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GAUSSIAN HARMONIC FORMS AND TWO-DIMENSIONAL SELF-SHRINKING SURFACES

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ABSTRACT. We consider two-dimensional self-shrinkers Σ^2 for the Mean Curvature Flow of polynomial volume growth immersed in $\mathbb{R}^n$. We look at closed one forms ω satisfying the Euler-Lagrange equation associated with minimizing the norm $\int_\Sigma dVe^{-|x|^2/4}|\omega|^2$ in their cohomology class. We call these forms Gaussian Harmonic one Forms (GHF).

Our main application of GHF's is to show that if Σ has genus ≥ 1, then we have a lower bound on the supremum norm of $|A|^2$. We also may give applications to the index of L acting on scalar functions of Σ and to estimates of the lowest eigenvalue η_0 of L if Σ satisfies certain curvature conditions.

INTRODUCTION

A two-dimensional submanifold Σ^2 immersed in $\mathbb{R}^n$ is considered a self-shrinker for the Mean Curvature Flow if it satisfies

$$H = \frac{x^N}{2} \tag{0.1}$$

(note our choice of normalization) where x is the position vector in $\mathbb{R}^n$. Self-shrinkers are related to the singularities of the Mean Curvature Flow [2]. For this paper, we will only consider Σ to be a complete two-dimensional self-shrinker of polynomial volume growth.

For a properly immersed self-shrinker Σ, we expect bounds on the curvature to affect the topology of Σ. Let A denote the second fundamental form of Σ. An argument, pointed out to the author by Professor Minicozzi, shows that if $|A|^2 \leq C|x|^2$ with $C < 1/8$ for $|x|$ large, then Σ has finite topology. For any two-dimensional Σ we have $|H|^2 \leq 2|A|^2$. Since we are on a self-shrinker, we then have that $|x^N|^2 \leq 8C|x|^2$. We then have that $|x^T|^2 \geq (1-8C)|x|^2 > 0$ for $|x|$ sufficiently large. Therefore, we may apply Morse Theory to find that the topology is constant for large enough $|x|$.

In Ros [6] and Urbano [7] harmonic one forms are used to study the index of a one-sided minimal surface of genus g. The harmonic forms are constructed by minimizing the L^2 norm $\int_\Sigma dV|\omega|^2$ in a cohomology class. From classical Riemann surface theory [3, p. 42], we can associate $2g$ linearly independent L^2 harmonic one forms to a surface of genus g. Ros [6] and Urbano [7] consider the duals to these

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harmonic one forms as vectors in Euclidean space. They study the Jacobi operator $\triangle_\Sigma + |A|^2$ acting on the coordinate functions of these vectors.

We consider a parallel situation for a self-shrinker Σ. We examine closed one forms ω minimizing $\int_\Sigma dV e^{-|x|^2/4}|\omega|^2$ in their cohomology class. Any closed one form satsifying the Euler-Lagrange Equation for this minimization will be called a Gaussian Harmonic one Form (GHF). Note that this norm does not come from a conformal change of Σ. The measure $e^{-|x|^2/4}dV$ is related to the variational characterization of self-shrinkers in Colding-Minicozzi [2]. A difference between this and the harmonic case is that we only get g linearly independent GHF, because the Euler-Lagrange condition for GHF is not preserved by a rotation. For simplicity in notation, we will define $d\mu \equiv e^{-|x|^2/4}dV$. We will also denote $L^2_\mu(\Sigma)$ to be the L^2 space associated with the measure $d\mu$.

For a normal vector field N and tangent vector fields X, Y we define the Second Fundamental Form by $A^N(X,Y) = \langle \nabla^E_X N, Y\rangle$. We also define $B(X,Y) = -\nabla^N_X Y$ and $H = B(i,i)$. Our sign convention is chosen to be consistent with that of Colding-Minicozzi [2].

We will use K to refer to the curvature of Σ. For an orthonormal frame N_α in the normal bundle $N\Sigma$, Gauss' Equation gives us that $K = A^{N_\alpha}_{11}A^{N_\alpha}_{22} - A^{N_\alpha}_{12}A^{N_\alpha}_{12}$.

For each N_α, we may diaganolize A^{N_α} with eigenvalues $\kappa_{\alpha i}$ for a frame $\{e_{\alpha i}\}$ in $T\Sigma$. Note, that from Gauss' Equation we have that $K = \frac{1}{2}(A^{N_\alpha}_{ii}A^{N_\alpha}_{jj} - A^{N_\alpha}_{ij}A^{N_\alpha}_{ij})$, and so $K = \sum_\alpha \kappa_{\alpha 1}\kappa_{\alpha 2}$. We denote the components of a tangential vector field X with respect to $e_{\alpha i}$ as $X_{\alpha i}$.

We will have need to work with vectors in $T\Sigma$, $N\Sigma$, and in $\mathbb{R}^n$. We will use the indices $\{i, j, \dots\}$ for vectors or forms considered intrinsically part of Σ, $\{a, b, \dots\}$ when considering vectors in $\mathbb{R}^n$, and $\{\alpha, \beta, \dots\}$ when considering vectors in $N\Sigma$.

The vectors ∂_a will denote a standard basis of orthonormal vectors for $\mathbb{R}^n$. We will make use of the exterior derivative of the coordinate function x^a along Σ and will denote this by dx^a. That is, dx^a is not the exterior derivative in $\mathbb{R}^n$. Also, for any vector $V \in \mathbb{R}^n$ let V^T be the projection onto the tangent space of Σ. For any one form ω on Σ we shall denote the vector dual to ω considered to be sitting in $\mathbb{R}^n$ as W.

As in Colding-Minicozzi [2], we define the operators $\mathcal{L}$ and L on scalar functions of Σ as

$$\mathcal{L}u \equiv \triangle u - \frac{1}{2}\nabla_{x^T} u \tag{0.2}$$

$$Lu \equiv \mathcal{L}u + \frac{1}{2}u + |A|^2 u. \tag{0.3}$$

Note that $\mathcal{L} = -\nabla^*\nabla$ for the measure $d\mu$. That is, $\int_\Sigma d\mu\, f\mathcal{L}g = -\int_\Sigma d\mu\, \langle\nabla f, \nabla g\rangle$ for any $f, g \in C^\infty_0(\Sigma)$. In particular, $\mathcal{L}$ is a symmetric operator. The operator L is associated with the second variation related to the variational characterization of self-shrinkers in Colding-Minicozzi [2].

Since we will be working with vectors in $T\Sigma$, $N\Sigma$, and $\mathbb{R}^n$ we will use ∇^E to denote the Euclidean connection, $\nabla^\Sigma = (\nabla^E)^T$ for the connection on Σ, and $\nabla^N = (\nabla^E)^N$ for the normal connection.

We also define the operators $\mathcal{L}^E$ and L^E on $\mathbb{R}^n$ valued vector fields V along Σ by

$$\mathcal{L}^E V \equiv \mathrm{Tr}_\Sigma \nabla^{2,E} V - \frac{1}{2}\nabla^E_{x^T} V \tag{0.4}$$

$$= \nabla^{2,E}_{i,i} V - \frac{1}{2}\nabla^E_{x^T} V \tag{0.5}$$

$$L^E V \equiv \mathcal{L}^E V + \frac{1}{2} V + |A|^2 V. \tag{0.6}$$

Note that for $V = V^a \partial_a$ we have $\mathcal{L}^E V = (\mathcal{L} V^a)\partial_a$ and $L^E V = (L V^a)\partial_a$.

We will likewise use the ∇^Σ connection to define $\mathcal{L}^\Sigma$ and L^Σ on tensors intrinsic to Σ.

In the first section of the paper, we establish some computations for GHF's on Σ. The main results of our computations are (1.8) and (1.9).

For the second section, we apply our computations for GHF's to prove two results in the general co-dimension case. The first is a type of "gap theorem" for the genus of a self-shrinker.

Theorem 2.1. *If Σ is a two-dimensional orientable self-shrinker of polynomial volume growth immersed in $\mathbb{R}^n$ with genus ≥ 1, then*

$$\sup_{x \in \Sigma, |v|=1} A^{N_\beta}(v,i) A^{N_\beta}(i,v) \geq 1/2. \tag{0.7}$$

Remark. In the case of $\Sigma \to \mathbb{R}^3$ we have that $\sup\limits_{x\in\Sigma,|v|=1} A^{N_\beta}(v,i)A^{N_\beta}(i,v) = \sup\limits_{x\in\Sigma,i} \kappa_i^2$ where the κ_i are the principal curvatures of Σ. Theorem 3.1 should also be compared with a result of Cao-Li [1], where they prove a gap theorem for self-shrinkers $\Sigma^k \subset \mathbb{R}^n$ of any co-dimension. They show that if the square norm $|A|^2 \leq 1/2$ (here we have renormalized their result for our definition of self-shrinker), then Σ^k is a round sphere, a cylinder, or a hyperplane.

Next, we show a lower bound on the index of L based on the genus of Σ if Σ satisfies an appropriate curvature condition.

Theorem 2.2. *Let Σ be a two-dimensional orientable self-shrinker of polynomial volume growth immersed in $\mathbb{R}^n$ with genus g and principal curvatures $\kappa_{\alpha i}$ for a normal frame $\{N_\alpha\}$. If $\sup\limits_{p\in\Sigma} \inf\limits_{\{N_\alpha(p)\}} \sum\limits_\alpha |\kappa_{\alpha 1}^2 - \kappa_{\alpha 2}^2| \leq \delta < 1$, then the index of L acting on scalar functions of Σ has a lower bound given by*

$$\mathrm{Index}_\Sigma(L) \geq \frac{g}{n}. \tag{0.8}$$

Remark. Note, that in the case of $\Sigma \subset \mathbb{R}^3$, we get that $\sup\limits_{p\in\Sigma} \inf\limits_{\{N_\alpha(p)\}} \sum\limits_\alpha |\kappa_{\alpha 1}^2 - \kappa_{\alpha 2}^2| = \sup |\kappa_1^2 - \kappa_2^2|$.

In the third section, we make applications to the lowest eigenvalue of L acting on scalar functions of Σ. Our first result is for the case where Σ isn't necessarily compact.

Theorem 3.1. *Let Σ be a two-dimensional orientable self-shrinker of polynomial volume growth immersed in $\mathbb{R}^n$ with genus ≥ 1. The lowest eigenvalue of L acting on scalar functions on Σ has upper bounds given by*

$$\eta_0 \leq -1 + \sup_{x\in\Sigma, |v|=1} A^{N_\beta}(v,i)A^{N_\beta}(i,v) \tag{0.9}$$

$$\eta_0 \leq -1 + \sup_{p\in\Sigma} \inf_{\{N_\alpha(p)\}} \sum_\alpha |\kappa_{\alpha 1}^2 - \kappa_{\alpha 2}^2|. \tag{0.10}$$

Remark. From Theorem 3.1 we see that the best upper bound (0.9) can give us is $\eta_0 \leq -1/2$. In the co-dimension one case $\Sigma \to \mathbb{R}^3$, Colding-Minicozzi [2] were able to show that $\eta_0 \leq -1$ by showing that $LH = H$ and that H is in the appropriate weighted space. So, we see that the estimate of Theorem 3.2 isn't optimal for the co-dimension one case. For the cases of higher co-dimension, one still has $L^N H = H$, but this doesn't say anything about the eigenvalues of L on scalar functions unless there exists a global parallel normal.

Then, we show an estimate for the lowest eigenvalue of L for compact Σ. We get a better bound since the compactness of Σ gives us that η_0 is realized by an eigenfunction. This allows us to use $\mathcal{L}^\Sigma$ instead of $\mathcal{L}^E$ to give a better estimate.

Theorem 3.2. *Let Σ be a two-dimensional orientable compact self-shrinker immersed in $\mathbb{R}^n$ with genus $g \geq 1$ and principal curvatures $\kappa_{\alpha i}$ in a normal frame $\{N_\alpha\}$. Let η_0 be the lowest eigenvalue of L acting on scalar functions on Σ. We have that*

$$\eta_0 \leq -3/2 + \sup_{p\in\Sigma} \inf_{\{N_\alpha(p)\}} \sum_\alpha |\kappa_{\alpha 1}^2 - \kappa_{\alpha 2}^2|. \tag{0.11}$$

Remark. As before, we note that Colding-Minicozzi [2] have shown that in the co-dimension one case that $\eta_0 \leq -1$. Therefore, we see that (0.11) is only optimal for $\Sigma^2 \subset \mathbb{R}^3$ if $|\kappa_1^2 - \kappa_2^2| < 1/2$.

1. Gaussian Harmonic one Forms

Let ω_0 be a closed C_0^∞ one form on Σ, i.e., $d\omega_0 = 0$. The classical harmonic form cohomologous to ω_0 is constructed by minimizing the $L^2(\Sigma)$ norm $\int_\Sigma |\omega|^2$ in the cohomology class of ω_0 [5]. A form ω is defined to be harmonic if and only if it is closed ($\nabla^\Sigma\omega$ is symmetric) and co-closed ($\nabla^\Sigma\omega$ is traceless) [6].

Instead of using harmonic one forms to represent the cohomology class of ω_0 as in Ros [6] and Urbano [7], we will use the form minimizing the $L^2_\mu(\Sigma)$ norm $\int_\Sigma d\mu\, |\omega|^2$ where $\omega = \omega_0 + df$ and $d\mu = e^{-|x|^2/4}dV$. To find the Euler-Lagrange Equation, assume the minimum is achieved by a closed form ω. Also, let δ be the dual to d with respect to the regular Euclidean surface measure dV. As in Jost[5], the minimization gives us that $0 = \delta(e^{-|x|^2/4}\omega) = -\mathrm{div}(e^{-|x|^2/4}\omega)$. Using this, we obtain the Euler-Lagrange Equation:

$$\mathrm{Tr}_\Sigma \nabla^\Sigma \omega = \frac{1}{2}\omega(x^T). \tag{1.1}$$

We make a definition:

Definition 1. A form ω will be called a Gaussian Harmonic Form (GHF) if and only if ω is closed ($\nabla^\Sigma\omega$ is symmetric) and ω is Gaussian co-closed (satisfies 1.1).

We have the following result:

Lemma 1.1. *Let ω be a GHF on a self-shrinker Σ^2 with dual vector $W \in T\Sigma$. We have that*

$$(\mathcal{L}^\Sigma \omega)(v) = \frac{1}{2}\omega(v) - A^{N_\alpha}(W, e_i)A^{N_\alpha}(e_i, v). \tag{1.2}$$

Proof. We have a Weitzenbock formula $\triangle^\Sigma \omega = -(\delta d + d\delta)\omega + K\omega$ where $\delta d + d\delta$ is the Hodge Laplacian [5]. Now $\delta\omega = -\text{div}\omega = -\frac{1}{2}\omega(x^T)$ and $d\omega = 0$. Therefore $-(\delta d + d\delta)\omega = \frac{1}{2}d(\omega(x^T)) = \frac{1}{2}\nabla^\Sigma(\omega(x^T))$. Then, using a Leibniz rule, we get

$$(\triangle^\Sigma \omega)(v) = K\omega + \frac{1}{2}\omega(\nabla^\Sigma_v x^T) + \frac{1}{2}\nabla^\Sigma \omega(x^T, v). \tag{1.3}$$

Now, we use that for a self-shrinker, $\nabla^\Sigma_v x^T = v - 2A^H(v, e_i)e_i$ and that $\mathcal{L}^\Sigma = \triangle^\Sigma - \frac{1}{2}\nabla^\Sigma_{x^T}$ to get

$$(\mathcal{L}^\Sigma \omega)(v) = K\omega(v) + \frac{1}{2}\omega(v) - A^H(v, W). \tag{1.4}$$

Decomposing in a normal basis $\{N_\alpha\}$, we have that

$$KW - \sum_i A^H(W, e_i)e_i = \sum_\alpha \sum_i (\kappa_{\alpha 1}\kappa_{\alpha 2}W_{\alpha i}e_{\alpha i} - (\kappa_{\alpha 1} + \kappa_{\alpha 2})\kappa_{\alpha i}W_{\alpha i}e_{\alpha i}) \tag{1.5}$$

$$= \sum_\alpha \sum_i -\kappa_{\alpha i}^2 W_{\alpha i}e_{\alpha i} \tag{1.6}$$

$$= -A^{N_\alpha}(W, e_i)A^{N_\alpha}(e_i, e_j)e_j. \tag{1.7}$$

□

Note that the dual form of equation (1.2) is

$$\mathcal{L}^\Sigma W = \langle \mathcal{L}^\Sigma \omega, dx^a \rangle \partial_a = \frac{1}{2}W - A^{N_\beta}(W, e_j)A^{N_\beta}(e_j, e_k)e_k. \tag{1.8}$$

Lemma 1.2. *Let ω be a GHF on a self-shrinker Σ^2. Let W be the vector field dual to ω. We have*

$$\mathcal{L}^E W = -2\langle \nabla^\Sigma \omega, A^{N_\beta} \rangle N_\beta + \frac{1}{2}W - 2A^{N_\beta}(W, e_j)A^{N_\beta}(e_j, e_k)e_k. \tag{1.9}$$

Proof. First, we compute $\langle \nabla^\Sigma \omega, \nabla^\Sigma dx^a \rangle \partial_a$. Note that $\nabla^\Sigma \partial_a^T = -\langle \partial_a, N_\beta \rangle \nabla^\Sigma N_\beta$. Hence $\nabla^\Sigma dx^a = -{N_\beta}^a A^{N_\beta}$, and so we get

$$\langle \nabla^\Sigma \omega, \nabla^\Sigma dx^a \rangle \partial_a = -\langle \nabla^\Sigma \omega, A^{N_\beta} \rangle N_\beta. \tag{1.10}$$

Next, we compute $\langle \omega, \mathcal{L}^\Sigma dx^a \rangle \partial_a$. Fix a point $p \in \Sigma$. Use a tangential frame $\{e_j\}$ and a normal frame $\{N_\beta\}$ such that $\nabla^\Sigma e_j(p) = 0$ and $\nabla^N N_\beta(p) = 0$. Using the Codazzi Equation, we get

$$(\triangle^\Sigma dx^a)(e_j) = -A^{N_\beta}(\partial_a^T, e_k)A^{N_\beta}(e_k, e_j) - (\nabla^N_{e_j} H)^a \tag{1.11}$$

$$= -A^{N_\beta}(\partial_a^T, e_k)A^{N_\beta}(e_k, e_j) - \frac{{N_\beta}^a}{2} A^{N_\beta}(x^T, W). \tag{1.12}$$

Therefore,

$$\langle \omega, \mathcal{L}^\Sigma dx^a \rangle \partial_a = -A^{N_\beta}(W, e_j)A^{N_\beta}(e_j, e_k)e_k. \tag{1.13}$$

Now, for $W = \langle \omega, dx^a \rangle \partial_a$, we have

$$\mathcal{L}^E W = (\mathcal{L}\langle \omega, dx^a \rangle)\partial_a \tag{1.14}$$

$$= \langle \mathcal{L}^\Sigma \omega, dx^a \rangle \partial_a + 2\langle \nabla^\Sigma \omega, \nabla^\Sigma dx^a \rangle \partial_a + \langle \omega, \mathcal{L}^\Sigma dx^a \rangle \partial_a. \tag{1.15}$$

Using equations (1.2), (1.10), and (1.13), we get (1.9). □

2. Applications

Theorem 2.1. *If Σ is a two-dimensional orientable self-shrinker of polynomial volume growth immersed in $\mathbb{R}^n$ with genus ≥ 1, then*

$$\sup_{x \in \Sigma, |v|=1} A^{N_\beta}(v,i) A^{N_\beta}(i,v) \geq 1/2. \tag{2.1}$$

Proof. Assume Σ has genus $g \geq 1$. Parallel to the result for classical Riemann surfaces [3], we then have g linearly independent GHF in $L^2_\mu(\Sigma)$. Let ω be one of these GHF and W be the dual to ω.

Consider any $\phi \in C_0^\infty(\Sigma)$. We have

$$0 \leq \int_\Sigma d\mu\, |\nabla^\Sigma(\phi W)|^2 = -\int_\Sigma d\mu\, \langle \phi W, \mathcal{L}^\Sigma(\phi W)\rangle \tag{2.2}$$

$$= -\int_\Sigma d\mu\, |W|^2 \phi \mathcal{L}\phi - \frac{1}{2}\int_\Sigma d\mu\, \langle \nabla^\Sigma \phi^2, \nabla^\Sigma |W|^2 \rangle - \int_\Sigma d\mu\, \phi^2 \langle W, \mathcal{L}^\Sigma W \rangle. \tag{2.3}$$

Now, using integration by parts, we have that

$$-\frac{1}{2}\int_\Sigma d\mu\, \langle \nabla^\Sigma \phi^2, \nabla^\Sigma |W|^2 \rangle = \int_\Sigma d\mu\, |W|^2 \phi \mathcal{L}\phi + \int_\Sigma d\mu\, |W|^2 |\nabla^\Sigma \phi|^2. \tag{2.4}$$

Putting this into (2.3), we get that

$$0 \leq \int_\Sigma d\mu\, |W|^2 |\nabla \phi|^2 - \int_\Sigma d\mu\, \phi^2 \langle W, \mathcal{L}^\Sigma W \rangle. \tag{2.5}$$

Then, using (1.8) we have that

$$0 \leq \int_\Sigma d\mu\, |W|^2 |\nabla \phi|^2 + \int_\Sigma d\mu\, \phi^2 A^{N_\beta}(W,i) A^{N_\beta}(i,W) - \frac{1}{2}\int_\Sigma d\mu\, \phi^2 |W|^2. \tag{2.6}$$

Let $M = \sup_{x \in \Sigma, |v|=1} A^{N_\beta}(v,i) A^{N_\beta}(i,v)$. Using standard cut-off functions of increasing domain and $|\nabla^\Sigma \phi|^2 \leq 1$, we get

$$0 \leq (M - \frac{1}{2}) \int_\Sigma d\mu\, |W|^2. \tag{2.7}$$

Since $W \not\equiv 0$, we get the theorem. □

In our next application, we will need to make use of the inequality

$$2A^{N_\alpha}(W, e_i) A^{N_\alpha}(e_i, W) - |A|^2 |W|^2 \leq |W|^2 \sup_{p \in \Sigma} \inf_{\{N_\alpha(p)\}} \sum_\alpha |\kappa^2_{\alpha 1} - \kappa^2_{\alpha 2}|. \tag{2.8}$$

This can be realized by diagonalizing A^{N_α} for each vector in a normal frame $\{N_\alpha\}$. We then get

$$2A^{N_\alpha}(W,e_i)A^{N_\alpha}(e_i,W)-|A|^2|W|^2 = \sum_\alpha 2(\sum_i \kappa_{\alpha i}^2 W_{\alpha i}^2)-(\kappa_{\alpha 1}^2+\kappa_{\alpha 2}^2)(W_{\alpha 1}^2+W_{\alpha 2}^2) \tag{2.9}$$

$$= \sum_\alpha \sum_{i\neq j}(\kappa_{\alpha i}^2 - \kappa_{\alpha j}^2)W_{\alpha i}^2 \tag{2.10}$$

$$\leq \sum_\alpha |\kappa_{\alpha 1}^2 - \kappa_{\alpha 2}^2||W|^2. \tag{2.11}$$

Note that we are free to choose $\{N_\alpha\}$ at any point, so therefore we get (2.8).

As in Fischer-Colbrie [4], the index of the operator L acting on scalar functions on $B_R \subset \Sigma$ is increasing in R for any exhaustion of Σ by B_R. The index of L on Σ is defined to be $\mathrm{Index}_\Sigma(L) = \sup_R(\mathrm{Index}_{B_R}(L))$.

Following the work of Ros [6] and Urbano [7] on the Jacobi operator on minimal surfaces, we may give lower bounds for the index of L if we have a condition on the principal curvatures. That is, we have

Theorem 2.2. *Let Σ be a two-dimensional orientable self-shrinker of polynomial volume growth immersed in $\mathbb{R}^n$ with genus g and principal curvatures $\kappa_{\alpha i}$ for a normal frame $\{N_\alpha\}$. If $\sup_{p\in\Sigma}\inf_{\{N_\alpha(p)\}}\sum_\alpha |\kappa_{\alpha 1}^2 - \kappa_{\alpha 2}^2| \leq \delta < 1$, then the index of L acting on scalar functions of Σ has a lower bound given by*

$$\mathrm{Index}_\Sigma(L) \geq \frac{g}{n}. \tag{2.12}$$

Proof. We may assume $\mathrm{Index}_\Sigma(L) = J < \infty$. As in Fischer-Colbrie [4], there exist $L^2_\mu(\Sigma)$ functions $\psi_1,\ldots,\psi_J$ such that if $f \in C_0^\infty(\Sigma)$ and $\int_\Sigma d\mu\, f\psi_i = 0$ for all i, then $-\int_\Sigma d\mu\, fLf \geq 0$.

Similar to Farkas-Kra [3, p. 42], we have g linearly independent $L^2_\mu(\Sigma)$ GHF's ω_i with dual vectors W_i. For now, consider the case that $g < \infty$ and define $V = \mathrm{span}\{W_i\}$ where we are considering W_i to be vector fields with values in $\mathbb{R}^n$. Consider any $\phi \in C_0^\infty(\Sigma)$. Similar to the calculation in Theorem 2.1, we have that for any $W \in V$ that

$$-\int_\Sigma d\mu\,\langle \phi W, L^E(\phi W)\rangle = \int_\Sigma d\mu\,|W|^2|\nabla\phi|^2 - \int_\Sigma d\mu\,\phi^2\langle W, L^E W\rangle. \tag{2.13}$$

Using (1.9) and (2.8), we get that

$$\int_\Sigma d\mu\,\phi^2\langle W, L^E W\rangle \geq \int_\Sigma d\mu\,\phi^2|W|^2(1-\sup_{p\in\Sigma}\inf_{\{N_\alpha(p)\}}\sum_\alpha|\kappa_{\alpha 1}^2-\kappa_{\alpha 2}^2|) \tag{2.14}$$

$$\geq (1-\delta)\int_\Sigma d\mu\,\phi^2|W|^2. \tag{2.15}$$

Since $\dim V < \infty$, by using a standard cut-off function of large enough domain and $|\nabla\phi|^2 \leq 1$, we may ensure that $\dim V = \dim \phi V$ and that $-\int_\Sigma d\mu\,\langle\phi W, L^E(\phi W)\rangle < 0$ for all $0 \neq \phi W \in \phi V$.

We consider the linear map $F : \phi V \to \mathbb{R}^{nJ}$ given by

$$F(\phi W) = (\int_\Sigma d\mu\,\phi W\psi_1,\ \ldots\ ,\int_\Sigma d\mu\,\phi W\psi_J). \tag{2.16}$$

For any $\phi W \in \phi V$, if $\phi W \in \mathrm{Ker} F$, then each of its coordinate functions is orthogonal to every ψ_i. Therefore, $-\int_\Sigma d\mu \, \langle \phi W, L^E \phi W \rangle \geq 0$. Hence, we have that $\phi W \equiv 0$. Therefore, $\mathrm{Ker} F = 0$.

So $g = \dim V \leq nJ$, and we have the conclusion for the case of $g < \infty$. For the case of $g = \infty$, it is clear that the above argument shows that $3J \geq m$ for all integers m. □

3. Applications to Lowest Eigenvalue η_0

For any non-compact manifold Σ the operator L on scalar functions may not have a nice spectrum, but we may still define the lowest eigenvalue of L by

$$\eta_0 \equiv \inf_{\phi \in C_0^\infty(\Sigma)} \frac{\int_\Sigma d\mu \, (|\nabla \phi|^2 - |A|^2 \phi^2 - \frac{1}{2}\phi^2)}{\int_\Sigma d\mu \, \phi^2}. \tag{3.1}$$

We get upper bounds for η_0.

Theorem 3.1. *Let Σ be a two-dimensional orientable self-shrinker of polynomial volume growth immersed in $\mathbb{R}^n$ with genus ≥ 1. The lowest eigenvalue of L acting on scalar functions on Σ has two upper bounds given by*

$$\eta_0 \leq -1 + \sup_{x \in \Sigma, |v|=1} A^{N_\beta}(v,i) A^{N_\beta}(i,v) \tag{3.2}$$

$$\eta_0 \leq -1 + \sup_{p \in \Sigma} \inf_{\{N_\alpha(p)\}} \sum_\alpha |\kappa_{\alpha 1}^2 - \kappa_{\alpha 2}^2|. \tag{3.3}$$

Proof. Let $\phi \in C_0^\infty(\Sigma)$ and ω be any non-zero GHF on Σ with daul vector field W. Also, let $M_p = \sup_{v \in T_p\Sigma, |v|=1} A^{N_\beta}(v,i) A^{N_\beta}(i,v)$. Note that M_p depends on $p \in \Sigma$ and is not the supremum over Σ. Consider the tangent vector field ϕW. Plugging the coordinate functions of ϕW into the definition of η_0 we get

$$\eta_0 \int_\Sigma d\mu \, \phi^2 |W|^2 \leq \int_\Sigma d\mu \, (|\nabla^E(\phi W)|^2 - |A|^2 \phi^2 |W|^2 - \frac{1}{2}\phi^2 |W|^2). \tag{3.4}$$

Note that our expression involves the Euclidean connection ∇^E. As in the proof of Theorem 2.1, we have that

$$\int_\Sigma d\mu \, (|\nabla^E(\phi W)|^2 = \int_\Sigma d\mu \, |\nabla \phi|^2 |W|^2 - \int_\Sigma d\mu \, \phi^2 \langle W, \mathcal{L}^E W \rangle. \tag{3.5}$$

Now using (1.9) we have that $-\int_\Sigma d\mu \, \phi^2 \langle W, \mathcal{L}^E W\rangle \leq \int_\Sigma d\mu \, \phi^2 |W|^2 (2M_p - 1/2)$. Then, using standard cut-off functions of increasing domain and $|\nabla^\Sigma \phi|^2 \leq 1$, we get that

$$\eta_0 \int_\Sigma d\mu \, |W|^2 \leq \int_\Sigma d\mu \, |W|^2 (2M_p - |A|^2 - 1). \tag{3.6}$$

Using that $M_p - |A|^2(p) \leq 0$, we get (3.2).

Similarly, if we choose an orthonormal basis $\{N_\alpha\}$ and analyze our terms differently, then from (1.9) we get

$$\eta_0 \int_\Sigma d\mu \, \phi^2 |W|^2 \leq \int_\Sigma d\mu \, \phi^2 (2A^{N_\alpha}(W, e_i) A^{N_\alpha}(e_i, W) - |A|^2 |W|^2 - |W|^2). \tag{3.7}$$

So, using (2.8) we get (3.3). □

For the case of compact Σ with genus ≥ 1 we may give another bound for the lowest eigenvalue of L since it is actually realized by a funtion $u > 0$. This allows us to use $\mathcal{L}^{\Sigma}$, instead of $\mathcal{L}^{E}$, to get better bounds.

Theorem 3.2. *Let Σ be a two-dimensional orientable compact self-shrinker immersed in $\mathbb{R}^n$ with genus $g \geq 1$ and principal curvatures $\kappa_{\alpha i}$ in a normal frame $\{N_\alpha\}$. Let η_0 be the lowest eigenvalue of L acting on scalar functions on Σ. We have that*

$$\eta_0 \leq -3/2 + \sup_{p\in\Sigma} \inf_{\{N_\alpha(p)\}} \sum_\alpha |\kappa_{\alpha 1}^2 - \kappa_{\alpha 2}^2|. \tag{3.8}$$

Proof. Let u be the eigenfunction for the lowest eigenvalue η_0. Note that by standard theory, $u > 0$. Let ω be a GHF with W its dual vector field. Consider the equation

$$\int_\Sigma d\mu\, |W|^2 Lu = -\eta_0 \int_\Sigma d\mu\, |W|^2 u. \tag{3.9}$$

We perform integration by parts on the left hand side; use that

$$L|W|^2 \geq 2\langle W, \mathcal{L}^{\Sigma} W\rangle + (|A|^2 + \frac{1}{2})|W|^2,$$

(1.8), and (2.8) to get

$$-\eta_0 \int_\Sigma d\mu\, |W|^2 u \geq \int_\Sigma d\mu\, \frac{3}{2}|W|^2 u - \int_\Sigma d\mu\, |W|^2 u \sup_{p\in\Sigma} \inf_{\{N_\alpha(p)\}} \sum_\alpha |\kappa_{\alpha 1}^2 - \kappa_{\alpha 2}^2|. \tag{3.10}$$

Since $|W|^2 u \geq 0$ and $|W|^2 u \not\equiv 0$ we get (3.8). ☐

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References

[1] Huai-Dong Cao and Haizhong Li, *A gap theorem for self-shrinkers of the mean curvature flow in arbitrary codimension*, Calc. Var. Partial Differential Equations **46** (2013), no. 3-4, 879–889, DOI 10.1007/s00526-012-0508-1. MR3018176

[2] Tobias H. Colding and William P. Minicozzi II, *Generic mean curvature flow I: generic singularities*, Ann. of Math. (2) **175** (2012), no. 2, 755–833, DOI 10.4007/annals.2012.175.2.7. MR2993752

[3] H. M. Farkas and I. Kra, *Riemann surfaces*, 2nd ed., Graduate Texts in Mathematics, vol. 71, Springer-Verlag, New York, 1992. MR1139765 (93a:30047)

[4] D. Fischer-Colbrie, *On complete minimal surfaces with finite Morse index in three-manifolds*, Invent. Math. **82** (1985), no. 1, 121–132, DOI 10.1007/BF01394782. MR808112 (87b:53090)

[5] Jürgen Jost, *Riemannian geometry and geometric analysis*, 6th ed., Universitext, Springer, Heidelberg, 2011. MR2829653

[6] Antonio Ros, *One-sided complete stable minimal surfaces*, J. Differential Geom. **74** (2006), no. 1, 69–92. MR2260928 (2007g:53008)

[7] Francisco Urbano, *Second variation of one-sided complete minimal surfaces*, Rev. Mat. Iberoam. **29** (2013), no. 2, 479–494, DOI 10.4171/RMI/727. MR3047425

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