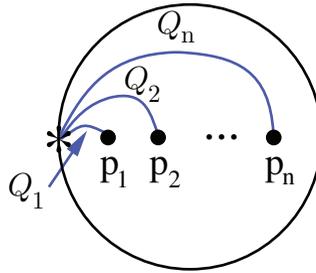


FIGURE 1. A collection of admissible arcs $Q_1, \dots, Q_n \subset D_n$ whose complement is a disk.

Corollary 3.11. *If $\psi(\widehat{\sigma}) \neq 0$ and $\psi(m(\widehat{\sigma})) \neq 0$, then $\sigma = \mathbb{1}$.*

Proof. By Proposition 3.10, σ is both right- and left-veering. By Lemma 3.9, $\sigma = \mathbb{1}$. □

Remark 3.12. As the converse of Corollary 3.11 is a straightforward calculation, Corollary 3.11 provides a direct solution to the word problem in B_n . Explicitly, given w, w' two words in the elementary Artin generators representing braids σ, σ' , calculate Plamenevskaya’s invariant, ψ , of the closure of $w^{-1}w'$ and of the closure of $m(w^{-1})m(w')$. The result of both calculations is nonzero iff $\sigma = \sigma'$.

3.2. Proof of Theorem 3.1.

Proof of Theorem 3.1.(a). Suppose $\text{SKh}(\widehat{\sigma}) \cong \text{SKh}(\widehat{\mathbb{1}})$. According to Roberts [41], the invariant $\psi(\widehat{\sigma})$ is the “image of the generator of the bottommost k grading of $\text{SKh}(\widehat{\sigma})$ under the spectral sequence from $\text{SKh}(\widehat{\sigma})$ to $\text{Kh}(\widehat{\sigma})$.” What this really means is that there exists a cycle $c \in \text{CKh}(\widehat{\sigma})$ which (1) represents the transverse invariant $\psi(\widehat{\sigma})$ in $\text{Kh}(\widehat{\sigma})$ and (2) represents the generator of the bottommost k grading of $\text{SKh}(\widehat{\sigma})$. By direct computation, $\text{SKh}(\widehat{\sigma})$ is supported in a single homological grading, which implies that the spectral sequence from $\text{SKh}(\widehat{\sigma})$ to $\text{Kh}(\widehat{\sigma})$ collapses immediately. It follows that $\psi(\widehat{\sigma}) \neq 0$. The symmetry of sutured Khovanov homology under taking mirrors [41, Lem. 2] implies that $\text{SKh}(m(\widehat{\sigma}))$ is also supported in a single homological grading. It follows that $\psi(m(\widehat{\sigma})) \neq 0$ as well. Corollary 3.11 then tells us that $\sigma = \mathbb{1}$, as desired. □

The proof of Theorem 3.1.(b) is very similar in spirit to that of Theorem 3.1.(a). The analogue of Proposition 3.10 (using a different notion of “right-veering”, cf. Remark 3.3) was proved in [3].

Proof of Theorem 3.1.(b). For $\sigma \in B_n$, Baldwin, Vela-Vick and Vértesi define in [3] a class $\widehat{t}(\widehat{\sigma}) \in \widehat{\text{HFL}}(m(\widehat{\sigma}))$ which is an invariant of the transverse link represented by $\widehat{\sigma}$ (and agrees, for transverse *knots*, with the invariants defined in [28, 39]). We show below that if σ is a pure braid and $\widehat{\text{HFL}}(\widehat{\sigma} \cup B) \cong \widehat{\text{HFL}}(\widehat{\mathbb{1}} \cup B)$, then $\widehat{t}(\widehat{\sigma}) \neq 0$ and $\widehat{t}(\widehat{\sigma^{-1}}) \neq 0$. An analogue of Corollary 3.11 then implies that σ is equal to $\mathbb{1}$.

Suppose σ is a pure braid and $\widehat{\text{HFL}}(\widehat{\sigma} \cup B) \cong \widehat{\text{HFL}}(\widehat{\mathbb{1}} \cup B)$. We first compute the latter. Let us denote $\mathbb{1}$ by $\mathbb{1}_n$ to indicate that it is the trivial braid on n -strands. Note that $\widehat{\mathbb{1}}_n \cup B$ is isotopic to link gotten by taking the connected sum of the

positive Hopf link $\widehat{\mathbb{1}}_1 \cup B$ with itself n times along the component B . As computed in [38, Sec. 12] (cf. also [30, Sec. 4]),

$$\widehat{\text{HFL}}(\widehat{\mathbb{1}}_1 \cup B) \cong (V_1 \otimes V_2)[1/2, 1/2].$$

Here, V_i is the triply-graded vector space $\mathbb{F} \oplus \mathbb{F}$ whose first and second summands are supported in Maslov gradings 0 and -1 and Alexander bi-gradings $(0, 0)$ and $-e_i$, where e_i is the i th standard basis vector in \mathbb{R}^2 . The $[1/2, 1/2]$ indicates that we have shifted the Alexander bi-grading by $(1/2, 1/2)$. Our initial assumption and the Künneth formula in [38] imply that

$$(4) \quad \widehat{\text{HFL}}(\widehat{\sigma} \cup B) \cong \widehat{\text{HFL}}(\widehat{\mathbb{1}}_n \cup B) \cong \widehat{\text{HFL}}(\widehat{\mathbb{1}}_1 \cup B)^{\otimes n} \cong (V_1^{\otimes n} \otimes V_2^{\otimes n})[n/2, n/2].$$

To compute $\widehat{t}(\widehat{\sigma})$, we need to know the link Floer homology of $m(\widehat{\sigma} \cup -B)$. From (4) together with the formulae in (2) and (3), it follows that

$$\widehat{\text{HFL}}(m(\widehat{\sigma} \cup -B)) \cong (V_1^{\otimes n} \otimes V_2^{\otimes n})[n/2, n/2]$$

as well. Lemma 2.4 then implies that there is a spectral sequence with E_1 term

$$\widehat{\text{HFL}}(m(\widehat{\sigma} \cup -B))[-n/2, 0] \cong (V_1^{\otimes n} \otimes V_2^{\otimes n})[0, n/2]$$

and whose E_∞ term (ignoring the $A_{m(-B)}$ grading) is isomorphic to $\widehat{\text{HFL}}(m(\widehat{\sigma})) \otimes V$, where V is the bi-graded vector space $\mathbb{F} \oplus \mathbb{F}$ whose summands are supported in Maslov gradings 0 and -1 and $A_{m(\widehat{\sigma})}$ grading 0. To reduce clutter, we introduce the notation

$$H(\widehat{\sigma}, B) := \widehat{\text{HFL}}(m(\widehat{\sigma} \cup -B))[-n/2, 0],$$

$$H(\widehat{\sigma}) := \widehat{\text{HFL}}(m(\widehat{\sigma})) \otimes V.$$

Note that the top Maslov graded piece H_{top} (in Maslov grading $-n$) of the portion of $H(\widehat{\sigma}, B)$ in the bottommost $A_{m(-B)}$ grading $-n/2$ has rank one. The element $\widehat{t}(\widehat{\sigma})$ can be characterized in terms of “the image” in $H(\widehat{\sigma})$ of the generator of H_{top} under the spectral sequence from $H(\widehat{\sigma}, B)$ to $H(\widehat{\sigma})$. Again, this means that there exists a cycle c in the complex for $H(\widehat{\sigma})$ which both represents $\widehat{t}(\widehat{\sigma})$ in homology and also represents the generator of $H_{top} \subset H(\widehat{\sigma}, B)$. In particular, $\widehat{t}(\widehat{\sigma})$ is non-zero iff H_{top} survives under this spectral sequence [3, Corollary 6.9].

Note that $H(\widehat{\sigma}, B)$ is supported in $A_{m(\widehat{\sigma})}$ gradings $-n \leq A_{m(\widehat{\sigma})} \leq 0$. Since $H(\widehat{\sigma})$ is supported in $A_{m(\widehat{\sigma})}$ gradings that are symmetric about the origin (as discussed in Subsection 2.2) it must be that the portion of $H(\widehat{\sigma}, B)$ in negative $A_{m(\widehat{\sigma})}$ gradings dies in the spectral sequence. Moreover, the portion in $A_{m(\widehat{\sigma})}$ grading 0 has rank 2^n . Therefore,

$$\text{rk}(H(\widehat{\sigma})) \leq 2^n.$$

Now, Lemma 2.5 implies that there is a spectral sequence whose E_1 term is $H(\widehat{\sigma})$ and whose E_∞ term is a rank $2^{|m(\widehat{\sigma})|}$ vector space. Since σ is a pure braid, $|m(\widehat{\sigma})| = n$. Thus,

$$\text{rk}(H(\widehat{\sigma})) \geq 2^n.$$

It follows that $\text{rk}(H(\widehat{\sigma})) = 2^n$. It must therefore be the case that the entire portion of $H(\widehat{\sigma}, B)$ in $A_{m(\widehat{\sigma})}$ grading 0 survives in the spectral sequence from $H(\widehat{\sigma}, B)$ to $H(\widehat{\sigma})$. Since H_{top} is contained in this portion we may deduce that $\widehat{t}(\widehat{\sigma})$ is non-zero. Hence, σ is right-veering by [3, Theorem 1.4].

Note that $\widehat{\sigma^{-1}} \cup B$ is isotopic to $m(-\widehat{\sigma} \cup B)$. The formulae in (2) and (3) therefore imply that

$$\widehat{\text{HFL}}(\widehat{\sigma^{-1}} \cup B) \cong \widehat{\text{HFL}}(\widehat{\sigma} \cup B) \cong \widehat{\text{HFL}}(\widehat{1} \cup B)$$

as well. It follows that $\widehat{t}(\widehat{\sigma^{-1}})$ is also non-zero. As before, this implies that σ^{-1} is right-veering and, hence, that σ is left-veering. Since σ is both right- and left-veering, it is equal to 1 . \square

The invariants $\text{SKh}(\widehat{\sigma})$ and $\widehat{\text{HFL}}(\widehat{\sigma} \cup B)$ can be used to solve the word problem in B_n as follows. Suppose w and w' are words in the generators $\sigma_1, \dots, \sigma_n$ and their inverses representing the braids $\sigma(w)$ and $\sigma(w')$ in B_n . Let $\sigma = \sigma(w) \cdot (\sigma(w'))^{-1}$ and note that

$$(5) \quad \sigma = 1 \iff \sigma(w) = \sigma(w').$$

In particular, if σ is not a pure braid, then $\sigma(w) \neq \sigma(w')$. If σ is a pure braid, compute $\text{SKh}(\widehat{\sigma})$ or $\widehat{\text{HFL}}(\widehat{\sigma} \cup B)$ (which can be done combinatorially) and apply Theorem 3.1 with (5) in mind.

4. TRANSVERSE MIRROR INVARIANCE: NO SOLUTION TO CONJUGACY PROBLEM

In this section, we show that $\text{SKh}(\widehat{\sigma})$ and $\widehat{\text{HFL}}(\widehat{\sigma} \cup B)$ cannot always distinguish non-conjugate braids. This fact is made precise in Corollary 4.4 of Theorem 4.3 below. First, some definitions and remarks.

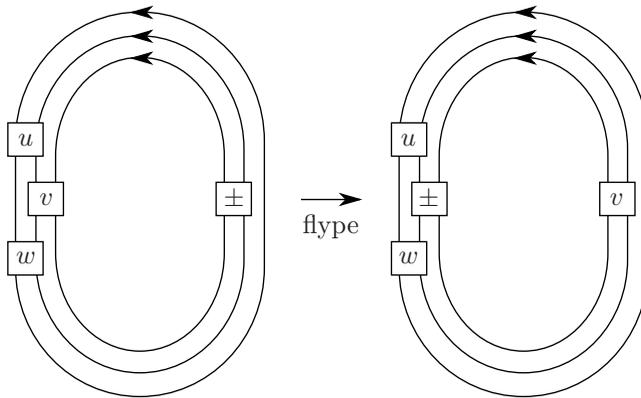


FIGURE 2. Left, a 3-braid closure $\widehat{\sigma}$. Each of the boxes labeled u, v, w represents some number of positive or negative half twists. The box labeled \pm represents a single positive or negative half twist. Right, the closed braid obtained from $\widehat{\sigma}$ by a flype. It is oriented isotopic to $\widehat{\sigma}^r$ in $A \times I$.

Definition 4.1. Let $\sigma \in B_n$ and suppose $\sigma = \sigma(w)$ for some word w in the generators $\sigma_1, \dots, \sigma_n$ and their inverses. Then the *reverse* of σ is the braid $\sigma^r = \sigma(w^r)$, where w^r is the word obtained from w by reversing the order of its letters.

Remark 4.2. Thought of as transverse links in the tight contact structure on S^3 , the closure $\widehat{\sigma}^r$ is the *transverse mirror* of $\widehat{\sigma}$, as defined by Ng in [34, Defn. 4.5].

Theorem 4.3. *Suppose $\sigma \in B_n$. Then*

- (a) $SKh(\widehat{\sigma}) \cong SKh(\widehat{\sigma}^r)$,
- (b) $\widehat{HFL}(\widehat{\sigma} \cup B) \cong \widehat{HFL}(\widehat{\sigma}^r \cup B)$.

Proof of Theorem 4.3.(a). Let D and D^r denote the projections of $\widehat{\sigma}$ and $\widehat{\sigma}^r$ onto $A \times \{1/2\}$ as described in Subsection 2.1. There is an obvious bijection between oriented resolutions of D and D^r which preserves the (i, j, k) triple-grading and commutes with the differentials in the complexes $CKh(D)$ and $CKh(D^r)$. (The projection D^r is exactly what you see by looking at D from below and reversing orientations.) It follows that $SKh(\widehat{\sigma}) \cong SKh(\widehat{\sigma}^r)$. □

Proof of Theorem 4.3.(b). Note that if $\mathbb{L} = \widehat{\sigma} \cup B$, then $\widehat{\sigma}^r \cup B$ is oriented isotopic to $-\mathbb{L}$ in S^3 . It follows from (1) and (2) that $\widehat{HFL}(\mathbb{L}) \cong \widehat{HFL}(-\mathbb{L})$ for any oriented link \mathbb{L} . □

Corollary 4.4. *There exist infinitely many pairs $(\sigma, \sigma') \in B_3 \times B_3$ such that $\sigma \not\sim \sigma'$ but*

$$SKh(\widehat{\sigma}) \cong SKh(\widehat{\sigma}') \quad \text{and} \quad \widehat{HFL}(\widehat{\sigma} \cup B) \cong \widehat{HFL}(\widehat{\sigma}' \cup B).$$

Proof. Suppose σ and σ' are 3-braids for which $\widehat{\sigma}$ and $\widehat{\sigma}'$ are related by a *flype*, as described in [9, Fig. 1.2] and illustrated in Figure 2. Then $\widehat{\sigma}'$ is clearly oriented isotopic to $\widehat{\sigma}^r$ in $A \times I$. In particular, $\widehat{\sigma}'$ and $\widehat{\sigma}$ are transverse mirrors, a fact first observed by Ng in [34]. Theorem 4.3 then implies that

$$SKh(\widehat{\sigma}) \cong SKh(\widehat{\sigma}') \quad \text{and} \quad \widehat{HFL}(\widehat{\sigma} \cup B) \cong \widehat{HFL}(\widehat{\sigma}' \cup B).$$

On the other hand, Birman and Menasco have enumerated infinitely many conjugacy classes of 3-braids admitting *non-degenerate* flypes; i.e. flypes which do not preserve conjugacy class [9, Tab. 2]. □

Remark 4.5. Let $H(\widehat{\sigma})$ denote either of the theories $SKh(\widehat{\sigma})$ or $\widehat{HFL}(\widehat{\sigma} \cup B)$. Given Theorem 4.3 and Corollary 4.4, one wonders, for braids σ, τ in B_n , whether $H(\widehat{\sigma}) \cong H(\widehat{\tau})$ implies that $\sigma \sim \tau$ or $\sigma \sim \tau^r$. This is almost certainly too optimistic. It is more reasonable to ask the following.

Question 4.6. Does $H(\widehat{\sigma}^k) \cong H(\widehat{\tau}^k)$ for all integers $k \geq 0$ imply that $\sigma \sim \tau$ or $\sigma \sim \tau^r$? What if we also assume that σ and τ are alternating braids?

This question is motivated to some extent by work of González-Meneses, who proves in [15] that if $\sigma^k = \tau^k$ for some k then $\sigma \sim \tau$. The notion that the answer to Question 4.6 might be “yes” for alternating braids is inspired by Greene’s recent result that if D_1, D_2 are reduced, alternating planar diagrams for knots K_1, K_2 in S^3 and $\widehat{HF}(\Sigma(S^3, K_1)) \cong \widehat{HF}(\Sigma(S^3, K_2))$, then D_1, D_2 are related by a sequence of Conway mutations [16].

ACKNOWLEDGEMENTS

The authors wish to thank Christian Blanchet, who asked the question that initiated this investigation, and Stephan Wehrli, for a number of interesting conversations. We owe a special debt of gratitude to Matt Hedden and Liam Watson, for sharing their paper [20]. In it, they show (using a simple but powerful argument that arose during conversations with Jeremy Van Horn-Morris) that if K is a Floer-simple fibered knot in a closed, oriented 3-manifold Y , then the monodromy

of K must be trivial. Our proofs of Theorem 3.1.(a) and 3.1.(b) were inspired by this idea. We are also grateful to Eddy Godelle for pointing us to the main result of [15] and to the anonymous referee for a number of valuable suggestions. Finally, we would like to thank Adam Lowrance, who—after hearing a talk given by the second author at the AMS sectional meeting in St. Louis—suggested that the computational complexity of the solution to the word problem described in Remark 3.12 deserves further investigation.

REFERENCES

- [1] Marta M. Asaeda, Józef H. Przytycki, and Adam S. Sikora, *Categorification of the Kauffman bracket skein module of I -bundles over surfaces*, *Algebr. Geom. Topol.* **4** (2004), 1177–1210 (electronic), DOI 10.2140/agt.2004.4.1177. MR2113902 (2006a:57010)
- [2] D. Auroux, J. E. Grigsby, and S. M. Wehrli, *Sutured Khovanov homology, Hochschild homology, and the Ozsváth-Szabó spectral sequence*, *math.GT/1303.1986*, 2013.
- [3] John A. Baldwin, David Shea Vela-Vick, and Vera Vértesi, *On the equivalence of Legendrian and transverse invariants in knot Floer homology*, *Geom. Topol.* **17** (2013), no. 2, 925–974, DOI 10.2140/gt.2013.17.925. MR3070518
- [4] J. Batson and C. Seed, *A link splitting spectral sequence in Khovanov homology*, *math.GT/1303.6240*, 2013.
- [5] Stephen Bigelow, *The Burau representation is not faithful for $n = 5$* , *Geom. Topol.* **3** (1999), 397–404, DOI 10.2140/gt.1999.3.397. MR1725480 (2001j:20055)
- [6] Stephen J. Bigelow, *Braid groups are linear*, *J. Amer. Math. Soc.* **14** (2001), no. 2, 471–486 (electronic), DOI 10.1090/S0894-0347-00-00361-1. MR1815219 (2002a:20043)
- [7] Joan S. Birman and Tara E. Brendle, *Braids: a survey*, *Handbook of knot theory*, Elsevier B. V., Amsterdam, 2005, pp. 19–103, DOI 10.1016/B978-044451452-3/50003-4. MR2179260 (2007a:57004)
- [8] Joan S. Birman and William W. Menasco, *Studying links via closed braids. V. The unlink*, *Trans. Amer. Math. Soc.* **329** (1992), no. 2, 585–606, DOI 10.2307/2153953. MR1030509 (92g:57010b)
- [9] Joan S. Birman and William W. Menasco, *A note on closed 3-braids*, *Commun. Contemp. Math.* **10** (2008), no. suppl. 1, 1033–1047, DOI 10.1142/S0219199708003150. MR2468377 (2010a:57006)
- [10] J.C. Cha and C. Livingston, *Knotinfo table of knot invariants*, 2006, Available from <http://www.indiana.edu/~knotinfo/>.
- [11] Tim D. Cochran, *Non-trivial links and plats with trivial Gassner matrices*, *Math. Proc. Cambridge Philos. Soc.* **119** (1996), no. 1, 43–53, DOI 10.1017/S0305004100073953. MR1356156 (96j:57005)
- [12] Patrick Dehornoy, Ivan Dynnikov, Dale Rolfsen, and Bert Wiest, *Ordering braids*, *Mathematical Surveys and Monographs*, vol. 148, American Mathematical Society, Providence, RI, 2008. MR2463428 (2009j:20046)
- [13] Benson Farb and Dan Margalit, *A primer on mapping class groups*, *Princeton Mathematical Series*, vol. 49, Princeton University Press, Princeton, NJ, 2012. MR2850125 (2012h:57032)
- [14] R. Fenn, M. T. Greene, D. Rolfsen, C. Rourke, and B. Wiest, *Ordering the braid groups*, *Pacific J. Math.* **191** (1999), no. 1, 49–74, DOI 10.2140/pjm.1999.191.49. MR1725462 (2000j:20064)
- [15] Juan González-Meneses, *The n th root of a braid is unique up to conjugacy*, *Algebr. Geom. Topol.* **3** (2003), 1103–1118 (electronic), DOI 10.2140/agt.2003.3.1103. MR2012967 (2004i:20069)
- [16] Joshua Evan Greene, *Lattices, graphs, and Conway mutation*, *Invent. Math.* **192** (2013), no. 3, 717–750, DOI 10.1007/s00222-012-0421-4. MR3049933
- [17] J. Elisenda Grigsby and Stephan M. Wehrli, *Khovanov homology, sutured Floer homology and annular links*, *Algebr. Geom. Topol.* **10** (2010), no. 4, 2009–2039, DOI 10.2140/agt.2010.10.2009. MR2728482 (2012c:57003)

- [18] *Topology and geometry in dimension three*, Contemporary Mathematics, vol. 560, American Mathematical Society, Providence, RI, 2011. Triangulations, invariants, and geometric structures; Papers from the Jacofest Conference in honor of William Jaco's 70th birthday held at Oklahoma State University, Stillwater, OK, June 4–6, 2010; Edited by Weiping Li, Loretta Bartolini, Jesse Johnson, Feng Luo, Robert Myers and J. Hyam Rubinstein. MR2866933 (2012g:57003)
- [19] Matthew Hedden and Yi Ni, *Khovanov module and the detection of unlinks*, *Geom. Topol.* **17** (2013), no. 5, 3027–3076. MR3190305
- [20] M. Hedden and L. Watson, *On the geography and botany of knot Floer homology*, arXiv: math.GT/1404.6913 (2014).
- [21] Ko Honda, William H. Kazez, and Gordana Matić, *Right-veering diffeomorphisms of compact surfaces with boundary*, *Invent. Math.* **169** (2007), no. 2, 427–449, DOI 10.1007/s00222-007-0051-4. MR2318562 (2008e:57028)
- [22] Mikhail Khovanov and Paul Seidel, *Quivers, Floer cohomology, and braid group actions*, *J. Amer. Math. Soc.* **15** (2002), no. 1, 203–271, DOI 10.1090/S0894-0347-01-00374-5. MR1862802 (2003d:53155)
- [23] Daan Krammer, *Braid groups are linear*, *Ann. of Math. (2)* **155** (2002), no. 1, 131–156, DOI 10.2307/3062152. MR1888796 (2003c:20040)
- [24] P. B. Kronheimer and T. S. Mrowka, *Khovanov homology is an unknot-detector*, *Publ. Math. Inst. Hautes Études Sci.* **113** (2011), 97–208, DOI 10.1007/s10240-010-0030-y. MR2805599
- [25] R. J. Lawrence, *Homological representations of the Hecke algebra*, *Comm. Math. Phys.* **135** (1990), no. 1, 141–191. MR1086755 (92d:16020)
- [26] R. Lipshitz, P. Ozsváth, and D. Thurston, *Bimodules in bordered Heegaard Floer homology*, *math.GT/1003.0598*, 2010.
- [27] Robert Lipshitz, Peter S. Ozsváth, and Dylan P. Thurston, *A faithful linear-categorical action of the mapping class group of a surface with boundary*, *J. Eur. Math. Soc. (JEMS)* **15** (2013), no. 4, 1279–1307, DOI 10.4171/JEMS/392. MR3055762
- [28] Paolo Lisca, Peter Ozsváth, András I. Stipsicz, and Zoltán Szabó, *Heegaard Floer invariants of Legendrian knots in contact three-manifolds*, *J. Eur. Math. Soc. (JEMS)* **11** (2009), no. 6, 1307–1363, DOI 10.4171/JEMS/183. MR2557137 (2010j:57016)
- [29] D. D. Long and M. Paton, *The Burau representation is not faithful for $n \geq 6$* , *Topology* **32** (1993), no. 2, 439–447, DOI 10.1016/0040-9383(93)90030-Y. MR1217079 (94c:20071)
- [30] Ciprian Manolescu, Peter Ozsváth, and Sucharit Sarkar, *A combinatorial description of knot Floer homology*, *Ann. of Math. (2)* **169** (2009), no. 2, 633–660, DOI 10.4007/annals.2009.169.633. MR2480614 (2009k:57047)
- [31] Ciprian Manolescu, Peter Ozsváth, Zoltán Szabó, and Dylan Thurston, *On combinatorial link Floer homology*, *Geom. Topol.* **11** (2007), 2339–2412, DOI 10.2140/gt.2007.11.2339. MR2372850 (2009c:57053)
- [32] John Atwell Moody, *The Burau representation of the braid group B_n is unfaithful for large n* , *Bull. Amer. Math. Soc. (N.S.)* **25** (1991), no. 2, 379–384, DOI 10.1090/S0273-0979-1991-16080-5. MR1098347 (92b:20041)
- [33] H. R. Morton, *The multivariable Alexander polynomial for a closed braid*, *Low-dimensional topology (Funchal, 1998)*, *Contemp. Math.*, vol. 233, Amer. Math. Soc., Providence, RI, 1999, pp. 167–172, DOI 10.1090/conm/233/03427. MR1701681 (2000e:57025)
- [34] Lenhard Ng, *Combinatorial knot contact homology and transverse knots*, *Adv. Math.* **227** (2011), no. 6, 2189–2219, DOI 10.1016/j.aim.2011.04.014. MR2807087 (2012j:53116)
- [35] Y. Ni, *Homological actions on sutured Floer homology*, *math.GT/1010.2808*, 2010.
- [36] Peter Ozsváth and Zoltán Szabó, *Holomorphic disks and genus bounds*, *Geom. Topol.* **8** (2004), 311–334, DOI 10.2140/gt.2004.8.311. MR2023281 (2004m:57024)
- [37] Peter Ozsváth and Zoltán Szabó, *Holomorphic disks and link invariants*, *math.GT/0512286*, 2005.
- [38] Peter Ozsváth and Zoltán Szabó, *Holomorphic disks, link invariants and the multi-variable Alexander polynomial*, *Algebr. Geom. Topol.* **8** (2008), no. 2, 615–692, DOI 10.2140/agt.2008.8.615. MR2443092 (2010h:57023)
- [39] Peter Ozsváth, Zoltán Szabó, and Dylan Thurston, *Legendrian knots, transverse knots and combinatorial Floer homology*, *Geom. Topol.* **12** (2008), no. 2, 941–980, DOI 10.2140/gt.2008.12.941. MR2403802 (2009f:57051)

- [40] Olga Plamenevskaya, *Transverse knots and Khovanov homology*, Math. Res. Lett. **13** (2006), no. 4, 571–586, DOI 10.4310/MRL.2006.v13.n4.a7. MR2250492 (2007d:57043)
- [41] Lawrence P. Roberts, *On knot Floer homology in double branched covers*, Geom. Topol. **17** (2013), no. 1, 413–467, DOI 10.2140/gt.2013.17.413. MR3035332

DEPARTMENT OF MATHEMATICS, 301 CARNEY HALL, BOSTON COLLEGE, CHESTNUT HILL, MASSACHUSETTS 02467

E-mail address: `john.baldwin@bc.edu`

DEPARTMENT OF MATHEMATICS, 301 CARNEY HALL, BOSTON COLLEGE, CHESTNUT HILL, MASSACHUSETTS 02467

E-mail address: `julia.grigsby@bc.edu`