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GENERAL Ω-THEOREMS FOR COEFFICIENTS OF L-FUNCTIONS

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ABSTRACT. We prove a general Ω-theorem for the coefficients of polynomial combinations of L-functions from the Selberg class. As a consequence, we show that the real and imaginary parts of any linear combination of coefficients of such L-functions have infinitely many sign changes, provided some simple necessary conditions are satisfied.

1. INTRODUCTION

Oscillation problems for the coefficients of L-functions and Ω-results for sums of such coefficients are a very classical subject in analytic number theory. For example, starting with degree 1, it is known that both real and imaginary parts of a complex primitive Dirichlet character $\chi(n)$ (mod q) with $q > 2$ change sign for infinitely many n's, and the same holds for $\chi(n)n^{i\theta}$ for all χ's and all $\theta \neq 0$. Passing to degree 2, sign changes and Ω-theorems related to the coefficients of modular forms, for example of the Ramanujan τ function, have been studied by several authors. Moreover, results of this type exist in the literature for the coefficients of some L-functions of higher degree, or even for the coefficients of certain classes of Dirichlet series. We quote here Balasubramanian-Ram Murty [1], Ram Murty [13], Kohnen-Sengupta [12], Pribitkin [17], [18] and Kohnen-Pribitkin [11] as a sample of the vast literature on the subject.

In this paper we first prove a general Ω-theorem for the sum of the coefficients of polynomial combinations of L-functions from the Selberg class $\mathcal{S}$ of Dirichlet series with functional equation and Euler product. In particular, our result holds for L-functions of any degree. Then, from the Ω-theorem we deduce a sign-changes result for certain linear combinations of the coefficients of such L-functions. We refer to the beginning of the next section for several definitions concerning the Selberg class, and to our survey papers [3], [7], [14], [15] and [16] for its basic properties.

Let $P \in \mathbb{C}[X_1, \dots, X_N]$ and $F_1, \dots, F_N \in \mathcal{S}$ be fixed and let

$$H_P(s) = P(F_1(s), \dots, F_N(s)) = \sum_{n=1}^{\infty} \frac{a_P(n)}{n^s},$$

the series being absolutely convergent for $\sigma > 1$. From the definition of the class $\mathcal{S}$, it follows that $H_P(s)$ is meromorphic over $\mathbb{C}$ with at most a pole at $s = 1$. Since the Selberg class is a semigroup, it may obviously happen that $H_P(s) \equiv 0$. Moreover, if

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$H_P(s) \not\equiv 0$ there exist functions $G_\nu \in \mathcal{S}$ and constants $c_\nu \in \mathbb{C} \setminus \{0\}$, $\nu = 1, \dots, M$, such that

$$H_P(s) = \sum_{\nu=1}^{M} c_\nu G_\nu(s). \tag{1.1}$$

Writing $d = \max_{\nu=1,\dots,M} d_{G_\nu}$ (d_F is the degree of $F \in \mathcal{S}$; see the beginning of the next section), we have

Theorem 1. *Let $H_P(s) \not\equiv 0$ and d be as above. Then as $x \to \infty$ we have*

$$\sum_{n \le x} a_P(n) = \Omega\big(x^{\frac{1}{2} - \frac{1}{2d}}\big)$$

and

$$\sum_{n \le x} |a_P(n)| = \Omega\big(x^{\frac{1}{2} + \frac{1}{2d}}\big).$$

We first remark that by the same method as in the proof of Corollary 2 of [8] (see pages 337–339), we may replace the right hand side of the two formulae in Theorem 1, respectively, by

$$xQ(\log x) + \Omega\big(x^{\frac{1}{2} \mp \frac{1}{2d}}\big),$$

with an arbitrary polynomial $Q \in \mathbb{C}[x]$. Hence Theorem 1 may be regarded as an extension of Corollary 2 in [8]. Moreover, Theorem 1 gives a partial solution to a general problem stated at the end of the Introduction of [9]. Indeed, given $F_1(s), \dots, F_N(s)$ in the extended Selberg class $\mathcal{S}^\sharp$ (see the next section), in [9] we asked for an exponent $\theta = \theta(d_{F_1}, \dots, d_{F_N}) > 0$ such that

$$\sum_{n \le x} |L(a_1(n), \dots, a_N(n))| = \Omega(x^\theta),$$

where the $a_\nu(n)$'s are the coefficients of the $F_\nu(s)$'s and $L(X_1, \dots, X_N)$ is a linear form such that $L(F_1(s), \dots, F_N(s)) \not\equiv 0$. Choosing $P(X_1, \dots, X_N) = L(X_1, \dots, X_N)$, Theorem 1 allows the choice

$$\theta = \frac{1}{2} + \frac{1}{2} \min_{\nu=1,\dots,N} \frac{1}{d_{F_\nu}},$$

but the functions $F_\nu(s)$ belong to the smaller class $\mathcal{S} \subset \mathcal{S}^\sharp$. Theorem 1 may therefore be regarded as an independence result for L-functions in $\mathcal{S}$.

Again let

$$L(X_1, \dots, X_N) = \sum_{\nu=1}^{N} \alpha_\nu X_\nu, \qquad \alpha_\nu \in \mathbb{C} \setminus \{0\}, \tag{1.2}$$

be a linear form. Note that $F, G \in \mathcal{S}$ are called conjugate if $G(s) = \overline{F}(s)$, and $F(s)$ is called real if $F(s) = \overline{F}(s)$ (i.e., the coefficients of $F(s)$ and $G(s)$ are conjugate, and the coefficients of $F(s)$ are real, respectively).

Theorem 2. *Let $F_1, \dots, F_N \in \mathcal{S}$ be entire, distinct and non-conjugate in pair, and assume that the linear form in (1.2) has $\alpha_\nu \in \mathbb{R}$ whenever $F_\nu(s)$ is real. Then the sequence $\Re L(a_1(n), \dots, a_N(n))$, $n \in \mathbb{N}$, changes sign infinitely many times. Moreover, if at least one of the $F_\nu(s)$ is not real, then also the sequence $\Im L(a_1(n), \dots, a_N(n))$ changes sign infinitely many times.*

Theorem 2 is a direct consequence of Theorem 1, and its proof is given after that of Theorem 1. Note that all the hypotheses of Theorem 2 are necessary. Indeed, if $N = 1$ and $F_1(s) = \zeta(s)$, then the theorem clearly fails. If $N = 2$ and $F_2(s) = F_1(s)$ or $F_2(s) = \overline{F_1}(s)$, then the theorem fails for $L(X, Y) = X - Y$. Moreover, if $N = 1$ and $F_1(s)$ is real, then the theorem fails with $L(X) = iX$. Finally, it is clear that an analogous result holds for the imaginary parts; in this case we assume that α_ν is purely imaginary if $F_\nu(s)$ is real. Note also that the results in Pribitkin [17] and Kohnen-Pribitkin [11] have non-empty intersection with the case $N = 1$ of Theorem 2. Indeed, [17] and [11] deal with general Dirichlet series with analytic continuation on an open set containing the real axis and having infinitely many zeros on the real axis. Additionally, Pribitkin [18] contains several results pertaining to the oscillatory nature of the coefficients of functions from the class of L-functions defined in Chapter 5 of Iwaniec-Kowalski [2], which overlaps with the Selberg class. Therefore, the results in [18] have an intersection with Theorem 2.

2. Proofs

We start with some definitions. A function $F(s)$ belongs to the Selberg class $\mathcal{S}$ if
i) $F(s)$ is an absolutely convergent Dirichlet series for $\sigma > 1$;
ii) $(s-1)^m F(s)$ is an entire function of finite order for some integer $m \geq 0$;
iii) $F(s)$ satisfies a functional equation of type $\Phi(s) = \omega\overline{\Phi}(1-s)$, where $|\omega| = 1$ and

$$\Phi(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) F(s) = \gamma(s) F(s),$$

say, with $r \geq 0$, $Q > 0$, $\lambda_j > 0$, $\Re\mu_j \geq 0$ (here and in the sequel we write $\overline{f}(s) = \overline{f(\overline{s})}$);
iv) the Dirichlet coefficients $a(n)$ of $F(s)$ satisfy $a(n) \ll n^\varepsilon$ for every $\varepsilon > 0$;
v) $\log F(s)$ is a Dirichlet series with coefficients $b(n)$ satisfying $b(n) = 0$ unless $n = p^m$, $m \geq 1$, and $b(n) \ll n^\vartheta$ for some $\vartheta < 1/2$.

The extended Selberg class $\mathcal{S}^\sharp$ is the larger class of the non-identically vanishing functions $F(s)$ satisfying only the first three axioms. *Degree, conductor, root number* and *ξ-invariant* of $F \in \mathcal{S}$ are defined respectively by

$$d_F = 2\sum_{j=1}^{r} \lambda_j, \qquad q_F = (2\pi)^{d_F} Q^2 \prod_{j=1}^{r} \lambda_j^{2\lambda_j},$$

$$\omega_F = \omega \prod_{j=1}^{r} \lambda_j^{-2i\Im\mu_j} \quad \text{and} \quad \xi_F = 2\sum_{j=1}^{r}(\mu_j - \frac{1}{2}) = \eta_F + i d_F \theta_F;$$

θ_F is the *internal shift* of $F(s)$. We also write $e(x) = e^{2\pi i x}$. Our main tool in the proof are the *non-linear twists* $F^\lambda(s, \alpha)$, defined for a function $F \in \mathcal{S}$ and positive α and λ by

$$F^\lambda(s, \alpha) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} e(-\alpha n^\lambda).$$

Of particular interest is the case of the *standard twist*, obtained by the choice $\lambda = 1/d_F$ and denoted simply by $F(s, \alpha)$. Let
(2.1)
$$\mathrm{Spec}(F) = \{\alpha > 0 : a(n_\alpha) \neq 0\} \text{ where } n_\alpha = q d^{-d} \alpha^d \text{ and } a(n_\alpha) = 0 \text{ if } n_\alpha \notin \mathbb{N},$$

where for simplicity we write $d = d_F$ and $q = q_F$. Spec(F) is called the *spectrum* of $F(s)$. The main properties of $F^\lambda(s,\alpha)$ are summarized as follows.

Lemma. *Let $F \in \mathcal{S}$ with $d_F > 0$ and let $\alpha > 0$. If $0 < \lambda < 1/d_F$, then $F^\lambda(s,\alpha)$ is entire. If $\lambda = 1/d_F$, then $F(s,\alpha)$ is entire if $\alpha \notin$ Spec(F), and it is meromorphic over $\mathbb{C}$ if $\alpha \in$ Spec(F). In the latter case, $F(s,\alpha)$ has at most simple poles at the points*

$$s_k = \frac{d+1}{2d} - \frac{k}{d} - i\theta_F, \qquad\qquad k = 0, 1, \ldots,$$

and

$$\mathrm{res}_{s=s_0} F(s,\alpha) = \frac{c_0(F)}{q^{s_0}} \frac{\overline{a(n_\alpha)}}{n_\alpha^{1-s_0}},$$

where $c_0(F) \neq 0$ is a certain constant depending only on $F(s)$.

Proof. These results are contained in Theorems 1 and 2 of [8] and Theorem 3 of [10]. ☐

In order to prove Theorem 1 we write $H_P(s)$ as in (1.1) and, for a given $\alpha > 0$ and d as in Theorem 1, we consider the twisted linear combination

$$H_P(s,\alpha) = \sum_{\nu=1}^{M} c_\nu G_\nu^{1/d}(s,\alpha). \tag{2.2}$$

Thanks to the Lemma, $H_P(s,\alpha)$ is meromorphic over $\mathbb{C}$. Denoting for simplicity by d_ν and θ_ν, respectively, degree and internal shift of $G_\nu(s)$, suppose that $d = d_{\nu_0}$ and denote by $\mathcal{F}$ the set of all ν such that

$$(d_\nu, \theta_\nu) = (d, \theta_{\nu_0}).$$

Since the twist $G_\nu^{1/d}(s,\alpha)$ coincides with the standard twist $G_\nu(s,\alpha)$ for $\nu \in \mathcal{F}$, we may rewrite (2.2) as

$$H_P(s,\alpha) = \sum_{\nu\in\mathcal{F}} c_\nu G_\nu(s,\alpha) + G(s,\alpha).$$

Let $s_0 = \frac{d+1}{2d} - i\theta_{\nu_0}$. In view of the Lemma and of the choice of d (recall that $0 < 1/d < 1/d_\nu$ or $\theta_\nu \neq \theta_{\nu_0}$ for $\nu \notin \mathcal{F}$) we have that $G(s,\alpha)$ is holomorphic at $s = s_0$, and $H_P(s,\alpha)$ has at most a simple pole at $s = s_0$. Arguing by contradiction, we now prove that $H_P(s,\alpha)$ has a simple pole at $s = s_0$ for some $\alpha > 0$.

Indeed, suppose that for every $\alpha > 0$

$$\sum_{\nu\in\mathcal{F}} c_\nu \mathrm{res}_{s=s_0} G_\nu(s,\alpha) = \mathrm{res}_{s=s_0} H_P(s,\alpha) = 0. \tag{2.3}$$

Denoting by $a_\nu(n)$ the coefficients of $G_\nu(s)$, once again thanks to the Lemma we have

$$\mathrm{res}_{s=s_0} G_\nu(s,\alpha) = \frac{c_0(G_\nu)}{q_\nu^{s_0}} \frac{\overline{a_\nu(n_\alpha)}}{n_\alpha^{1-s_0}}. \tag{2.4}$$

Therefore, given any $\beta > 0$ and choosing $\alpha = d(\beta/q_{\nu_0})^{1/d}$, from (2.3), (2.4) and the definition of n_α in (2.1) we obtain

$$\sum_{\nu\in\mathcal{F}} \tilde{c}_\nu a_\nu(\frac{q_\nu}{q_{\nu_0}}\beta) = 0 \tag{2.5}$$

with certain constants $\tilde{c}_\nu \neq 0$. Recalling the definition of $a(n_\alpha)$ in (2.1), multiplying both sides of (2.5) by β^{-s} and summing over all positive $\beta \in \mathbb{Q}$ we obtain

$$\sum_{\nu\in\mathcal{F}_0} \tilde{c}_\nu \Big(\frac{q_\nu}{q_{\nu_0}}\Big)^s G_\nu(s) \equiv 0, \tag{2.6}$$

where $\mathcal{F}_0 = \{\nu \in \mathcal{F} : q_\nu/q_{\nu_0} \in \mathbb{Q}\} \neq \emptyset$. Since $q_\nu/q_{\nu_0} \in \mathbb{Q}$, dividing (2.6) by A^s with a suitable positive integer A we obtain an identity of type

$$\sum_{\nu\in\mathcal{F}_0} \tilde{c}_\nu M_\nu^{-s} G_\nu(s) \equiv 0 \tag{2.7}$$

with $M_\nu \in \mathbb{N}$. Hence (2.7) is a non-trivial linear dependence relation of functions $G_\nu(s)$ from the Selberg class over Dirichlet polynomials (actually, Dirichlet monomials), a contradiction. Indeed, Kaczorowski-Molteni-Perelli [4] and [5] have shown that distinct functions from the Selberg class are linearly independent over p-finite Dirichlet series, thus in particular over Dirichlet polynomials. Therefore, $H_P(s,\alpha_0)$ has a simple pole at $s = s_0$ for a certain $\alpha_0 > 0$.

Now we conclude the proof of Theorem 1 arguing similarly as in Corollary 2 in [8], so we only give a sketch. First we may assume that $d > 1$, since otherwise all d_ν equal 1 and Theorem 1 is classical in view of the characterization of the degree 1 functions in $\mathcal{S}$ given in [6] (i.e., $\zeta(s)$ and $L(s+i\theta,\chi)$, $\theta \in \mathbb{R}$ and χ primitive). Arguing by contradiction we assume that

$$\sum_{n\le x} a_P(n) = o(x^{\frac{1}{2}-\frac{1}{2d}}).$$

Since $H_P(s,\alpha_0)$ has a simple pole at $s = s_0$ we deduce that as $\sigma \to 0^+$

$$|H_P(s_0+\sigma,\alpha_0)| \gg \frac{1}{\sigma}.$$

On the other hand, since $H_P(s)$ is holomorphic at $s = s_0$, by partial summation we get

$$|H_P(s_0+\sigma,\alpha_0)| \le |H_P(s_0+\sigma,\alpha_0) - e(-\alpha_0)H_P(s_0+\sigma)| + O(1) = o\Big(\frac{1}{\sigma}\Big)$$

as $\sigma \to 0^+$, a contradiction. The second assertion of Theorem 1 is proved as follows. Suppose that

$$\sum_{n\le x} |a_P(n)| = o(x^{\frac{1}{2}+\frac{1}{2d}}).$$

Then by partial summation we have that $\sum_{n=1}^{\infty} \frac{|a_P(n)|}{n^\sigma}$ is convergent for $\sigma > \Re(s_0) = \sigma_0$; moreover, $H_P(s,\alpha_0)$ has a pole at $s = s_0$. Therefore, as $\sigma \to 0^+$ by partial summation we have

$$\frac{1}{\sigma} \ll \Big|\sum_{n=1}^{\infty} \frac{a_P(n)e(-\alpha_0 n^{1/d})}{n^{s_0+\sigma}}\Big| \le \sum_{n=1}^{\infty} \frac{|a_P(n)|}{n^{s_0+\sigma}} \ll \int_1^\infty \sum_{n\le x} |a_P(n)| \frac{\mathrm{d}x}{x^{\sigma_0+1+\sigma}} = o\Big(\frac{1}{\sigma}\Big),$$

a contradiction. Theorem 1 therefore follows. □

In order to prove the first statement of Theorem 2 we consider the linear form

$$L^*(X_1,\dots,X_{N^*}), \qquad N^* = 2N - M,$$

where M is the number of ν such that $F_\nu(s)$ is real (we may assume that these are the ν's from 1 to M), $F_\nu(s) = \overline{F_{\nu-M}}(s)$ for $\nu = N+1,\dots,N^*$ and

$$L^*(X_1,\dots,X_{N^*}) = \sum_{\nu=1}^{N^*} \alpha_\nu^* X_\nu, \qquad \alpha_\nu^* = \begin{cases} \alpha_\nu, & \nu = 1,\dots,M,\\ \frac{1}{2}\alpha_\nu, & \nu = M+1,\dots,N,\\ \frac{1}{2}\overline{\alpha_{\nu+M-N}}, & \nu = N+1,\dots,N^*. \end{cases}$$

Therefore, for $\sigma > 1$ we have

$$L^*(F_1(s),\dots,F_{N^*}(s)) = \sum_{n=1}^{\infty} \frac{\Re L(a_1(n),\dots,a_N(n))}{n^s} = H(s), \tag{2.8}$$

and we must show that the coefficients of $H(s)$ have infinitely many sign changes.

We argue by contradiction, hence assuming that $\Re L(a_1(n),\dots,a_N(n))$ has constant sign for $n \geq n_0$. Thus, in view of (2.8) and of the hypothesis that the $F_\nu(s)$ are entire, by Landau's theorem the Dirichlet series of $H(s)$ is everywhere convergent. As a consequence, its coefficients satisfy

$$\Re L(a_1(n),\dots,a_N(n)) \ll n^{-A}$$

for every $A > 0$. But this contradicts the second assertion of Theorem 1, and the first assertion of Theorem 2 is proved.

To prove the second statement of Theorem 2 we proceed in a completely analogous way, considering the linear form $-iL^*(X_1,\dots,X_{N^*})$ with

$$\alpha_\nu^* = \begin{cases} \alpha_\nu, & \nu = 1,\dots,M,\\ \frac{1}{2}\alpha_\nu, & \nu = M+1,\dots,N,\\ -\frac{1}{2}\overline{\alpha_{\nu+M-N}}, & \nu = N+1,\dots,N^*. \end{cases}$$

This concludes the proof of Theorem 2. $\square$

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