

HYPERKÄHLER FLOER THEORY AS INFINITE DIMENSIONAL HAMILTONIAN SYSTEM

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ABSTRACT. We reformulate the equation characterizing the critical points of the hypersymplectic action functional as solutions of a Hamiltonian system on the iterated loop space. The intent is to gain more insight into dynamics of hyperkähler Floer theory.

1. INTRODUCTION

Floer theory was devised by Floer ([F1], [F2], [F3]) at the end of the 1980s with the intent to solve Arnold’s Conjecture on the number of fixed points of a Hamiltonian diffeomorphism. Since then, Floer theory developed into a powerful tool in symplectic geometry, but its techniques are not limited to symplectic geometry. For instance, in order to study 3-dimensional manifolds, Ozsváth & Szabó associated a suitable even-dimensional manifold to a 3-dimensional manifold and imitated Floer theoretic methods. More generally, apart from the resulting Heegaard Floer theory, various Floer theories for 3-manifolds such as Knot Floer homology, Seiberg-Witten Floer homology, Embedded Contact homology, etc. exist.

On hyperkähler manifolds, Floer theory and an analogue of Arnold’s Conjecture was established by Hohloch & Noetzel & Salamon ([HNS1], [HNS2]) and reproved and generalized by Ginzburg & Hein in [GH]. The important difference from the previous Floer settings is that hyperkähler Floer theory is not based on the Cauchy-Riemann, resp. pseudo-holomorphic equation, but on a ‘triholomorphic’ equation called the Cauchy-Riemann-Fueter equation. In classical Floer theory, the critical points of the symplectic action functional are 1-periodic Hamiltonian solutions. In hyperkähler Floer homology, the critical points of the hypersymplectic action functional are certain ‘triholomorphic 3-manifolds’.

The aim of this short note is to reformulate the ‘triholomorphic 3-manifolds’ appearing in hyperkähler Floer homology as 1-periodic solutions of a suitable Hamiltonian system on the (iterated) loop space in case the ‘triholomorphic 3-manifold’ is a torus. In a subsequent work, we intend to use this approach to come up with a nice geometric interpretation for the index of the ‘triholomorphic 3-manifolds’ since their index in Hohloch & Noetzel & Salamon [HNS1] is given abstractly by the spectral flow. In classical Floer theory, the Conley-Zehnder index provides a nice geometric interpretation of the index and we hope to find an analogue using the Hamiltonian setting for hyperkähler Floer homology.

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2. HYPERKÄHLER FLOER THEORY

In this section, we recall the setting for hyperkähler Floer theory in case the 3-dimensional M is a torus.

A manifold is *symplectic* if it carries a nondegenerate closed 2-form. Such a 2-form is called a *symplectic form*. Finite dimensional symplectic manifolds are even dimensional.

Definition 1. A manifold X is *hyperkähler* if there are three complex structures I_1, I_2 and I_3 and a metric $\langle \cdot, \cdot \rangle$ such that $I_1 I_2 = -I_2 I_1 = I_3$ and $\langle \cdot, \cdot \rangle = \langle I_i \cdot, I_i \cdot \rangle$ and $\omega_i := \langle I_i \cdot, \cdot \rangle$ are symplectic forms for $1 \leq i \leq 3$.

In other words, $\langle \cdot, \cdot \rangle$ is Kähler w.r.t. I_1, I_2 and I_3 . Finite dimensional hyperkähler manifolds are $4n$ -dimensional and the quaternions \mathbb{H} with complex structures i, j and k are an example. The only compact 4-dimensional hyperkähler manifolds are the 4-torus and K3-surfaces. In Berger’s classification, hyperkähler manifolds appear as those with holonomy group $Sp(n)$. They are Ricci-flat and thus Calabi-Yau such that in particular the first Chern class vanishes. Flat compact hyperkähler manifolds are $4n$ -tori modulo a finite group action.

Now consider the 3-torus $\mathbb{T}^3 = (\mathbb{R}/\mathbb{Z})^3 = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ with standard coordinates $t = (t_1, t_2, t_3)$, standard vector fields $\partial_1 := \frac{\partial}{\partial t_1}, \partial_2 := \frac{\partial}{\partial t_2}, \partial_3 := \frac{\partial}{\partial t_3}$ and volume form $\sigma := dt_1 \wedge dt_2 \wedge dt_3$. Let $(a_{ij}) \in GL(3, \mathbb{R})$ be a constant matrix and set $v_1 := \sum_{k=1}^3 a_{1k} \partial_k, v_2 := \sum_{k=1}^3 a_{2k} \partial_k$ and $v_3 := \sum_{k=1}^3 a_{3k} \partial_k$. Note that the Lie derivative $\mathcal{L}_{v_i} \sigma$ vanishes for $1 \leq i \leq 3$, i.e. the (constant) vector fields v_1, v_2, v_3 are volume preserving. Let $(X, I_1, I_2, I_3, \langle \cdot, \cdot \rangle)$ be a hyperkähler manifold and set $\mathcal{F} := \{f \in C^\infty(\mathbb{T}^3, X) \mid f \text{ contractible}\}$. As will be explained in more detail later in Section 3, its universal cover can be considered as

$$\tilde{\mathcal{F}} = \left\{ (f, [F^1], [F^2], [F^3]) \left| \begin{array}{l} f \in \mathcal{F}, \\ F^1 \in C^\infty(\mathbb{D} \times \mathbb{S}^1 \times \mathbb{S}^1, X), \quad F^1|_{\mathbb{T}^3} = f, \\ F^2 \in C^\infty(\mathbb{S}^1 \times \mathbb{D} \times \mathbb{S}^1, X), \quad F^2|_{\mathbb{T}^3} = f, \\ F^3 \in C^\infty(\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{D}, X), \quad F^3|_{\mathbb{T}^3} = f \end{array} \right. \right\}$$

where \mathbb{D} is the closed unit disk in \mathbb{R}^2 . The equivalence class $[F^\lambda]$ is the homotopy class of F^λ relative to the boundary, i.e. $F^1(e^{2\pi i t_1}, t_2, t_3) = f(t_1, t_2, t_3)$, etc., and we abbreviate $F^1_{t_2 t_3} := F^1(\cdot, t_2, t_3)$, etc. For $1 \leq j, k \leq 3$, set

$$\mathcal{A}_{jk} : \tilde{\mathcal{F}} \rightarrow \mathbb{R}, \quad \mathcal{A}_{jk}(f) := - \int_0^1 \int_0^1 \int_{\mathbb{D}} (F^k_{t_\mu t_\nu})^* \omega_j dt_\mu dt_\nu.$$

Geometrically, \mathcal{A}_{jk} is the symplectic action w.r.t. ω_j of the loop $t_k \mapsto f(t)$ averaged over the other two variables t_μ, t_ν . We define the **hypersymplectic action functional** \mathcal{A} via

$$\mathcal{A} := \sum_{j,k=1}^3 a_{jk} \mathcal{A}_{jk} : \tilde{\mathcal{F}} \rightarrow \mathbb{R}.$$

Let $H : X \times \mathbb{T}^3 \rightarrow \mathbb{R}$ be a Hamiltonian function and define the **perturbed hypersymplectic action functional** \mathcal{A}_H via

$$\mathcal{A}_H(f) := \mathcal{A}(f) - \int_{\mathbb{T}^3} H(f(t), t) dt.$$

Note that $\mathbb{T}^3 \simeq \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$. The definition of the hypersymplectic action functional on a cover of the ‘torus loop space’ \mathcal{F} is motivated by the construction of the classical symplectic action functional on a cover of the standard loop space (see for example McDuff & Salamon [MS, p. 154]).

- Remark 2.* 1) The action functional descends to \mathcal{F} if and only if it is independent of the equivalence classes $(f, [F^1], [F^2], [F^3])$ of f which in turn is equivalent to X being **symplectically aspherical** w.r.t. ω_1, ω_2 and ω_3 , i.e. $0 = \int_{\mathbb{S}^2} \mathfrak{s}^* \omega_i$ for all smooth $\mathfrak{s} : \mathbb{S}^2 \rightarrow X$ and $1 \leq i \leq 3$.
 2) For flat X , the action functional descends to \mathcal{F} , i.e. $\mathcal{A}, \mathcal{A}_H : \mathcal{F} \rightarrow \mathbb{R}$.

Proof. Item one is clear. It remains to prove item two: If X is flat, its universal cover is $\tau : \mathbb{R}^{4n} \rightarrow X$ and any map $\mathfrak{s} : \mathbb{S}^2 \rightarrow X$ lifts to a map $\tilde{\mathfrak{s}} : \mathbb{S}^2 \rightarrow \mathbb{R}^{4n}$ with $\tau \circ \tilde{\mathfrak{s}} = \mathfrak{s}$. Moreover, all symplectic forms on \mathbb{R}^{4n} are exact, i.e. there is a 1-form α_i such that $\tau^* \omega_i = d\alpha_i$ and we get

$$0 = \int_{\partial \mathbb{S}^2} \tilde{\mathfrak{s}}^* \alpha_i = \int_{\mathbb{S}^2} \tilde{\mathfrak{s}}^* d\alpha_i = \int_{\mathbb{S}^2} \tilde{\mathfrak{s}}^* \tau^* \omega_i = \int_{\mathbb{S}^2} \mathfrak{s}^* \omega_i.$$

□

A vector field ξ along f can be seen as a section in the pullback bundle $\xi \in \Gamma^\infty(f^*TX)$. If we want to compute the critical points of, say, \mathcal{A}_{12} we have to consider a variation of f and F^2 in direction ξ , i.e. we consider $s \mapsto f^s(t_1, t_1, t_3)$ with $\frac{d}{ds}|_{s=0} f^s = \xi$ and $s \mapsto (F^2)^s(t_1, z, t_3)$ with $\frac{d}{ds}|_{s=0} (F^2)^s|_{\mathbb{T}^3} = \xi$ and we calculate using Cartan’s formula and Stokes’ theorem,

$$\begin{aligned} d\mathcal{A}_{12}(f, [F^1], [F^2], [F^3]) \cdot \xi &= - \int_0^1 \int_0^1 \int_0^1 \omega_1|_{f(t)}(\xi(t), \partial_2 f(t)) dt_1 dt_2 dt_3 \\ &= \int_{\mathbb{T}^3} \langle \xi, I_1 \partial_2 f \rangle dt. \end{aligned}$$

For $f \in \mathcal{F}$ and the vector fields v_1, v_2, v_3 on \mathbb{T}^3 , we set $\partial_{v_i} f := df(v_i)$ for $1 \leq i \leq 3$. Thus for $\mathcal{A} = \sum_{j,k=1}^3 a_{jk} \mathcal{A}_{jk}$, the critical points $\text{Crit}(\mathcal{A})$ are maps $f \in \tilde{\mathcal{F}}$ with

$$(3) \quad \partial f := I_1 \partial_{v_1} f + I_2 \partial_{v_2} f + I_3 \partial_{v_3} f = 0$$

and $f \in \text{Crit}(\mathcal{A}_H)$ satisfies

$$(4) \quad \partial_H f := \partial f - \text{grad } H(f) = 0$$

where the gradient $\text{grad } H(f)|_t := \text{grad } H(f(t), t)$ is taken w.r.t. the X -valued variable of H and the metric $\langle \cdot, \cdot \rangle$. It is some kind of Dirac-type equation. A solution is called *nondegenerate* if the linearized operator for this equation is bijective. By elliptic regularity, every $W^{1,p}$ solution of the above equations is in fact smooth (cf. Hohloch & Noetzel & Salamon [HNS1, Theorem 3.1]).

Since the notion of a compact *Cartan hypercontact manifold* will appear only in the next two (quoted) theorems, we omit the definition here and refer the interested reader to Hohloch & Noetzel & Salamon [HNS1], [HNS2] for details. Note that the hypersymplectic action functional looks different in the case of a Cartan hypercontact manifold, but its critical points are described by the same equation as above.

Theorem 5 (Hyperkähler Arnold Conjecture, [HNS1, HNS2]). *Let M be either a compact Cartan hypercontact 3-manifold (with Reeb vector fields v_i) or the 3-torus*

(with a constant frame v_i). Let X be a compact flat hyperkähler manifold. Then the space of solutions of $\partial_H f = 0$ is compact. Moreover, if the contractible solutions are all nondegenerate, then their number is bounded below by the sum of the \mathbb{Z}_2 -Betti numbers of X . In particular, $\partial_H f = 0$ has a contractible solution for every H .

Theorem 5 relies on the construction of Floer theory and its computation:

Theorem 6 ([HNS1, HNS2]). *Let M and X be as in Theorem 5 and fix a class $\tau \in \pi_0(C^\infty(M, X))$. Then, for a generic perturbation $H : X \times M \rightarrow \mathbb{R}$, there is a natural Floer homology group $HF_*(M, X, \tau; H)$ associated to a chain complex generated by the solutions of $\partial_H f = 0$. The Floer homology groups associated to different choices of H are naturally isomorphic. Moreover, for the component τ_0 of the constant maps there is a natural isomorphism $HF_*(M, X, \tau_0; H) \cong H_*(X; \mathbb{Z}_2)$.*

Theorem 5 inspired Ginzburg & Hein [GH] to reprove the hyperkähler Arnold Conjecture on compact flat hyperkähler manifolds using Conley & Zehnder’s method of finite dimensional approximation. They also established the degenerate version. Salamon [Sa] reformulated and relaxed certain conditions in the construction of Hohloch & Noetzel & Salamon [HNS1, HNS2].

3. THE INFINITE DIMENSIONAL HAMILTONIAN SYSTEM ON THE ITERATED LOOP SPACE

In this section, we will reformulate the solutions of (3) and (4) as 1-periodic solutions of a Hamiltonian system on the iterated loop space.

Let $(X, I_1, I_2, I_3, \langle \cdot, \cdot \rangle)$ be a $4n$ -dimensional hyperkähler manifold with symplectic forms $\omega_i := \langle I_i \cdot, \cdot \rangle$ for $1 \leq i \leq 3$. The space of smooth contractible loops of X is given by $\mathcal{L}(X) := \{c \in C^\infty(\mathbb{S}^1, X) \mid c \text{ contractible}\}$. Its universal cover can be seen as

$$\widetilde{\mathcal{L}}(X) = \{(c, [C]) \mid c \in C^\infty(\mathbb{S}^1, X), C \in C^\infty(\mathbb{D}, X), C|_{\mathbb{S}^1} = c\}$$

for the following reason: there exists the homotopy group relation $\pi_k(\mathcal{L}(X)) \simeq \pi_{k+1}(X)$ which reduces for $k = 1$ to $\pi_1(\mathcal{L}(X)) \simeq \pi_2(X)$. Moreover, the universal cover holds $\mathcal{L}(X) \simeq \widetilde{\mathcal{L}}(X)/\pi_1(\mathcal{L}(X))$ which simplifies to $\mathcal{L}(X) \simeq \widetilde{\mathcal{L}}(X)/\pi_2(X)$, i.e. $\pi_2(X)$ acts on $\widetilde{\mathcal{L}}(X)$ via building the connected sum of a 2-sphere with a disk which yields again a disk (see also McDuff & Salamon [MS], p. 154, for a ‘smaller’ covering space of the loop space used in classical Floer theory).

The twice iterated loop space can be seen as

$$\mathcal{L}(\mathcal{L}(X)) \simeq \mathcal{L}^2(X) = C^\infty(\mathbb{S}^1 \times \mathbb{S}^1, X)$$

and its universal cover is

$$\widetilde{\mathcal{L}}^2(X) = \left\{ (\gamma, [G^1], [G^2]) \left| \begin{array}{l} \gamma \in C^\infty(\mathbb{S}^1 \times \mathbb{S}^1, X), \\ G^1 \in C^\infty(\mathbb{D} \times \mathbb{S}^1, X), G^1|_{\mathbb{S}^1 \times \mathbb{S}^1} = \gamma, \\ G^2 \in C^\infty(\mathbb{S}^1 \times \mathbb{D}, X), G^2|_{\mathbb{S}^1 \times \mathbb{S}^1} = \gamma \end{array} \right. \right\}.$$

$\mathcal{L}^2(X)$ carries three symplectic forms Ω_i with $1 \leq i \leq 3$ given by

$$\Omega_i|_\gamma(v, w) := \int_0^1 \int_0^1 \omega_i|_{\gamma(s,r)}(v(s, r), w(s, r)) ds dr$$

where v and w are vector fields along γ , i.e. $v, w \in T_\gamma \widetilde{\mathcal{L}}^2(X) \simeq \Gamma^\infty(\gamma^*TX)$. Moreover, there are three complex structures \mathcal{I}_i defined via

$$(\mathcal{I}_i|_\gamma v)(s, r) := I_i|_{\gamma(s,r)} v(s, r)$$

for $1 \leq i \leq 3$. We have a metric

$$\langle\langle v, w \rangle\rangle|_\gamma := \Omega_i|_\gamma(\mathcal{I}_i v, w)$$

for $1 \leq i \leq 3$. Replacing γ by $(\gamma, [G^1], [G^2])$ defines these symplectic forms, complex structures and the metric also on $\widetilde{\mathcal{L}}^2(X)$.

Theorem 7. *Equations (3) and (4) can be written as solutions of a Hamiltonian system on $\widetilde{\mathcal{L}}^2(X)$. If X is flat, then the construction descends to $\mathcal{L}^2(X)$.*

Proof. Let $H : X \times \mathbb{T}^3 \rightarrow \mathbb{R}$ be a smooth Hamiltonian function and abbreviate $G_{t_i}^1 := G^1(\cdot, t_i)$ and $G_{t_i}^2 := G^2(t_i, \cdot)$. We define the (autonomous) Hamiltonian functions

$$\mathcal{H}_{jkl} : \widetilde{\mathcal{L}}^2(X) \times \mathbb{S}^1 \rightarrow \mathbb{R}, \quad \mathcal{H}_{jkl}((\gamma, [G^1], [G^2]), t_l) := -a_{jk} \int_0^1 \int_{\mathbb{D}} (G_{t_\mu}^k)^* \omega_j dt_\mu$$

where $\mu \neq k$. Let $\xi \in \Gamma^\infty(\gamma^*TX)$ be a vector field along γ and $s \mapsto \gamma^s$ a variation of γ with $\frac{d}{ds}|_{s=0} \gamma^s = \xi$ and $s \mapsto (G^k)^s$ a variation with $\frac{d}{ds}|_{s=0} (G^k)^s|_{\mathbb{S}^1 \times \mathbb{S}^1} = \xi$. We compute

$$\begin{aligned} d\mathcal{H}_{jkl}(\gamma, [G^1], [G^2]) \cdot \xi &= -a_{jk} \frac{d}{ds} \Big|_{s=0} \int_0^1 \int_{\mathbb{D}} ((G_{t_\mu}^k)^s)^* \omega_j dt_\mu \\ &= -a_{jk} \int_0^1 \int_0^1 \omega_j(\xi, \partial_k \gamma) dt_k dt_\mu \\ &= \int_0^1 \int_0^1 \langle \xi, a_{jk} I_j \partial_k \gamma \rangle dt_k dt_\mu = -\Omega_l(\xi, I_l(a_{jk} I_j \partial_k \gamma)). \end{aligned}$$

Thus the Hamiltonian vector field of \mathcal{H}_{jkl} w.r.t. Ω_l is $I_l(a_{jk} I_j \partial_k \gamma)$ and the Hamiltonian equation becomes

$$\partial_l f = I_l(a_{jk} I_j \partial_k f) \iff a_{jk} I_j \partial_k f + I_l \partial_l f = 0$$

for $f : \mathbb{R} \rightarrow \widetilde{\mathcal{L}}^2(X)$, $t_l \mapsto f_{t_l}(\cdot, \cdot)$. In particular for the linear independent, constant vectors $w_1 = (a_{11}, a_{12}, 0)$, $w_2 := (a_{21}, a_{22}, 0)$, $w_3 := (0, 0, 1)$ and the Hamiltonian $\mathcal{H}_{113} + \mathcal{H}_{123} + \mathcal{H}_{213} + \mathcal{H}_{223}$ we obtain

$$(8) \quad I_1 \partial_{w_1} f + I_2 \partial_{w_2} f + I_3 \partial_{w_3} f = 0.$$

Since $\det(w_1, w_2, w_3) \neq 0 \neq \det(v_1, v_2, v_3)$ and the w_i and v_i preserve the volume form $dt_1 \wedge dt_2 \wedge dt_3$, we can make a change of variables transforming (8) into

$$\partial f = I_1 \partial_{v_1} f + I_2 \partial_{v_2} f + I_3 \partial_{v_3} f = 0.$$

If we consider instead of \mathcal{H}_{jkl} the perturbed, nonautonomous Hamiltonian

$$\mathcal{H}_{jkl} - \int_0^1 \int_0^1 H(\gamma(t_1, t_2), (t_1, t_2, t_3)) dt_1 dt_2$$

the Hamiltonian equation becomes

$$a_{jk} I_j \partial_k f + I_l \partial_l f - \text{grad } H(f) = 0.$$

If we consider in particular the nonautonomous Hamiltonian

$$\mathcal{H}_{113} + \mathcal{H}_{123} + \mathcal{H}_{213} + \mathcal{H}_{223} - \int_0^1 \int_0^1 H(\gamma(t_1, t_2), (t_1, t_2, t_3)) dt_1 dt_2$$

we obtain

$$I_1 \partial_{w_1} f + I_2 \partial_{w_2} f + I_3 \partial_{w_3} f - \text{grad } H(f) = 0$$

which can be transformed into (4). Contractible 1-periodic solutions of this equation are elements of

$$\widetilde{\mathcal{L}}^3(X) = \left\{ [f, F^1, F^2, F^3] \left\{ \begin{array}{l} f \in C^\infty(\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1, X), \\ F^1 \in C^\infty(\mathbb{D} \times \mathbb{S}^1 \times \mathbb{S}^1, X), \quad F^1|_{\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1} = f, \\ F^2 \in C^\infty(\mathbb{S}^1 \times \mathbb{D} \times \mathbb{S}^1, X), \quad F^2|_{\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1} = f, \\ F^3 \in C^\infty(\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{D}, X), \quad F^3|_{\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1} = f \end{array} \right. \right\}$$

which we can identify with the space $\widetilde{\mathcal{F}}$ on which our original action functional \mathcal{A} lives. \square

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