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ON μ-STATISTICAL CONVERGENCE

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Abstract. The concept of μ-statistical convergence at a point for measurable functions in measurable space with a measure is introduced in this work. This concept is a generalization of a similar idea about the sequence of numbers. We also introduce the concept of μ-statistical fundamentality at a point, and the equivalence of these two concepts is proved. The concept of μ-statistical convergence at a point generalizes the usual one of the limit of a function at a point.

1. Introduction

Apparently the concept of statistical convergence of a sequence of numbers, as the generalization of the classical concept of a limit of a sequence, was first introduced in [1]. In [2], [3], [4], [5] the basic properties of statistically convergent sequences have been studied. Later there has appeared much research with various generalizations of this concept (see [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16]). In [17], [18], [19] this concept was used in the approximation theory by positive operators.

We consider in this work a measurable space with a measure on the real axis. We introduce the concept of a μ-statistical limit of measurable function at a point. Also is introduced the concept of a μ-statistical fundamentality at a point, and the equivalence of these two concepts is proved. This concept is a generalization of a similar one of statistical convergence for a sequence.

2. μ-statistical convergence

We will use the standard notation. N will be the set of all positive integers; R is the set of all real numbers; $\exists$ will mean "there exist(s)"; $\exists!$ will mean "there exists a unique"; $\Rightarrow$ will mean "it follows"; $\Leftrightarrow$ will mean equivalence.

Let $\mathscr{B}$ be a σ-algebra of Borel subsets in the segment $I \equiv [a, +\infty)$ and $(\mathscr{B}; \mu)$ be a measurable space with σ-finite measure $\mu : \mu(I) = +\infty$, where $a \in R$ is some number. Consider $\mathscr{B}$-measurable function $f : I \to R$. The measure of the set $e \in \mathscr{B}$ will be denoted by $|e|$, i.e. $|e| = \mu(e)$. Let $M \in \mathscr{B}$ be some set.

Definition 2.1. We say that the infinitely remote point ∞ is a point μ-statistical (μ-*stat*) density for M if and only if

$$\lim_{x\to\infty} \frac{|M \cap I_x|}{|I_x|} = 1, \text{ where } I_x \equiv [a, x].$$

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Let $\varepsilon > 0$ be an arbitrary number and let

$$A_\varepsilon^f \equiv \{x \in I : |f(x) - A| \geq \varepsilon\},$$

where $A \in R$ is some number.

Definition 2.2. We say that f has a *μ-st* (μ-statistical) limit A at infinity if and only if

$$\lim_{x\to\infty} \frac{\left|A_\varepsilon^f \cap I_x\right|}{|I_x|} = 0, \ \ \forall\varepsilon > 0, \tag{2.1}$$

and this limit will be denoted as μ-st $\lim_{x\to\infty} f(x) = A$.

Let $M^c \equiv I\backslash M$ be a complement of a set $M \subset I$. From the relation $I_x = \left[A_\varepsilon^f \cap I_x\right] \cup \left[\left(A_\varepsilon^f\right)^c \cap I_x\right]$ it directly follows that the condition (2.1) is equivalent to the following one:

$$\lim_{x\to\infty} \frac{\left|\left(A_\varepsilon^f\right)^c \cap I_x\right|}{|I_x|} = 1, \ \ \forall\varepsilon > 0. \tag{2.2}$$

Let us show that if there exists a *μ-stat* limit, then it is unique. Let $A \neq B$ and B also be a *μ-stat* limit for f. Take $0 < \varepsilon < \frac{1}{2}|A - B|$. Then it is clear that $\left(A_\varepsilon^f\right)^c \cap \left(B_\varepsilon^f\right)^c = \emptyset$. Consequently, from

$$\left[\left(A_\varepsilon^f\right)^c \cap I_x\right] \cup \left[\left(B_\varepsilon^f\right)^c \cap I_x\right] \subset I_x,$$

it follows that

$$\left|\left[\left(A_\varepsilon^f\right)^c \cap I_x\right]\right| + \left|\left[\left(B_\varepsilon^f\right)^c \cap I_x\right]\right| \leq |I_x|,$$

i.e.

$$\frac{\left|\left(A_\varepsilon^f\right)^c \cap I_x\right|}{|I_x|} + \frac{\left|\left(B_\varepsilon^f\right)^c \cap I_x\right|}{|I_x|} \leq 1.$$

Passing to the limit as $x \to \infty$ and using (2.2), we arrive at a contradiction.

It is easy to see that if $\lim_{x\to\infty} f(x) = A$, then $\exists\mu$-*st* lim and *μ-st* $\lim_{x\to\infty} f(x) = A$. The converse is not true. Let μ be a Lebesgue measure. Consider the Dirichlet function

$$D(x) = \begin{cases} 0, \ x \in I\backslash Q_I, \\ 1, \ x \in Q_I, \end{cases}$$

where Q_I are rational numbers from I. It is absolutely obvious that *μ-st* $\lim_{x\to\infty} D(x) = 0$, and $\lim_{x\to\infty} D(x)$ does not exist.

From

$$\lambda \neq 0 : \{x : |\lambda f(x) - \lambda A| \geq \varepsilon\} \Leftrightarrow \left\{x : |f(x) - A| \geq \frac{\varepsilon}{|\lambda|}\right\},$$

it directly follows that

$$\mu - st \lim_{x\to\infty} (\lambda f(x)) = \lambda(\mu - st \lim_{x\to\infty} f(x)).$$

Let

$$\mu - st \lim_{x\to\infty} f(x) = A; \mu - st \lim_{x\to\infty} g(x) = B.$$

Then it is clear that

$$\{x : |f(x) + g(x) - A - B| \geq \varepsilon\}$$
$$\subset \left[\left\{x : |f(x) - A| \geq \frac{\varepsilon}{2}\right\} \cup \left\{x : |g(x) - B| \geq \frac{\varepsilon}{2}\right\}\right].$$

Consequently

$$\left|\left\{x:\,|f+g-A-B|\geq\varepsilon\right\}\cap I_x\right|$$
$$\leq\left|\left\{x:\,|f-A|\geq\frac{\varepsilon}{2}\right\}\cap I_x\right|+\left|\left\{x:\,|g-B|\geq\frac{\varepsilon}{2}\right\}\cap I_x\right|.$$

Hence we obtain

$$\mu\text{-}st\lim_{x\to\infty}\left(f\left(x\right)+g\left(x\right)\right)=A+B.$$

Thus, $\mathscr{B}$ measurable functions with μ-*st* lim at infinity form a linear space, and we denote this space by $\mathscr{B}_{st}$. The set of all subsets of I, for which the infinity is the point of μ-*stat* density, will be denoted by I_{st}^{∞}. Assume that we have

$$\lim_{\substack{x\to\infty\\ x\in M}}f\left(x\right)=A\,,\ M\in I_{st}^{\infty}. \tag{2.3}$$

This implies that for $\forall\varepsilon>0$, $\exists x_0^{\varepsilon}\in I$:

$$\left|f\left(x\right)-A\right|<\varepsilon\,,\ \forall x\in\left[M\cap\left(x_0^{\varepsilon},\infty\right)\right].$$

It is absolutely obvious that for $\forall t\in I$:

$$\lim_{x\to\infty}\frac{\left|M_t\cap I_x\right|}{\left|I_x\right|}=1,\text{where }\ M_t\equiv M\cap I_t,I_t\equiv\left[a,t\right],$$

if $|I|=\mu\left(I\right)=+\infty$. The latter follows from the σ-finiteness of the measure μ. Consequently

$$\lim_{x\to\infty}\frac{\left|M_t^c\cap I_x\right|}{\left|I_x\right|}=0\,,\ \ \forall t\in I.$$

It is absolutely clear that $\left(A_{\varepsilon}^{f}\right)^c\supset M_{x_0^{\varepsilon}}$. As a result, we obtain

$$\frac{\left|\left(A_{\varepsilon}^{f}\right)^c\cap I_x\right|}{\left|I_x\right|}\geq\frac{\left|M_{x_0^{\varepsilon}}\cap I_x\right|}{\left|I_x\right|}\to1\,,\ x\to\infty.$$

Hence we obtain

$$\mu\text{-}st\lim_{x\to\infty}f\left(x\right)=A. \tag{2.4}$$

Thus, if the relation (2.3) holds, then the relation (2.4) is true.

Now, let the relation (2.4) hold. Hence, for $\forall\varepsilon>0$:

$$\lim_{x\to\infty}\frac{\left|\left(A_{\varepsilon}^{f}\right)^c\cap I_x\right|}{\left|I_x\right|}=1.$$

Put

$$M_n\equiv\left\{x\in I:\left|f\left(x\right)-A\right|<\frac{1}{n}\right\},\ n\in N.$$

It is clear that

$$\lim_{x\to\infty}\frac{\left|M_n\cap I_x\right|}{\left|I_x\right|}=1\,,\ \ \forall n\in N, \tag{2.5}$$

and $M_1\supset M_2\supset\ldots$. Take $\forall x_1\in M_1$ and let $\xi_1=x_1$. From (2.5) we obtain that $\exists x_2>\xi_1$, $x_2\in M_2$; $\xi_2=\max\left\{x_1;\,1\right\}$:

$$\frac{\left|M_2\cap I_x\right|}{\left|I_x\right|}\geq\frac{1}{2}\,,\ \ \forall x\geq x_2.$$

Continuing this process, we obtain a sequence $\left\{x_n\right\}_{n\in N}:x_n\in M_n$, $x_1<x_2\ldots$, and

$$\frac{\left|M_n\cap I_x\right|}{\left|I_x\right|}\geq\frac{n-1}{n}\,,\ \ \forall x\geq x_n\geq\xi_n,$$

where $\xi_n = \max\{x_{n-1};\, n-1\}$. Put

$$A_0 = [a,\, x_1)\ ,\ \ A_n = [x_n,\, x_{n+1}) \cap M_n\,,\ \forall n \in N.$$

Let $M \equiv \bigcup_{n=0}^{\infty} A_n$. Consider $M_{n_0} \cap I_x$, where $x \in [x_{n_0},\, x_{n_0+1})$ is some point. Let $y \in M_{n_0} \cap I_x \Rightarrow |f(y) - A| < \frac{1}{n_0}$. It is clear that $\exists k_0 \in \{0; \ldots; n_0\} : y \in [x_{k_0},\, x_{k_0+1})$, where $x_0 = a$. We have

$$|f(y) - A| < \frac{1}{k_0} \Rightarrow y \in A_{k_0} \Rightarrow y \in M \cap I_x.$$

Consequently, $M_{n_0} \cap I_x \subset M \cap I_x$ and

$$\frac{|M \cap I_x|}{|I_x|} \geq \frac{|M_{n_0} \cap I_x|}{|I_x|} \geq \frac{n_0 - 1}{n_0}\,.$$

As a result

$$\lim_{x \to \infty} \frac{|M \cap I_x|}{|I_x|} = 1\,.$$

Take $\forall \varepsilon > 0$. Since $x_n \to \infty$ as $n \to \infty$, we have $\exists n_0 : \frac{1}{n_0} \leq \varepsilon$. Let $x \geq x_{n_0}$ and $x \in M$. Consequently, $x \in A_{k_0}$, for some $k_0 \geq n_0$. We have

$$|f(x) - A| < \frac{1}{k_0} \leq \frac{1}{n_0} \leq \varepsilon.$$

Thus, $M \in I_{st}^{\infty}$ and

$$\lim_{\substack{x \to \infty \\ x \in M}} f(x) = A.$$

So we get the validity of

Theorem 2.3. *Let $\mu(I) = +\infty$, and $f : I \to R$ be some $\mathscr{B}$-measurable function. Then f has a μ-stat limit* (2.4) *if and only if there exists a set $M \in I_{st}^{\infty}$ such that relation* (2.3) *holds.*

We introduce the following

Definition 2.4. The sequence $\{a_n\}_{n \in N} \subset I$ is said to have a *st* lim at infinity if

$$\lim_{n \to \infty} \frac{r(A_\varepsilon \cap e_n)}{n} = 0\,,\ \forall \varepsilon > 0,$$

where $e_n \equiv \{1; \ldots; n\}$, $A_\varepsilon \equiv \{k \in N : |a_k| < \varepsilon\}$ and $r(M)$ is the number of elements M. For brevity's sake, this limit will be denoted as $a_n \overset{st}{\to} \infty$, $n \to \infty$.

Now let's give another definition of *st* lim for function f at infinity. First, we introduce the following concept of *st* lim for a sequence.

Definition 2.5. We say that *st* $\lim_{n \to \infty} a_n = a$ if and only if

$$\lim_{n \to \infty} \frac{r(a_\varepsilon \cap e_n)}{n} = 0\,,\ \forall \varepsilon > 0,$$

where $a_\varepsilon \equiv \{k \in N : |a_k - a| \geq \varepsilon\}$.

This limit will be denoted as $a_n \overset{st}{\to} a$, $n \to \infty$.

And now we give the following

Definition 2.6. We say that the function $f : I \to R$ has an *st* lim limit at infinity equal to A if $f(a_n) \overset{st}{\to} A$ with $\forall \{a_n\}_{n \in N} \subset I :\ a_n \overset{st}{\to} \infty$, and it will be denoted as *st* $\lim_{x \to \infty} f(x) = A$.

The question then arises: are Definitions 2.2 and 2.6 equivalent to each other? Consider the following function:

$$f(x) = \begin{cases} n, \, x = n \in N, \\ x^{-1}, \; x \in I_0 \backslash N, \end{cases}$$

where $I_0 \equiv [1, +\infty)$. Let μ be the Lebesgue measure on I_0. It is absolutely obvious that f has μ-*st* lim equal to zero at infinity in the sense of Definition 2.2, but does not have *st* lim in the sense of Definition 2.6.

Now let $\exists st \lim_{x\to\infty} f(x) = A$. Consequently, for $\forall \{a_n\}_{n\in N} \subset I : \; a_n \overset{st}{\to} \infty$ we have $f(a_n) \overset{st}{\to} A$. Assume that the relation $\mu - st \; \lim_{x\to\infty} f(x) = A$ does not hold. This means that there exists $\varepsilon_0 > 0$ such that the relation

$$\lim_{x\to\infty} \frac{\left|A^f_{\varepsilon_0} \cap I_x\right|}{|I_x|} = 0$$

does not hold. Thus, $\exists \delta_0 > 0$ and $\exists \{x_n\}_{n\in N} \subset E : \; x_1 < x_2 < \ldots, \; x_n \to \infty$:

$$\frac{\left|A^f_{\varepsilon_0} \cap I_{x_n}\right|}{|I_{x_n}|} \geq \delta_0,$$

i.e.

$$\left|A^f_{\varepsilon_0} \cap I_{x_n}\right| \geq \delta_0 \left|I_{x_n}\right| \to \infty, \; n \to \infty.$$

Then $\exists \{n_k\}$:

$$\left|A^f_{\varepsilon_0} \cap I_{x_{n_k}}\right| < \left|A^f_{\varepsilon_0} \cap I_{x_{n_{k+1}}}\right|,$$
$$\left|A^f_{\varepsilon_0} \cap I_{x_{n_k}}\right| \to +\infty, \; k \to \infty.$$

It immediately follows that

$$\exists \{a_k\} : a_k \in A^f_{\varepsilon_0} \cap I_{x_{n_k}}, \; a_k \in I_{x_{n_k}} \backslash I_{x_{n_{k-1}}}, \; a_k \to \infty.$$

Consequently, $|f(a_k) - A| \geq \varepsilon_0$, and we obtain that $a_k \overset{st}{\to} \infty$, but the relation $f(a_k) \overset{st}{\to} A$ does not hold. Thus, the following statement is true.

Statement 2.7. Let $\mu(I) = +\infty$. If $\exists st \lim\limits_{x\to\infty} f(x)$, then $\exists \mu$-*st* $\lim_{x\to\infty} f(x)$ and μ-*st* $\lim_{x\to\infty} f(x) = st \; \lim_{x\to\infty} f(x)$. The converse is generally not true.

3. μ-STATISTICAL FUNDAMENTALITY

Let us define the concept of μ-statistical fundamentality at infinity.

Definition 3.1. We say that the function $f : I \to R$ is μ-*stat* fundamental at infinity if $\forall \varepsilon > 0, \; \exists x_\varepsilon \in I$:

$$\lim_{x\to\infty} \frac{|X_\varepsilon \cap I_x|}{|I_x|} = 0, \tag{3.1}$$

where $X_\varepsilon \equiv \{x \in I : |f(x) - f(x_\varepsilon)| \geq \varepsilon\}$.

Let $\exists \mu$-*st* $\lim_{x\to\infty} f(x) = A$. Then by Theorem 2.3 $\exists M \in I^\infty_{st} : \lim_{\substack{x\to\infty \\ x\in M}} f(x) = A$. Consequently, for $\forall \varepsilon > 0, \; \exists x_\varepsilon \in M$:

$$|f(x) - A| < \varepsilon, \; \forall x \geq x_\varepsilon, \; x \in M.$$

We have

$$|f(x) - f(x_\varepsilon)| \leq |f(x) - A| + |f(x_\varepsilon) - A| < \varepsilon, \; \forall x \in M, \; x \geq x_\varepsilon.$$

Hence, from previous arguments we immediately obtain that if $\mu(I) = +\infty$, then

$$\lim_{x\to\infty} \frac{|X_\varepsilon^c \cap I_x|}{|I_x|} = 1,$$

and this in turn is equivalent to (3.1). Thus, if $\exists \mu\text{-}st \lim_{x\to\infty} f(x)$, then f is μ-*stat* fundamental at infinity.

Let $M_1\,,\, M_2 \in I_{st}^\infty$. We have

$$M_1 \cap M_2 = M_1 \cup M_2 \setminus [(M_2 \setminus M_1) \cup (M_1 \setminus M_2)]\,.$$

Consequently

$$M_1 \cap M_2 \cap I_x = [(M_1 \cup M_2) \cap I_x] \setminus [((M_2 \setminus M_1) \cup (M_1 \setminus M_2)) \cap I_x]\,. \tag{3.2}$$

As

$$((M_2 \setminus M_1) \cup (M_1 \setminus M_2)) \cap I_x = ((M_2 \setminus M_1) \cap I_x) \cup ((M_1 \setminus M_2) \cap I_x)\,,$$

taking into account relations

$$(M_2 \setminus M_1) \cap I_x \subset M_1^c \cap I_x$$

$$\Rightarrow \frac{|(M_2 \setminus M_1) \cap I_x|}{|I_x|} \le \frac{|M_1^c \cap I_x|}{|I_x|} \to 0\,,\ x \to \infty,$$

$$\frac{|(M_1 \setminus M_2) \cap I_x|}{|I_x|} \to 0\,,\ x \to \infty,$$

we obtain that

$$\frac{|((M_2 \setminus M_1)\,(M_1 \setminus M_2)) \cap I_x|}{|I_x|} \to 0\,,\ x \to \infty.$$

From $(M_1 \cap I_x) \subset (M_1 \cup M_2) \cap I_x$ and $M_1 \in I_{st}^\infty$ it follows that

$$\frac{|(M_1 \cup M_2) \cap I_x|}{|I_x|} \to 1\,,\ x \to \infty,$$

and, consequently, $M_1 \cup M_2 \in I_{st}^\infty$. Then from (3.2) we directly obtain

$$\frac{|M_1 \cap M_2 \cap I_x|}{|I_x|} = \frac{|(M_1 \cup M_2) \cap I_x|}{|I_x|}$$

$$-\frac{|((M_2 \setminus M_1) \cup (M_1 \setminus M_2)) \cap I_x|}{|I_x|} \to 1\,,\ x \to \infty,$$

i.e. $M_1 \cap M_2 \in I_{st}^\infty$. As a result, we get the validity of

Lemma 3.2. *If* $M_2 \in I_{st}^\infty\,,\ k = 1, 2,$ *then* $M_1 \cap M_2 \in I_{st}^\infty$.

Now, to the contrary, assume that f is μ-*stat* fundamental at infinity. Then for $\varepsilon_1 = 1\,,\ \exists x_1 \in I:$

$$\lim_{x\to\infty} \frac{|X_1 \cap I_x|}{|I_x|} = 0,$$

i.e.

$$X_1^c \equiv \{x \in I : |f(x) - f(x_1)| < 1\} \in I_{st}^\varepsilon.$$

Similarly, for $\varepsilon_2 = \frac{1}{2},\ \exists x_2 \in I : X_2^c \in I_{st}^\infty$, where

$$X_k^c \equiv \left\{x \in I : |f(x) - f(x_2)| < \frac{1}{k}\right\}.$$

By Lemma 3.2 we obtain $X_1^c \cap X_2^c \equiv Y_1 \in I_{st}^{\infty}$. Assume $\tilde{Y}_1 = \{f(x) : x \in Y_1\}$. It is absolutely clear that

$$\tilde{Y}_1 \subset I_2 \equiv \left[f(x_2) - \frac{1}{2},\, f(x_2) + \frac{1}{2}\right].$$

Similarly we define $X_4^c \equiv \left\{x \in I : |f(x) - f(x_4)| < \frac{1}{4}\right\}$ and consider $Y_2 = Y_1 \cap X_4^c$. Again by Lemma 3.2 we have $Y_2 \in I_{st}^{\infty}$. Let

$$\tilde{Y}_2 \subset I_4 \equiv \left(\left[f(x_4) - \frac{1}{4},\, f(x_4) + \frac{1}{4}\right] \cap I_2\right).$$

Continuing this process, we obtain a sequence of segments I_{2^n} and sets $\tilde{Y}_n \subset I_{2^n}$ with the following properties:

$$I_{2^n} \supset I_{2^{n+1}} \supset \ldots;\, d(I_{2^n}) \leq \frac{1}{2^{n-1}},$$

$$\tilde{Y}_n \equiv \{f(x) : x \in Y_n\} \subset I_{2^n},$$

$$Y_{n-1} \equiv X_{n-1}^c \cap X_n^c;\; Y_n \in I_{st}^{\infty},$$

$$\tilde{Y}_n \equiv \{f(x) : x \in Y_n\},$$

where $d(E)$ is the length of the segment E.

Obviously, $\exists! A \in \bigcap_{n=1}^{\infty} I_{2^n}$. Let us show that μ-*st* $\lim_{x\to\infty} f(x) = A$. Take $\forall \varepsilon > 0$. It is clear that $\exists n_0 \in N : I_{2^n} \subset \left(A - \frac{\varepsilon}{2},\, A + \frac{\varepsilon}{2}\right)$, $\forall n \geq n_0$. Thus,

$$\tilde{Y}_{n_0} \subset I_{2^{n_0}} \equiv \left\{f(x_{2^{n_0}}) - \frac{1}{2^{n_0}},\, f(x_{2^{n_0}}) + \frac{1}{2^{n_0}}\right\} \cap I_{2^{n_0-1}}.$$

On the other hand

$$\tilde{Y}_{n_0} \equiv \{f(x) : x \in Y_{n_0}\},$$

and, due to the structure of $Y_{n_0} \in I_{st}^{\infty}$, where $Y_{n_0} \in Y_{n_0-1} \cap X_{2^{n_0}}^c$, we have $X_{2^{n_0}}^c \equiv \left\{x \in I : |f(x) - f(x_{2^{n_0}})| < \frac{1}{2^{n_0}}\right\}$. Choose n_0 from the condition $\frac{1}{2^{n_0}} < \frac{\varepsilon}{2}$. We have

$$|f(x) - A| \leq |f(x) - f(x_{2^{n_0}})| + |f(x_{2^{n_0}}) - A| < |f(x) - f(x_{2^{n_0}})| + \frac{\varepsilon}{2}.$$

It directly follows that

$$x : |f(x) - f(x_{2^{n_0}})| < \frac{1}{2^{n_0}} \subset \left(A_{\varepsilon}^f\right)^c,$$

i.e. $X_{2^{n_0}}^c \subset \left(A_{\varepsilon}^f\right)^c$. As $X_{2^{n_0}}^c \in I_{st}^{\infty}$, it is clear that $\left(A_{\varepsilon}^f\right)^c \in I_{st}^{\infty}$. Thus, the relation μ-*st* $\lim_{x\to\infty} f(x) = A$ is true. So the following theorem is valid.

Theorem 3.3. *Let $\mu(I) = +\infty$. Then the function $f : I \to R$ is μ-stat fundamental at infinity if and only if $\exists \mu$-st $\lim_{x\to\infty} f(x)$.*

We give the following

Definition 3.4. The functions $f; g : I \to R$ are called μ-*stat* equivalent at infinity if $I_{f,g} \in I_{st}^{\infty}$, where $I_{f,g} = \{x \in I : f(x) = g(x)\}$ (denoted as $f \overset{st}{\sim} g$).

Assume that $\exists \mu$-*st* $\lim_{x\to\infty} f(x) = A$. Then, by Theorem 2.3, $\exists M \in I_{st}^{\infty}$ such that $\lim_{\substack{x\to\infty \\ x\in M}} f(x) = A$. Define

$$g(x) \equiv \begin{cases} f(x),\, x \in M, \\ A,\, x \in M^c. \end{cases}$$

It is absolutely clear that $M \subset I_{f,g}$, i.e. $f \overset{st}{\sim} g$. Obviously

$$\lim_{\substack{x\to\infty \\ x\in I}} g(x) = A. \tag{3.3}$$

To the contrary, it is clear that if (3.3) holds and $f \overset{st}{\sim} g$, then $\exists \mu\text{-}st \lim_{x\to\infty} f(x) = A$. Thus, the following theorem is true.

Theorem 3.5. *Let $\mu(I) = +\infty$. Then for the $\mathscr{B}$-measurable function $f : I \to R$ the following statements are equivalent to each other:*
a) $\exists \mu\text{-}st \lim_{x\to\infty} f(x)$;
b) f is μ-stat fundamental at infinity;
c) $\exists g : f \overset{st}{\sim} g$ and $\exists \lim_{x\to\infty} g(x)$.

4. μ-STAT CONTINUITY

Let R be the real axis, $\mathscr{R}$ be a σ-algebra of all Borel sets in R and $(R\,;\mathscr{R}\,;\mu)$ be some space with σ-finite measure μ and $\mu((-\infty, a)) = \mu((a, +\infty)) = +\infty$, $\forall a \in R$. Let $E_0 \subset \mathscr{R}$ be some set, and let $f : E_0 \to R$ be some $(R\,;\mathscr{R})$-measurable function. Let x_0 be a limit point of E_0. Let

$$E_0^{x_0} \equiv \left\{ x : \, x_0 + \frac{1}{x} \in E_0 \right\}.$$

Put

$$I_t^{x_0} \equiv \begin{cases} \left[a, \, (t - x_0)^{-1} \right], \, \frac{1}{t-x_0} \geq a, \\ \left[(t - x_0)^{-1}, \, a \right], \, \frac{1}{t-x_0} < a, \end{cases}$$

where $a \in R$ is an arbitrary fixed point. We introduce the following concept.

Definition 4.1. We say that the function f has a *μ-stat* (μ-statistical) limit at x_0, if

$$\lim_{t\to x_0} \frac{\left| A_{\varepsilon,x_0}^f \cap I_t^{x_0} \right|}{\left| I_t^{x_0} \right|} = 0\,, \; \varepsilon > 0,$$

where

$$A_{\varepsilon,x_0}^f \equiv \left\{ x \in E_0^{x_0} : \left| f\left(x_0 + \frac{1}{x} \right) - A \right| \geq \varepsilon \right\}.$$

This will be denoted as $\mu\text{-}st \lim_{x\to x_0} f(x) = A$.

It is absolutely clear that if $\exists \lim_{x\to x_0} f(x)$, then $\exists \mu\text{-}st \lim_{x\to x_0} f(x)$, and these two are equal. Moreover, if $\exists \mu\text{-}st \lim_{x\to x_0} f_k(x)$, $k = 1, 2$, then there are the following limits:

$$\mu - st \lim_{x\to x_0} (\lambda f_1(x)) = \lambda \left(\mu - st \lim_{x\to x_0} f_1(x) \right), \; \forall \lambda \in R;$$
$$\mu - st \lim_{x\to x_0} (f_1 + f_2) = \mu - st \lim_{x\to x_0} f_1 + \mu - st \lim_{x\to x_0} f_2.$$

Before we continue, we need some more concepts. Let $M \in \mathscr{R}$ and $x_0 \in R$ be a limit for M. Put

$$M_{x_0} \equiv \left\{ x \in R : \, x_0 + \frac{1}{x} \in M \right\}.$$

Definition 4.2. We say that x_0 is a point of *μ-stat* density for the set M if

$$\lim_{t\to x_0} \frac{|M_{x_0} \cap I_t^{x_0}|}{|I_t^{x_0}|} = 1.$$

By $\mathscr{R}_{st}^{x_0}$ we denote the subset $\mathscr{R}$ for the elements of which the point x_0 is a point of *μ-stat* density.

Using the results of previous section, we immediately obtain the validity of the following statements.

Theorem 4.3. *Let $\mu\left(E_0^{x_0}\right) = +\infty$, and let $f : E_0 \to R$ be a function. Then $\exists \mu$-st $\lim_{x\to x_0} f(x)$ if and only if $\exists M \in \mathscr{R}_{st}^{x_0} : \exists \lim\limits_{\substack{x\to x_0\\ x\in M}} f(x)$, and these limits are equal.*

Let us introduce the concept of *μ-stat* fundamentality at the point x_0. Let a Borel measurable function $f : E_0 \to R$ be given and x_0 be a limit point of E_0. Let

$$A^{x_\varepsilon} \equiv \{x \in E_0 : |f(x) - f(x_\varepsilon)| \geq \varepsilon\}$$

and

$$A_{x_0}^{x_\varepsilon} \equiv \left\{x \in R : x_0 + \frac{1}{x} \in A^{x_\varepsilon}\right\}.$$

Definition 4.4. We say that the function $f : E_0 \to R$ is *μ-stat* fundamental at x_0, if for $\forall \varepsilon > 0$, $\exists x_\varepsilon \in E_0$:

$$\lim_{\substack{t\to x_0\\ t\in E_0}} \frac{\left|A_{x_0}^{x_\varepsilon} \cap I_t^{x_0}\right|}{|I_t^{x_0}|} = 0.$$

Definition 4.5. The functions f, $g : E_0 \to R$ are called *μ-stat* equivalent at x_0, if $I_{f,g} \in E_{st}^{x_0}$.

Using Theorem 3.5, we obtain the following

Theorem 4.6. *Let $\mu\left(E_0^{x_0}\right) = +\infty$, and let $f : E_0 \to R$ be a given Borel measurable function $f : E_0 \to R$. Then the following statements are equivalent to each other:*

a) $\exists \mu$-st $\lim\limits_{x\to x_0} f(x)$;

b) f is μ-stat fundamental at x_0;

c) $\exists\, g : f \overset{st}{\sim} g$ at x_0.

Quite naturally we come to the following

Definition 4.7. Let $f : E_0 \to R$ and $x_0 \in E_0$ be a limit point of E_0. f is said to be *μ-stat* continuous at x_0 in E_0, if *μ-st* $\lim\limits_{\substack{x\to x_0\\ x\in E_0}} f(x) = f(x_0)$. f is said to be *μ-stat* continuous in the set $M \subset E_0$, if it is *μ-stat* continuous at every point in M.

It is absolutely clear that if f and g are *μ-stat* continuous at x_0 (in M), then λf and $(f + g)$ are also *μ-stat* continuous at x_0 (in M). Denote by $C^{st}(M)$ the set of all *μ-stat* continuous functions in M. Then $C^{st}(M)$ is a linear space.

Example 4.8. Let $E \equiv [1, +\infty)$, $\mathscr{B}$ be a σ-algebra of Borel subsets of E. As μ, we take a discrete measure

$$\mu(A) = \sum_{k=1}^{\infty} \chi_A(k),$$

where $\chi_A(\cdot)$ is the characteristic function of the set $A \in \mathscr{B}$. Then it is not difficult to see that in this case the μ-statistical limit of the function $f : E \to R$ at infinity coincides with the statistical limit of the corresponding sequence $\{f(n)\}_{n\in N}$.

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