PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 143, Number 8, August 2015, Pages 3547–3559
S 0002-9939(2015)12516-6
Article electronically published on March 19, 2015

CONDENSER CAPACITY, EXPONENTIAL BLASCHKE PRODUCTS AND UNIVERSAL COVERING MAPS

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(Communicated by Pamela Gorkin)

ABSTRACT. Let B be an exponential Blaschke product, let C be a compact subset of the unit disc $\mathbb{D}$ with positive logarithmic capacity and let $K_n = B^{-1}(C) \cap \{z \in \mathbb{D} : |z| \leq 1 - 2^{-n}\}$. We give a sharp estimate for the rate of growth of the capacity of the condensers $(\mathbb{D}, K_n)$. Also, we examine a similar problem for universal covering maps of multiply connected Greenian domains and we give a precise formula in the case of doubly connected domains.

1. INTRODUCTION

A classical object of study in the geometric theory of functions is the distortion of sets, with respect to several geometric quantities (e.g. length, area, harmonic measure, Hausdorff measure, logarithmic capacity), under holomorphic functions. There are several well-known distortion results, expressed as monotonicity or invariance principles, that have important applications in complex analysis and potential theory; see e.g. [1, 17, 21]. We mention here some results about inverse images under holomorphic functions.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk in the complex plane and let φ be an inner function on $\mathbb{D}$; that is, φ is a bounded holomorphic function such that its radial boundary values have modulus 1 almost everywhere with respect to arc length measure m on $\partial\mathbb{D}$. The monograph [14] contains several basic facts about these objects. The following old result of Löwner concerns the arc length measure on the unit circle. If $A \subset \partial\mathbb{D}$ is a Borel set, φ is an inner function with $\varphi(0) = 0$ and

$$\varphi^{-1}(A) = \{e^{i\theta} \in \partial\mathbb{D} : \lim_{r\to 1} \varphi(re^{i\theta}) \text{ exists and belongs to } A\},$$

then $m(\varphi^{-1}(A)) = m(A)$. To prove this, let u be the harmonic measure of A and let v be the harmonic measure of $\varphi^{-1}(A)$ on $\mathbb{D}$. Then, from the definition of inner functions, the bounded harmonic function $u(\varphi(z)) - v(z)$, $z \in \mathbb{D}$, has radial limits equal to zero almost everywhere on $\partial\mathbb{D}$. Therefore, $u(\varphi(z)) = v(z)$, $z \in \mathbb{D}$, and the equality $m(\varphi^{-1}(A)) = m(A)$ follows from the relations $m(A) = u(0)$, $m(\varphi^{-1}(A)) = v(0)$ and the assumption $\varphi(0) = 0$. More generally, if f is holomorphic on $\mathbb{D}$ with $|f(z)| < 1$ and $f(0) = 0$, and if E is a Borel subset of

$$\{e^{i\theta} \in \partial\mathbb{D} : \lim_{r\to 1} f(re^{i\theta}) \text{ exists and has modulus } 1\},$$

Received by the editors October 11, 2013 and, in revised form, March 17, 2014.
2010 *Mathematics Subject Classification.* Primary 30C85, 30J10, 30C80, 31A15.
Key words and phrases. Condenser capacity, Lindelöf Principle, exponential Blaschke products, universal covering maps.
This work was supported by grants from NSERC (Canada) and FRQNT (Québec).

then $m(E) \leq m(f(E))$; see [1, p. 12], [21, p. 322]. Moreover, if f is univalent, Pommerenke proved that

$$c(E) \leq \frac{c(E)}{\sqrt{|f'(0)|}} \leq c(f(E)),$$

where c denotes the logarithmic capacity; see [17, pp. 341–350] for the proof and for more results about boundary behavior and logarithmic capacity. Also, the above results have been extended by several authors for Hausdorff measures, Riesz capacities and Hausdorff dimension; see [9] and references therein.

In the present article we consider condenser capacity. A *condenser* in the complex plane $\mathbb{C}$ is a pair (D, K) where D is a proper subdomain of $\mathbb{C}$, and K is a compact subset of D. Let h be the solution of the generalized Dirichlet problem on $D \setminus K$ with boundary values 0 on ∂D and 1 on ∂K. The function h is the *equilibrium potential* of the condenser (D, K). The *capacity* of (D, K) is

$$\mathrm{Cap}(D, K) = \int_{D\setminus K} |\nabla h(z)|^2 dm_2(z),$$

where m_2 is the two-dimensional Lebesgue measure. The problem of determining the asymptotic behavior of the capacity of a condenser when its plates $\mathbb{C} \setminus D$ and K approach each other has been studied by many authors and for several types of condensers; see [4, 8, 11, 20] and references therein.

In this article we examine the asymptotic behavior of the capacity of inverse images of condensers under Blaschke products and universal covering maps. Let f be a holomorphic function on D such that the condenser $(f(D), f(K))$ has positive capacity. We have the following inequality: if $V_f(K)$ is the number of preimages of $f(K)$ in K (see Section 2 for the precise definition), then (see [16, Theorem 3.1])

$$\mathrm{Cap}(f(D), f(K)) \leq \frac{\mathrm{Cap}(D, K)}{V_f(K)}.$$

In particular, let φ be an inner function on $\mathbb{D}$ and let $(\mathbb{D}, C)$ be a condenser with positive capacity. Suppose that every point of C has infinitely many preimages with respect to φ and let $\varphi^{-1}(C) = \cup_{n=1}^{\infty} K_n$, where $\{K_n\}$ is an increasing sequence of compact subsets of $\mathbb{D}$. Then $\lim_{n\to+\infty} V_\varphi(K_n) = +\infty$ and from the above inequality we obtain

$$\lim_{n\to+\infty} \mathrm{Cap}(\mathbb{D}, K_n) \geq \lim_{n\to+\infty} V_\varphi(K_n)\mathrm{Cap}(\mathbb{D}, C) = +\infty. \tag{1.1}$$

Our purpose is to examine the asymptotic behavior of the capacity of the condensers $(\mathbb{D}, K_n)$.

A Blaschke product B is called an *exponential Blaschke product* if there exists a constant $M \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ the set

$$B^{-1}(0) \cap R_n,$$

where $R_n = \{z \in \mathbb{D} : 2^{-n-1} < 1 - |z| \leq 2^{-n}\}$, counting multiplicities contains at most M points; see [5] for an equivalent characterization of exponential Blaschke products. For this class of inner functions our first result is as follows: if B is an exponential Blaschke product, $(\mathbb{D}, C)$ is a condenser with positive capacity and $K_n = B^{-1}(C) \cap T_n$ where $T_n = \{z \in \mathbb{D} : |z| \leq 1 - 2^{-n}\}$, then (see Theorem 3.3)

$$\mathrm{Cap}(\mathbb{D}, K_n) = \mathcal{O}(n), \qquad \text{as } n \to +\infty.$$

Note that $\lim_{n\to+\infty}\operatorname{Cap}(\mathbb{D},K_n)=+\infty$ by (1.1). Moreover, we give an explicit example of an exponential Blaschke product to show that the growth estimate $\mathcal{O}(n)$ is sharp. We also provide counterexamples with interpolating Blaschke products to show that the above growth estimate is not true in general without the assumption that B is exponential.

We will also consider a similar problem for universal covering maps. Let D be a doubly connected Greenian domain in the complex plane and let $f:\mathbb{D}\mapsto D$ be a universal covering map of D; see e.g. [7,15]. Suppose that E is a connected compact subset of a fundamental neighborhood of D such that the condenser (D,E) has positive capacity. Then, if E_0, $E_1,\dots$ is a certain enumeration of the connected components of $f^{-1}(E)$ and $K_n=\bigcup_{i=0}^{n-1}E_i$, our second result is the following precise formula for the rate of growth of the capacity of the condensers $(\mathbb{D},K_n)$ (see Theorem 4.4):

$$\operatorname{Cap}(D,E)=\lim_{n\to\infty}\frac{\operatorname{Cap}(\mathbb{D},K_n)}{n}.$$

For general multiply connected Greenian domains D we give an upper and a lower bound for the ratio $\frac{\operatorname{Cap}(\mathbb{D},K_n)}{n}$, for every $n\in\mathbb{N}$.

2. Background material

2.1. **Green energy and equilibrium measure.** Let (D,K) be a condenser. If D is a Greenian domain, the *Green equilibrium energy* of (D,K) is defined by

$$I(D,K)=\inf_{\mu}\iint G_D(z,w)d\mu(z)d\mu(w),$$

where $G_D(x,y)$ is the Green function of D and the infimum is taken over all probability Borel measures μ supported on K. When $I(D,K)<+\infty$, the unique probability Borel measure μ_K for which the above infimum is attained is the *Green equilibrium measure*. The function

$$U^D_{\mu_K}(z)=\int G_D(z,w)d\mu_K(w),\qquad z\in D,$$

is the *Green equilibrium potential* of (D,K). It is true that $U^D_{\mu_K}=I(D,K)$ on K except on a set of zero logarithmic capacity and

$$U^D_{\mu_K}\le I(D,K)\tag{2.1}$$

on D; see e.g. [12, p. 174]. From the formula (see [12, p. 97])

$$\iint G_D(z,w)d\mu(z)d\mu(w)=\frac{1}{2\pi}\int_D|\nabla U^D_\mu(z)|^2dm_2(z),$$

we obtain that

$$\operatorname{Cap}(D,K)=\frac{2\pi}{I(D,K)}.\tag{2.2}$$

Finally, condenser capacity satisfies the *strong subadditivity property* (see [2, p. 136]): if (D,K_1) and (D,K_2) are two condensers, then

$$\operatorname{Cap}(D,K_1\cup K_2)+\operatorname{Cap}(D,K_1\cap K_2)\le\operatorname{Cap}(D,K_1)+\operatorname{Cap}(D,K_2).$$

For more information about potential theory see e.g. [2,12,18,21].

2.2. BL_1 **holomorphic functions.** Let f be a non-constant holomorphic function on a Greenian domain D such that $f(D)$ is Greenian. We denote by $m(a)$ the *multiplicity* of the zero of $f(z) - f(a)$ at $a \in D$ and by

$$\mathrm{v}(y) = \sum_{f(a)=y} m(a)$$

the *valency* of f at $y \in f(D)$. The following inequality is known as the Lindelöf Principle (see e.g. [10]):

$$G_{f(D)}(y_0, f(z)) \geq \sum_{f(a)=y_0} m(a) G_D(a, z), \tag{2.3}$$

where $z \in D$ and $y_0 \in f(D)$. For fixed $y_0 \in f(D)$, if equality holds in (2.3) for a point $z \in D$ with $f(z) \neq y_0$, then it holds for every point in D. Following [10, p. 447], we will denote by BL_1 the class of holomorphic functions for which equality holds in (2.3) for a point $y_0 \in f(D)$ and for every $z \in D$. If

$$\sup_{y \in f(D)} \mathrm{v}(y) = +\infty$$

and $f \in BL_1$, we will say that f is an *infinite* BL_1 *function.*

Theorem 2.1 ([10, p. 470])**.** *Let f be an infinite BL_1 function on a Greenian domain D. Then equality holds in* (2.3) *and* $\mathrm{v}(y_0) = +\infty$ *for every $y_0 \in f(D)$ except on an F_σ set of zero logarithmic capacity.*

Theorem 2.1 is a generalization of Frostman's theorem (see [14, Theorem 2.5, p. 35]) about inner functions on the unit disc to BL_1 functions on general Greenian domains. There is more on this in Section 3. From the definition of Blaschke products, it follows that they are BL_1 functions for $y_0 = 0$. Also, universal covering maps of Greenian domains are examples of BL_1 functions. We note that, for universal covering maps, the exceptional set of zero logarithmic capacity in Theorem 2.1 is empty (see [13, Lemma 3]). For a characterization of the equality cases in Lindelöf's Principle see [3, Theorem 3].

2.3. **A condenser capacity inequality.** Let (D, K) be a condenser, let f be a non-constant holomorphic function on the domain D such that the condenser $(f(D), f(K))$ has positive capacity, let ν be the Green equilibrium measure of $(f(D), f(K))$ and let $E := \mathrm{supp}(\nu) \setminus f(\{a \in K : m(a) \geq 2\})$. For every $y \in E$, let $N_f(y, K)$ be the cardinality of the set $\{x \in K : f(x) = y\}$. If $V_f(K) := \min_{y \in E} N_f(y, K)$, then

$$\mathrm{Cap}(f(D), f(K)) \leq \frac{\mathrm{Cap}(D, K)}{V_f(K)}; \tag{2.4}$$

see [16, Theorem 3.1].

3. Exponential Blaschke products and condenser capacity

Let φ be an inner function and let $a \in \mathbb{D}$. The function $\varphi_a := \frac{\varphi - a}{1 - \bar{a}\varphi}$ is called a Frostman shift of φ. According to Frostman's theorem, φ_a is a Blaschke product for every $a \in \mathbb{D}$ except on a set of zero logarithmic capacity. The following result says that exponential Blaschke products are Frostman shift invariant and the exceptional set in Frostman's theorem for an exponential Blaschke product is empty.

Theorem 3.1 ([5]). *If B is an exponential Blaschke product, then $B_a = \frac{B-a}{1-\bar{a}B}$ is an exponential Blaschke product for every $a \in \mathbb{D}$.*

Let B be a Blaschke product. For every $a \in \mathbb{D}$ and $n \in \mathbb{N}$, let $A_B(a,n)$ denote the number of the points in the set

$$B^{-1}(a) \cap \{z \in \mathbb{D} : 2^{-n-1} \leq 1 - |z| \leq 2^{-n}\},$$

counting multiplicities. From Theorem 3.1 it follows that for every exponential Blaschke product B and for every $a \in \mathbb{D}$,

$$M(a) := \sup_{n\in\mathbb{N}} A_B(a,n) < +\infty.$$

We will need the following strengthened version.

Lemma 3.2. *Let B be an exponential Blaschke product and let $M(a)$, $a \in \mathbb{D}$, be as above. Then for every compact subset K of $\mathbb{D}$,*

$$\sup_{a\in K} M(a) < +\infty. \tag{3.1}$$

Proof. We will use the following characterization of exponential Blaschke products (see [5, Theorem 1]). If B is a Blaschke product, then B is an exponential Blaschke product if and only if B' belongs to the weak Hardy space H^1_w, that is, if and only if there exists $C < +\infty$ such that for every $r \in (0,1)$ and for every $\lambda > 0$,

$$m(\{e^{i\theta} : |B'(re^{i\theta})| > \lambda\}) \leq \frac{C}{\lambda}.$$

From Theorem 3.1 it follows that for every $a \in \mathbb{D}$

$$C(a) = \inf\{t \in \mathbb{R} : m(\{e^{i\theta} : |B'_a(re^{i\theta})| > \lambda\}) \leq \frac{t}{\lambda}, \forall r \in (0,1), \lambda > 0\}$$

is finite. Then, from the proof of [5, Theorem 1], we obtain that the inequality (3.1) follows from the inequality

$$\sup_{a\in K} C(a) < +\infty. \tag{3.2}$$

To prove (3.2), let $\rho < 1$ such that $K \subset \{z \in \mathbb{D} : |z| < \rho\}$. For $a \in \mathbb{D}$ we have

$$B'_a(z) = \Big(\frac{B(z)-a}{1-\bar{a}B(z)}\Big)' = \frac{1-|a|^2}{(1-\bar{a}B(z))^2}B'(z).$$

Therefore, for $|a| < \rho$, we have

$$|B'_a(z)| \leq \frac{1+\rho}{1-\rho}|B'(z)|. \tag{3.3}$$

From inequality (3.3) it follows that for every $r \in (0,1)$ and for every $\lambda > 0$,

$$\{e^{i\theta} : |B'_a(re^{i\theta})| > \lambda\} \subset \Big\{e^{i\theta} : |B'(re^{i\theta})| > \frac{\lambda(1-\rho)}{(1+\rho)}\Big\}$$

and

$$\begin{aligned} m(\{e^{i\theta} : |B'_a(re^{i\theta})| > \lambda\}) &\leq m\Big(\Big\{e^{i\theta} : |B'(re^{i\theta})| > \frac{\lambda(1-\rho)}{(1+\rho)}\Big\}\Big) \\ &\leq \frac{(C(0)+1)(1+\rho)}{\lambda(1-\rho)}. \end{aligned}$$

Therefore, $C(a) \leq \frac{(C(0)+1)(1+\rho)}{1-\rho}$ for every $a \in K$ and (3.2) follows. $\square$

Also, we will need the following inequality. If $\{z_n\}$ are the zeroes of an exponential Blaschke product B, then (see [5]) there exists a constant $A = A(M(0))$ such that for every $r \in (0,1)$

$$\sum_{|z_n|\geq r} \log\frac{1}{|z_n|} \leq A\log\frac{1}{r}. \tag{3.4}$$

In the following result we study the asymptotic behavior of the capacity of the inverse image of a condenser under an exponential Blaschke product.

Theorem 3.3. *Let B be an exponential Blaschke product, let $(\mathbb{D}, C)$ be a condenser with positive capacity and let $K_n = B^{-1}(C)\cap T_n$ where $T_n = \{z\in\mathbb{D} : |z| \leq 1-2^{-n}\}$. Then*

$$\mathrm{Cap}(\mathbb{D}, K_n) = \mathcal{O}(n), \qquad \text{as } n\to+\infty.$$

Proof. From Lemma 3.2 we have

$$M = \sup_{y\in C} M(y) < +\infty.$$

For every $n\in\mathbb{N}$, let μ_n be the equilibrium measure of the condenser $(\mathbb{D}, K_n)$ and consider the measure

$$\nu_n(A) = \mu_n(B^{-1}(A)), \qquad A\subset C \text{ Borel measurable.}$$

Then ν_n is a probability Borel measure on C, for every $n\in\mathbb{N}$. Therefore,

$$I(\mathbb{D}, C) \leq \iint G_{\mathbb{D}}(x,y)d\nu_n(x)d\nu_n(y), \qquad \text{for every } n\in\mathbb{N}.$$

Let $N_n(y) = B^{-1}(y)\cap T_n$, $y\in C$, $n\in\mathbb{N}$. Fix $k_0\in\mathbb{N}$. Since B is a BL_1 function and $B(\mathbb{D}) = \mathbb{D}$,

$$\begin{aligned}
\iint G_{\mathbb{D}}(x,y)d\nu_n(x)d\nu_n(y) &= \iint G_{\mathbb{D}}(B(z),y)d\mu_n(z)d\nu_n(y)\\
&= \iint \sum_{a\in N_{n+k_0}(y)} G_{\mathbb{D}}(z,a)d\mu_n(z)d\nu_n(y)\\
&+ \iint \sum_{a\in B^{-1}(y)\setminus N_{n+k_0}(y)} G_{\mathbb{D}}(z,a)d\mu_n(z)d\nu_n(y).
\end{aligned}$$

For the first integral, from inequality (2.1) and the definition of M, we have

$$\begin{aligned}
\iint \sum_{a\in N_{n+k_0}(y)} G_{\mathbb{D}}(z,a)d\mu_n(z)d\nu_n(y) &= \int \sum_{a\in N_{n+k_0}(y)} \int G_{\mathbb{D}}(z,a)d\mu_n(z)d\nu_n(y)\\
&= \int \sum_{a\in N_{n+k_0}(y)} U^{\mathbb{D}}_{\mu_n}(a)d\nu_n(y)\\
&\leq \int \sum_{a\in N_{n+k_0}(y)} I(\mathbb{D}, K_n)d\nu_n(y)\\
&\leq M(n+k_0)I(\mathbb{D}, K_n). \qquad (3.5)
\end{aligned}$$

For the second integral, we note that for every $a \in B^{-1}(y) \setminus N_{n+k_0}(y)$, $y \in C$, the function $z \mapsto G_{\mathbb{D}}(z,a)$ is a positive harmonic function on the disc $\{z \in \mathbb{D} : |z| < 1 - 2^{-n-k_0}\}$. From Harnack's inequality (see [18, p. 14]),

$$G_{\mathbb{D}}(z,a) \le \frac{1 - 2^{-n-k_0} + |z|}{1 - 2^{-n-k_0} - |z|} G_{\mathbb{D}}(0,a).$$

So, for $z \in K_n$,

$$G_{\mathbb{D}}(z,a) \le \frac{2}{2^{-n}(1 - 2^{-k_0})} G_{\mathbb{D}}(0,a). \tag{3.6}$$

From the inequalities (3.4) and (3.6), we obtain that

$$\begin{aligned} &\iint \sum_{a \in B^{-1}(y) \setminus N_{n+k_0}(y)} G_{\mathbb{D}}(z,a) d\mu_n(z) d\nu_n(y) \\ \le\ &\iint \frac{2}{2^{-n}(1-2^{-k_0})} \sum_{a \in B^{-1}(y) \setminus N_{n+k_0}(y)} G_{\mathbb{D}}(0,a) d\mu_n(z) d\nu_n(y) \\ \le\ &\frac{2A}{2^{-n}(1-2^{-k_0})} \log\Big(\frac{1}{1-2^{-n-k_0}}\Big). \end{aligned} \tag{3.7}$$

From the inequalities (3.5) and (3.7), we obtain

$$I(\mathbb{D}, C) \le MnI(\mathbb{D}, K_n) + Mk_0 I(\mathbb{D}, K_n) + \frac{2A}{2^{-n}(1-2^{-k_0})} \log\Big(\frac{1}{1-2^{-n-k_0}}\Big).$$

By (1.1) it follows that $\lim_{n \to +\infty} I(\mathbb{D}, K_n) = 0$. Therefore, letting $n \to +\infty$,

$$I(\mathbb{D}, C) \le \liminf_{n \to +\infty} MnI(\mathbb{D}, K_n) + \frac{2A}{(1-2^{-k_0})} 2^{-k_0}.$$

Since k_0 was arbitrary, letting $k_0 \to +\infty$, we obtain

$$I(\mathbb{D}, C) \le \liminf_{n \to +\infty} MnI(\mathbb{D}, K_n)$$

or

$$\limsup_{n \to +\infty} \frac{\mathrm{Cap}(\mathbb{D}, K_n)}{n} \le M\mathrm{Cap}(\mathbb{D}, C),$$

and the conclusion follows. □

Remark 3.4. Let B be an exponential Blaschke product, let $(\mathbb{D}, C)$ be a condenser with positive capacity and let K_n and M be as above. Suppose that every point of C has at least one pre-image in $\{z \in \mathbb{D} : 2^{-n-1} \le 1 - |z| \le 2^{-n}\}$. Then $V_B(K_n) \ge n$ and from inequality (2.4) we obtain

$$\begin{aligned} 0 < \mathrm{Cap}(\mathbb{D}, C) &\le \liminf_{n \to +\infty} \frac{\mathrm{Cap}(\mathbb{D}, K_n)}{V_B(K_n)} \\ &\le \limsup_{n \to +\infty} \frac{\mathrm{Cap}(\mathbb{D}, K_n)}{n} \\ &\le M\mathrm{Cap}(\mathbb{D}, C). \end{aligned} \tag{3.8}$$

Consider, for example, the exponential Blaschke product with zeroes $z_n = 1 - 3 \cdot 2^{-n-2}$ and let C be a closed disk with center at the origin and sufficiently small radius. Inequality (3.8) shows that, in general, we cannot replace $\mathcal{O}(n)$ in Theorem 3.3 by $\mathcal{O}(n^t)$ for some $t < 1$.

We now give an example of an interpolating Blaschke product which is not an exponential Blaschke product, for which the conclusion of Theorem 3.3 is not valid. Let E be a compact subset of $\mathbb{D}$ with zero logarithmic capacity which contains at least two points and let $f : \mathbb{D} \mapsto \mathbb{D} \setminus E$ be a universal covering map of $\mathbb{D} \setminus E$. Then f is an inner function (see [6, p. 37]); in fact, if $0 \notin E$, then $f^{-1}(0)$ is a uniformly separated sequence and f is an interpolating Blaschke product. For example, take $E = \{-\frac{1}{2}, \frac{1}{2}\}$. Let C be a connected compact subset of a fundamental neighborhood of $\mathbb{D}\setminus E$ with positive logarithmic capacity and let $K_n = f^{-1}(C) \cap T_n$. Then $V_f(K_n)$ equals the number of connected components of $f^{-1}(C)$ that are contained in K_n. It is well known that the rate of growth of the number of connected components of $f^{-1}(C)$ is exponential; in particular, for every $p \in \mathbb{N}$, we have $\lim_{n\to+\infty} \frac{V_f(K_n)}{n^p} = +\infty$. Therefore,

$$\begin{aligned} \lim_{n\to+\infty} \frac{\mathrm{Cap}(\mathbb{D}, K_n)}{n^p} &= \lim_{n\to+\infty} \frac{\mathrm{Cap}(\mathbb{D}, K_n)}{V_f(K_n)} \frac{V_f(K_n)}{n^p} \\ &\geq \lim_{n\to+\infty} \mathrm{Cap}(\mathbb{D}, C) \frac{V_f(K_n)}{n^p} = +\infty. \end{aligned}$$

4. Universal covering maps and condenser capacity

We will need some notation from the theory of universal covering maps. Let D be a multiply connected Greenian domain in the complex plane and let $f : \mathbb{D} \mapsto D$ be a universal covering map of D. We refer to [7] and [15] for information about universal covering mappings and other related notions that we will use. Let $u_0 \in D$ and let $V \subset D$ be a fundamental neighborhood of u_0; that is,

$$f^{-1}(V) = \bigcup_i A_i,$$

where A_i are disjoint connected open subsets of $\mathbb{D}$ such that the restriction of f to A_i is one to one and onto V. Also, E will be a connected compact subset of V and we will denote by E_i the connected components of $f^{-1}(E)$. We make the connectedness assumption on E only for simplicity in the statements of the following results; otherwise, one can define $E_i = f^{-1}(E) \cap A_i$ for a disconnected compact set E. We will denote by $\mathrm{Deck}(f)$ the group of covering transformations of f, that is, the conformal automorphisms T of $\mathbb{D}$ such that $f \circ T = f$.

Theorem 4.1. *Let D, E and f be as above, let $E_1, E_2, \dots,$ be an enumeration of the connected components of $f^{-1}(E)$ and let $K_n = \bigcup_{i=1}^n E_i$. Then, for every $n \in \mathbb{N}$,*

$$\mathrm{Cap}(D, E) \leq \frac{\mathrm{Cap}(\mathbb{D}, K_n)}{n} \leq \mathrm{Cap}(\mathbb{D}, E_1). \tag{4.1}$$

Proof. We have $V_f(K_n) = n$ and from inequality (2.4)

$$\mathrm{Cap}(D, E) \leq \frac{\mathrm{Cap}(\mathbb{D}, K_n)}{n},$$

for every $n \in \mathbb{N}$.

Let $m \in \mathbb{N}$. There exists a covering transformation T such that $T(E_1) = E_m$. From the conformal invariance of condenser capacity,

$$\mathrm{Cap}(\mathbb{D}, E_1) = \mathrm{Cap}(T(\mathbb{D}), T(E_1)) = \mathrm{Cap}(\mathbb{D}, E_m).$$

Therefore, from the strong subadditivity property of condenser capacity,

$$\frac{\mathrm{Cap}(\mathbb{D}, K_n)}{n} \leq \frac{\sum_{i=1}^{n} \mathrm{Cap}(\mathbb{D}, E_i)}{n} = \frac{n\mathrm{Cap}(\mathbb{D}, E_1)}{n} = \mathrm{Cap}(\mathbb{D}, E_1),$$

for every $n \in \mathbb{N}$. $\square$

Suppose that the Greenian domain D is doubly connected. Let γ_1 be a simple closed curve in D with initial point in E which is not null-homotopic. For every $n \geq 1$, let γ_n be the curve obtained by tracing the curve γ a number n times and let γ_{-n} be the curve obtained by tracing the curve γ a number n times in the opposite direction. Let $E_0 \subset \mathbb{D}$ be a connected component of $f^{-1}(E)$. For every $k \in \mathbb{Z} \setminus \{0\}$, let $\tilde{\gamma}_k$ be the unique lifting of γ_k with initial point in E_0 and let E_k be the unique connected component of $f^{-1}(E)$ which contains the end point of $\tilde{\gamma}_k$. Then the components E_k, $k \in \mathbb{Z}$, are mutually disjoint and $f^{-1}(E) = \bigcup_{k\in\mathbb{Z}} E_k$, since the fundamental group of D has one generator. We will need the following lemma.

Lemma 4.2. *Let D, E, f and E_k be as above and let h_k be the restriction of f to E_k, $k \in \mathbb{Z}$. Then the sequence of functions*

$$g_n(z, y) = G_D(f(z), y) - \sum_{k=-n}^{n} G_{\mathbb{D}}(z, h_k^{-1}(y)), \qquad n \in \mathbb{N},$$

converges uniformly to the zero function on $E_0 \times E$.

Proof. Let $G_D(x, y) = \log \frac{1}{|x-y|} + H_D(x, y)$. Then

$$\begin{aligned} g_n(z, y) &= \log \frac{1}{|f(z) - y|} + H_D(f(z), y) - \sum_{k=-n}^{n} \log \frac{|1 - \bar{z}h_k^{-1}(y)|}{|z - h_k^{-1}(y)|} \\ &= \log \frac{|z - h_0^{-1}(y)|}{|f(z) - y|} + H_D(f(z), y) - \log |1 - \bar{z}h_0^{-1}(y)| \\ &\quad - \sum_{0\neq k=-n}^{n} \log \frac{|1 - \bar{z}h_k^{-1}(y)|}{|z - h_k^{-1}(y)|}. \end{aligned}$$

Since f has non-vanishing derivative on $\mathbb{D}$, the function $(z, y) \mapsto \log \frac{|z-h_0^{-1}(y)|}{|f(z)-y|}$ is locally bounded on an open neighborhood Ω of $E_0 \times E$. Therefore, $(z, y) \mapsto g_n(z, y)$ is locally bounded on Ω and is harmonic as a function of each variable separately. From [2, Corollary 3.3.7, p. 72] we obtain that g_n is harmonic on Ω. Therefore, g_n is a monotone sequence of positive harmonic functions which converges to the zero function on Ω. From [2, Theorem 1.5.8, p. 17] we obtain that g_n converges locally uniformly on Ω and the conclusion follows. $\square$

We will consider the sequence F_n of compact subsets of $\mathbb{D}$, defined in the following manner:

$$F_0 = E_0,\ F_1 = E_1,\ F_2 = E_{-1},\ F_3 = E_2,\ F_4 = E_{-2}, \ldots .$$

More precisely, $F_n = E_{\phi^{-1}(n)}$, $n \in \mathbb{N}$, where

$$\phi(k) = \begin{cases} 2k - 1, & k > 0; \\ -2k, & k \leq 0. \end{cases}$$

Also, we will consider the compact sets $K_n = \bigcup_{i=0}^{n} F_i$, $n \in \mathbb{N}$.

Lemma 4.3. *Let D, E, f, F_n and K_n be as above and let f_n be the restriction of f to F_n, $n \in \mathbb{N}$. Then for every $\epsilon > 0$ there exist $m_0 \in \mathbb{N}$ such that for every $n \in \mathbb{N}$*

$$\sup\left\{G_D(f(z),u) - \sum_{i=0}^{n+m_0} G_{\mathbb{D}}(z, f_i^{-1}(u)) : z \in K_n,\, u \in E\right\} \le \epsilon. \tag{4.2}$$

Proof. Let E_k, $k \in \mathbb{Z}$, be the connected components of $f^{-1}(E)$ defined as above and let h_k be the restriction of f to E_k, $k \in \mathbb{Z}$. Note that $h_k = f_{\phi(k)}$.

Fix $\epsilon > 0$. From Lemma 4.2, there exists $m \in \mathbb{N}$ such that

$$\sup\left\{G_D(f(z),y) - \sum_{k=-m}^{m} G_{\mathbb{D}}(z, h_k^{-1}(y)) : z \in E_0,\, y \in E\right\} \le \epsilon. \tag{4.3}$$

For every $k \in \mathbb{Z}$, let $T_k \in \mathrm{Deck}(f)$ be the covering transformation such that $T_k(E_0) = E_k$. Then, $T_k(E_\lambda) = E_{\lambda+k}$ for every $\lambda \in \mathbb{Z}$ (see [15, Theorem 10.4.2, p. 492]). Let $m_0 = 2m$ and $n_0 \in \mathbb{N}$. Then, from the conformal invariance of the Green function and the fiber preserving property of covering transformations, we obtain that if $n \le n_0$ and $k = \phi^{-1}(n)$,

$$\begin{aligned}
& \sup\left\{G_D(f(z),y) - \sum_{i=0}^{n_0+m_0} G_{\mathbb{D}}(z, f_i^{-1}(y)) : z \in F_n,\, y \in E\right\} \\
= \; & \sup\left\{G_D(f(z),y) - \sum_{i=0}^{n_0+m_0} G_{\mathbb{D}}(z, f_i^{-1}(y)) : z \in E_k,\, y \in E\right\} \\
\le \; & \sup\left\{G_D(f(z),y) - \sum_{j=k-m}^{k+m} G_{\mathbb{D}}(z, h_j^{-1}(y)) : z \in E_k,\, y \in E\right\} \\
= \; & \sup\left\{G_D(f(T_k(z)),y) - \sum_{j=k-m}^{k+m} G_{\mathbb{D}}(T_k(z), h_j^{-1}(y)) : z \in E_0,\, y \in E\right\} \\
= \; & \sup\left\{G_D(f(z),y) - \sum_{j=k-m}^{k+m} G_{\mathbb{D}}(z, h_{-k+j}^{-1}(y)) : z \in E_0,\, y \in E\right\} \\
= \; & \sup\left\{G_D(f(z),y) - \sum_{j=-m}^{m} G_{\mathbb{D}}(z, h_j^{-1}(y)) : z \in E_0,\, y \in E\right\} \overset{(4.3)}{\le} \epsilon.
\end{aligned}$$

Since the above inequality is true for every $n \le n_0$, we obtain

$$\sup\left\{G_D(f(z),y) - \sum_{i=0}^{n_0+m_0} G_{\mathbb{D}}(z, f_i^{-1}(y)) : z \in K_{n_0},\, y \in E\right\} \le \epsilon.$$

Since n_0 was arbitrary, (4.2) follows. $\square$

We continue with the following result which gives the precise rate of growth of condenser capacity under inverse images of universal covering maps of doubly connected domains.

Theorem 4.4. *Let D, E, f and K_n be as above, $n \in \mathbb{N}$. Then*

$$\mathrm{Cap}(D,E) = \lim_{n\to\infty} \frac{\mathrm{Cap}(\mathbb{D}, K_n)}{n+1}. \tag{4.4}$$

Proof. Let F_n be as above and let f_n be the restriction of f to F_n, $n \in \mathbb{N}$.

We have $V_f(K_n) = n+1$ and from inequality (2.4)

$$\operatorname{Cap}(D,E) \le \frac{\operatorname{Cap}(\mathbb{D},K_n)}{n+1},$$

for every $n \in \mathbb{N}$. Equivalently, $(n+1)I(\mathbb{D},K_n) \le I(D,E)$. Therefore,

$$\limsup_{n\to\infty}(n+1)I(\mathbb{D},K_n) \le I(D,E) \tag{4.5}$$

and $\lim_{n\to\infty} I(\mathbb{D},K_n) = 0$.

For every $n \in \mathbb{N}$, let μ_n be the Green equilibrium measure of $(\mathbb{D},K_n)$ and consider the measure

$$\nu_n(A) = \mu_n(f^{-1}(A)), \qquad A \subset E \text{ Borel measurable.}$$

Then ν_n is a probability Borel measure with support in E for every $n \in \mathbb{N}$. Let ν_{n_k} be a subsequence of ν_n which converges weak star to a measure ν. Then ν is a probability Borel measure with support in E.

Fix $\epsilon > 0$. From Lemma 4.3, there exist $m_0 \in \mathbb{N}$ such that for every $n \in \mathbb{N}$

$$\sup\Big\{G_D(f(z),u) - \sum_{i=0}^{n+m_0} G_{\mathbb{D}}(z, f_i^{-1}(u)) : z \in K_n,\, u \in E\Big\} \le \epsilon.$$

From the definition of Green equilibrium energy and the lower-semicontinuity of energy in measure (see [12, pp. 78–79]) we obtain

$$\begin{aligned}
I(D,E) &\le \iint G_D(v,u)d\nu(v)d\nu(u)\\
&\le \liminf_{k\to\infty}\iint G_D(v,u)d\nu_{n_k}(v)d\nu_{n_k}(u)\\
&= \liminf_{k\to\infty}\iint G_D(f(z),u)d\mu_{n_k}(z)d\nu_{n_k}(u)\\
&\le \liminf_{k\to\infty}\iint \Big(\sum_{i=0}^{n_k+m_0} G_{\mathbb{D}}(z,f_i^{-1}(u)) + \epsilon\Big)d\mu_{n_k}(z)d\nu_{n_k}(u)\\
&= \liminf_{k\to\infty}\int \sum_{i=0}^{n_k+m_0}\int G_{\mathbb{D}}(z,f_i^{-1}(u))d\mu_{n_k}(z)d\nu_{n_k}(u) + \epsilon\\
&= \liminf_{k\to\infty}\int \sum_{i=0}^{n_k+m_0} U^{\mathbb{D}}_{\mu_{n_k}}(f_i^{-1}(u))d\nu_{n_k}(u) + \epsilon\\
&\le \liminf_{k\to\infty}\int \sum_{i=0}^{n_k+m_0} I(\mathbb{D},K_{n_k})d\nu_{n_k}(u) + \epsilon\\
&= \liminf_{k\to\infty}(n_k+1+m_0)I(\mathbb{D},K_{n_k}) + \epsilon\\
&= \liminf_{k\to\infty}(n_k+1)I(\mathbb{D},K_{n_k}) + \lim_{k\to\infty} m_0 I(\mathbb{D},K_{n_k}) + \epsilon\\
&= \liminf_{k\to\infty}(n_k+1)I(\mathbb{D},K_{n_k}) + \epsilon\\
&\le \limsup_{k\to\infty}(n_k+1)I(\mathbb{D},K_{n_k}) + \epsilon \overset{(4.5)}{\le} I(D,E)+\epsilon.
\end{aligned}$$

Since ϵ was arbitrary, we obtain that $\lim_{k\to\infty}(n_k+1)I(\mathbb{D},K_{n_k}) = I(D,E)$ and

$$I(D,E) = \iint G_D(v,u)d\nu(v)d\nu(u).$$

Therefore, ν is the Green equilibrium measure of (D,E). From the uniqueness of the Green equilibrium measure we obtain that the sequence $\{\nu_n\}$ has a unique weak star accumulation point. So, $\{\nu_n\}$ converges weak star to ν and

$$\lim_{n\to\infty}(n+1)I(\mathbb{D},K_n) = I(D,E),$$

which is equivalent to (4.4). □

Acknowledgement

The authors profoundly thank Professor Thomas Ransford for interesting discussions on the subject.

References

[1] Lars V. Ahlfors, *Conformal invariants*, AMS Chelsea Publishing, Providence, RI, 2010. Topics in geometric function theory; Reprint of the 1973 original; With a foreword by Peter Duren, F. W. Gehring and Brad Osgood. MR2730573 (2011m:30001)

[2] David H. Armitage and Stephen J. Gardiner, *Classical potential theory*, Springer Monographs in Mathematics, Springer-Verlag London, Ltd., London, 2001. MR1801253 (2001m:31001)

[3] Dimitrios Betsakos, *Lindelöf's Principle and Estimates for Holomorphic Functions Involving Area, Diameter or Integral Means*, Comput. Methods Funct. Theory **14** (2014), no. 1, 85–105, DOI 10.1007/s40315-014-0049-z. MR3194314

[4] A. Bonnafé, *Estimates and asymptotic expansions for condenser p-capacities. The anisotropic case of segments*, preprint, 2013.

[5] Joseph A. Cima and Artur Nicolau, *Inner functions with derivatives in the weak Hardy space*, Proc. Amer. Math. Soc. **143** (2015), no. 2, 581–594, DOI 10.1090/S0002-9939-2014-12305-7. MR3283646

[6] E. F. Collingwood and A. J. Lohwater, *The theory of cluster sets*, Cambridge Tracts in Mathematics and Mathematical Physics, No. 56, Cambridge University Press, Cambridge, 1966. MR0231999 (38 #325)

[7] John B. Conway, *Functions of one complex variable. II*, Graduate Texts in Mathematics, vol. 159, Springer-Verlag, New York, 1995. MR1344449 (96i:30001)

[8] V. N. Dubinin, *Generalized condensers and the asymptotics of their capacities under a degeneration of some plates* (Russian, with Russian summary), Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **302** (2003), no. Anal. Teor. Chisel i Teor. Funkts. 19, 38–51, 198–199, DOI 10.1007/s10958-005-0319-4; English transl., J. Math. Sci. (N. Y.) **129** (2005), no. 3, 3835–3842. MR2023031 (2004m:31001)

[9] José L. Fernández and Domingo Pestana, *Distortion of boundary sets under inner functions and applications*, Indiana Univ. Math. J. **41** (1992), no. 2, 439–448, DOI 10.1512/iumj.1992.41.41025. MR1183352 (93k:30014)

[10] Maurice Heins, *On the Lindelöf principle*, Ann. of Math. (2) **61** (1955), 440–473. MR0069275 (16,1011g)

[11] Reiner Kühnau, *Randeffekte beim elektrostatischen Kondensator* (German, with Russian summary), Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **254** (1998), no. Anal. Teor. Chisel i Teor. Funkts. 15, 132–144, 246–247, DOI 10.1023/A:1011385226334; English transl., J. Math. Sci. (New York) **105** (2001), no. 4, 2210–2219. MR1691600 (2000k:30035)

[12] N. S. Landkof, *Foundations of modern potential theory*, Springer-Verlag, New York-Heidelberg, 1972. Translated from the Russian by A. P. Doohovskoy; Die Grundlehren der mathematischen Wissenschaften, Band 180. MR0350027 (50 #2520)

[13] D. E. Marshall, *The Uniformization Theorem*, available at `https://www.math.washington.edu/~marshall/preprints/uniformizationII.pdf`

[14] Javad Mashreghi, *Derivatives of inner functions*, Fields Institute Monographs, vol. 31, Springer, New York; Fields Institute for Research in Mathematical Sciences, Toronto, ON, 2013. MR2986324

[15] Terrence Napier and Mohan Ramachandran, *An introduction to Riemann surfaces*, Cornerstones, Birkhäuser/Springer, New York, 2011. MR3014916

[16] Michael Papadimitrakis and Stamatis Pouliasis, *Condenser capacity under multivalent holomorphic functions*, Comput. Methods Funct. Theory **13** (2013), no. 1, 11–20, DOI 10.1007/s40315-012-0004-9. MR3089940

[17] Christian Pommerenke, *Univalent functions*, Vandenhoeck & Ruprecht, Göttingen, 1975. With a chapter on quadratic differentials by Gerd Jensen; Studia Mathematica/Mathematische Lehrbücher, Band XXV. MR0507768 (58 #22526)

[18] Thomas Ransford, *Potential theory in the complex plane*, London Mathematical Society Student Texts, vol. 28, Cambridge University Press, Cambridge, 1995. MR1334766 (96e:31001)

[19] Walter Rudin, *Real and complex analysis*, 3rd ed., McGraw-Hill Book Co., New York, 1987. MR924157 (88k:00002)

[20] Yan Soibelman, *Asymptotics of a condenser capacity and invariants of Riemannian submanifolds*, Selecta Math. (N.S.) **2** (1996), no. 4, 653–667, DOI 10.1007/PL00001386. MR1443187 (98j:31004)

[21] M. Tsuji, *Potential theory in modern function theory*, Maruzen Co., Ltd., Tokyo, 1959. MR0114894 (22 #5712)

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