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ON ESTIMATES FOR WEIGHTED BERGMAN PROJECTIONS

P. CHARPENTIER, Y. DUPAIN, AND M. MOUNKAILA

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Abstract. In this note we show that the weighted L^2-Sobolev estimates obtained by P. Charpentier, Y. Dupain & M. Mounkaila for the weighted Bergman projection of the Hilbert space $L^2(\Omega, d\mu_0)$ where Ω is a smoothly bounded pseudoconvex domain of finite type in $\mathbb{C}^n$ and $\mu_0 = (-\rho_0)^r\, d\lambda$, with λ the Lebesgue measure, $r \in \mathbb{Q}_+$ and ρ_0 a special defining function of Ω, are still valid for the Bergman projection of $L^2(\Omega, d\mu)$ where $\mu = (-\rho)^r\, d\lambda$, with ρ any defining function of Ω and $r \in \mathbb{R}_+$. In fact a stronger directional Sobolev estimate is established. Moreover similar generalizations (for $r \in \mathbb{Q}_+$) are obtained for weighted L^p-Sobolev and Lipschitz estimates in the case of the pseudoconvex domain of finite type in $\mathbb{C}^2$ (or, more generally, when the rank of the Levi form is $\geq n-2$).

1. Introduction

Let Ω be a smoothly bounded domain in $\mathbb{C}^n$. A nonnegative measurable function ν on Ω is said to be an admissible weight if the space of holomorphic functions square integrable for the measure $\nu d\lambda$ ($d\lambda$ being the Lebesgue measure) is a closed subspace of the Hilbert space $L^2(\nu d\lambda)$ of square integrable functions on Ω (see, for example, [PW90]). In complex analysis, ν being admissible, the regularity of the Bergman projection associated to $\nu d\lambda$ (i.e. the orthogonal projection of $L^2(\nu d\lambda)$ onto the subspace of holomorphic functions) is a fundamental question. It has been intensively studied when $\nu \equiv 1$ and especially when Ω is pseudoconvex.

If η is a smooth strictly positive function on $\overline{\Omega}$, it is well known that the regularity properties of the Bergman projections of the Hilbert spaces $L^2(\eta\nu d\lambda)$ and $L^2(\nu d\lambda)$ can be very different. For example in [Koh72] J. J. Kohn proved that if Ω is pseudoconvex, then for any integer k there exists $t > 0$ such that the Bergman projection of $L^2\left(e^{-t|z|^2} d\lambda\right)$ maps the Sobolev space $L^2_k(\Omega)$ into itself, and, in [Bar92] D. Barrett (see also [Chr96]) showed that on certain smooth Diederich-Fornaess worm domains the Bergman projection of $L^2(\Omega) = L^2\left(e^{t|z|^2} e^{-t|z|^2} d\lambda\right)$ is not L^2-Sobolev regular.

In this paper we show that some of the (weighted) estimates obtained in [CDM14] for pseudoconvex domains of finite type remain true when the weight is multiplied by a function which is smooth and strictly positive in $\overline{\Omega}$. This shows that the corresponding estimates, obtained in [CDM14] for the Bergman projection of

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$L^2\left((-\rho_0)^r d\lambda\right)$, where ρ_0 is a special defining function of Ω and r a nonnegative rational number, are valid for the Bergman projection of $L^2\left((-\rho)^r d\lambda\right)$ where ρ is any defining function of the domain. Moreover, for the L^2-Sobolev estimates we extend the results to the weights $(-\rho)^r d\lambda$ with $r \in \mathbb{R}_+$ and we show that these Bergman projections satisfy a stronger directional Sobolev estimate. For L^p-Sobolev and Lipschitz estimates, we extend the results of [CDM14] (for domains whose Levi form have a rank $\geq n-2$) to the weights $(-\rho)^r d\lambda$ only when $r \in \mathbb{Q}_+$.

2. Notation and main results

Throughout the entire paper $d\lambda$ denotes the Lebesgue measure. Let D be a $\mathcal{C}^1$-smoothly bounded open set in $\mathbb{C}^l$. Recall that d is said to be a defining function of D if it is a real function in $\mathcal{C}^1\left(\mathbb{C}^l\right)$ such that $D = \left\{\zeta \in \mathbb{C}^l \text{ s. t. } d(\zeta) < 0\right\}$ and ∇d does not vanish on ∂D.

Let ν be an admissible weight on D.

For $1 \leq p < +\infty$ we denote by $L^p(D, \nu d\lambda)$ the L^p space for the measure $\nu d\lambda$. When $\nu \equiv 1$ we write, as usual, $L^p(D)$.

We denote by P_ν^D the orthogonal projection of the Hilbert space $L^2(D, \nu d\lambda)$ (i.e. for the scalar product $\langle f, g\rangle = \int_D f\overline{g}\nu d\lambda$) onto the closed subspace of holomorphic functions. If $\nu \equiv 1$ we simply write P^D. In this paper, P_ν^D is called the (weighted) *Bergman projection of* $L^2(D, \nu d\lambda)$.

For $k \in \mathbb{N}$ and $1 < p < +\infty$, we define the weighted Sobolev space $L_k^p(D, \nu d\lambda)$ by

$$L_k^p(D, \nu d\lambda) = \Bigg\{ u \in L^p(D, \nu d\lambda) \text{ such that } \|u\|^p_{L_k^p(D,\nu d\lambda)} = \sum_{|\alpha| \leq k} \|D^\alpha u\|^p_{L^p(D,\nu d\lambda)} < +\infty \Bigg\}.$$

If $\nu \equiv 1$ this space is the classical Sobolev space $L_k^p(D)$.

Let ψ be any function in $\mathcal{C}^1(\overline{D})$. We denote by T_ψ the vector field

$$T_\psi = \sum_i \frac{\partial \psi}{\partial \overline{z_i}} \frac{\partial}{\partial z_i} - \frac{\partial \psi}{\partial z_i} \frac{\partial}{\partial \overline{z_i}}.$$

In particular, if d is a defining function of D, T_d is a vector field tangent to d (i.e. $T_d d \equiv 0$) which is transverse to the complex tangent space to d near the boundary of D.

Following terminology introduced in [HMS14] a vector field T with coefficients in $\mathcal{C}^0(\overline{D})$ is said to be *tangential and complex transversal* to $\partial\Omega$ if it can be written $T = aT_d + L$ where $a \in \mathcal{C}^0(\overline{D})$ is nowhere vanishing on ∂D and $L = L_1 + \overline{L_2}$ where L_1 and L_2 are $(1,0)$-type vector fields tangential to ∂D. Note that this definition is independent of the choice of the defining function d (see section 4 before Theorem 4.1). Then, for all nonnegative integers k, and $1 < p < +\infty$, D and T being sufficiently regular, we denote by $L_{k,T}^p(D, \nu d\lambda)$ the weighted directional

Sobolev space

$$L^p_{k,T}(D,\nu d\lambda)=\left\{u\in L^p(D,\nu d\lambda)\text{ such that }\right.$$
$$\left.\|u\|^p_{L^p_{k,T}(D,\nu d\lambda)}=\sum_{l\leq k}\left\|T^l u\right\|^p_{L^p(D,\nu d\lambda)}<+\infty\right\}.$$

Our first result extends Theorem 2.2 of [CDM14] and establishes a weighted directional estimate similar to the one obtained by A.-K. Herbig, J. D. McNeal and E. J. Straube in [HMS14, Theorem 1.1] for the standard Bergman projection[1]:

Theorem 2.1. *Let Ω be a smooth bounded pseudoconvex domain of finite type in $\mathbb{C}^n$. Let ρ be a smooth defining function of Ω. Let $r\in\mathbb{R}_+$ be a nonnegative real number and let $\eta\in\mathcal{C}^\infty(\overline{\Omega})$ be strictly positive. Let T be a $\mathcal{C}^\infty(\overline{\Omega})$ vector field tangential and complex transversal to $\partial\Omega$. Define $\omega=\eta(-\rho)^r$. Then, for any integer k, P^Ω_ω maps continuously the weighted directional Sobolev space $L^2_{k,T}(\Omega,\omega d\lambda)$ into $L^2_k(\Omega,\omega d\lambda)$.*

Note that r is allowed to be 0, and, in that case, $\eta(-\rho)^r$ is not of the form $(-\widehat{\rho})^r$.

Corollary. *In the conditions stated in the theorem, P^Ω_ω continuously maps*

$$\bigcap_{k\in\mathbb{N}}L^2_{k,T}(\Omega,\omega_0 d\lambda)$$

into $\mathcal{C}^\infty(\overline{\Omega})$.

Our second result is inspired by Theorem 1.10 of [HMS14]:

Theorem 2.2. *Let Ω, η and ω be as in Theorem 2.1. Let $f\in L^2(\Omega,\omega d\lambda)$ such that $\overline{f}$ is holomorphic and let $h\in\mathcal{C}^\infty(\overline{\Omega})$. Then $P^\Omega_\omega(fh)\in\mathcal{C}^\infty(\overline{\Omega})$.*

The proofs are done in section 4.

Our other result is a partial generalization of Theorem 2.1 of [CDM14] for domains in $\mathbb{C}^2$, or, more generally, when the rank of the Levi form is $\geq n-2$. It extends the result obtained by A. Bonami and S. Grellier in [BG95]:

Theorem 2.3. *Let Ω, η and ω be as in Theorem 2.1 with $r\in\mathbb{Q}_+$. Assume moreover that, at every point of $\partial\Omega$, the rank of the Levi form is $\geq n-2$. Then:*

(1) *For $1<p<+\infty$ and $k\in\mathbb{N}$, P^Ω_ω continuously maps the Sobolev space $L^p_k(\Omega,\omega d\lambda)$ into itself;*
(2) *For $\alpha<1$, P^Ω_ω continuously maps the Lipschitz space $\Lambda_\alpha(\Omega)$ into itself.*

Theorem 2.3 is proved in section 5 as a special case of a stronger directional L^p_k estimate (Theorem 5.1).

The general scheme of the proofs of these results is as follows. Recall that in [CDM14] we obtain the estimates in the above theorems for the projections $P^\Omega_{\omega_0}$ where $\omega_0=(-\rho_0)^r$, with ρ_0 the following special defining function of Ω: using a celebrated theorem of K. Diederich & J. E. Fornæss ([DF77, Theorem 1]), ρ_0 is

[1]We thank the referee for pointing out that the theorem which was stated originally only for $r\in\mathbb{Q}_+$ could be extended to all $r\in\mathbb{R}_+$ using the results of [Har07] and [Str97].

chosen so that there exists $t \in]0,1[$ such that $-(-\rho_0)^t$ is strictly plurisubharmonic in Ω.

Then we obtain the results for P_ω^Ω comparing P_ω^Ω and $P_{\omega_0}^\Omega$: essentially (see Proposition 3.1 for the exact formula) the difference between $P_\omega^\Omega(u)$ and $P_{\omega_0}^\Omega(u)$ is equal to the solution v, orthogonal to holomorphic functions in $L^2(\Omega, \omega_0 d\lambda)$, of the equation $\overline{\partial} v = P_\omega^\Omega(u)\overline{\partial}\left(\frac{\omega}{\omega_0}\right)$ (note that $\overline{\partial}\left(\frac{\omega}{\omega_0}\right)$ is smooth up to the boundary). Then we prove that the operator giving that solution satisfies some estimates with gain in weighted Sobolev scale and Lipschitz scale (see (1) of the remark at the end of section 4). As there is no regularity theory for the $\overline{\partial}$-equation with such weights in finite type domains, following the ideas developed in [CDM14], we write this solution using the standard $\overline{\partial}$-Neumann problem of a domain in $\mathbb{C}^{n+m}$ which is sufficiently smooth. This leads to some technical difficulties, one of them being solved using directional estimates for the standard Bergman projection.

Remark.

(1) As noted in [CDM14], the restriction $r \geq 0$ is not natural (the natural scale for r should be $]-1,+\infty[$). The restriction $r \in \mathbb{Q}_+$ in Theorem 2.3 is due to the fact that our method uses estimates with gain for solutions of the $\overline{\partial}$-equation in a domain $\widetilde{\Omega}$ in $\mathbb{C}^{n+m}$ ($n+m \geq 3$) whose Levi form is locally diagonalizable, which are proved in [FKM90] and [Koe04] for $\mathcal{C}^\infty$-smooth domains, and, thus, $\widetilde{\Omega}$ must be constructed with a smooth defining function.

(2) Local weighted estimates of P_ω^Ω can also be obtained when only one point of $\partial\Omega$ is supposed to be of finite type. For example, assuming that $z_0 \in \partial\Omega$ is a point of finite type, then $(z_0, 0) \in \partial\widetilde{\Omega}$ is of finite type (see [CDM14, Section 3.4]). Thus local results for the $\overline{\partial}$-Neumann problem of $\widetilde{\Omega}$, based on subelliptic estimates, immediately imply that $P_{\omega_0}^\Omega$ satisfies local weighted L^2-Sobolev estimates near z_0, and the methods developed in the next sections can be adapted to obtain the same estimates for P_ω^Ω.

As ρ and ρ_0 are two smooth defining functions of Ω, there exists a function φ smooth and strictly positive on $\overline{\Omega}$ such that $\rho = \varphi\rho_0$. Then, there exists a function $\eta_1 \in \mathcal{C}^\infty(\overline{\Omega})$, $\eta_1 > 0$, such that $\omega = \eta_1 (-\rho_0)^r$.

Thus, from now on, ρ_0 and ω_0 are fixed as above and, to simplify the notation, we write $\omega = \eta(-\rho_0)^r$ where η is a strictly positive function in $\mathcal{C}^1(\overline{\Omega})$.

3. Comparing P_ω^Ω and $P_{\omega_0}^\Omega$

This comparison is based on the following simple formula (similar formulas have been used, for example, in [BS90] and [BC00]):

Proposition 3.1. *With the previous notation for D and P_ν^D, let η be a strictly positive function in $\mathcal{C}^1\left(\overline{D}\right)$ (so that $\eta\nu$ is an admissible weight). Let $L^2_{(0,1)}(D, \nu d\lambda)$ be the space of $(0,1)$-forms with coefficients in $L^2(D, \nu d\lambda)$. Assume there exists a linear operator from $L^2_{(0,1)}(D, \nu d\lambda) \cap \ker\overline{\partial}$ into $L^2(D, \nu d\lambda)$ solving the $\overline{\partial}$-equation (i.e. the $\overline{\partial}$ problem is solvable in $L^2(D, \nu d\lambda)$) and denote by A_ν the continuous linear operator from $L^2_{(0,1)}(D, \nu d\lambda) \cap \ker\overline{\partial}$ into $L^2(D, \nu d\lambda)$ such that, for $f \in L^2_{(0,1)}(D, \nu d\lambda) \cap \ker\overline{\partial}$, $A_\nu(f)$ is orthogonal to holomorphic functions in $L^2(D, \nu d\lambda)$ and $\overline{\partial} A_\nu(f) = f$. Then, for all $u \in L^2(D, \nu d\lambda)$ we have*

$$\eta P_{\eta\nu}^D(u) = P_\nu^D(\eta u) + A_\nu\left(P_{\eta\nu}^D(u)\overline{\partial}\eta\right).$$

Proof. This is almost immediate: from the second hypothesis on A_ν both sides of the formula have the same $\overline{\partial}$, and, from the first hypothesis, both sides have the same scalar product, in $L^2(D,\nu d\lambda)$, against holomorphic functions. □

We use this formula in the context developed in [CDM14].

Suppose $r>0$. For $h(w)=|w|^{2q}$, $w\in\mathbb{C}^m$, $r={}^m/_q$, q being a real number ≥ 1 or $h(w)=\sum|w_i|^{2q_i}$, $w_i\in\mathbb{C}$, $r=\sum{}^1/_{q_i}$, $q_i\in\mathbb{N}^*$, and ρ_0 and ω_0 as introduced in the preceding section, we consider the domain in $\mathbb{C}^{n+m}$ defined by

$$\widetilde{\Omega}=\left\{(z,w)\in\mathbb{C}^n\times\mathbb{C}^m,\text{ s. t. }\widetilde{\rho}(z,w)=\rho_0(z)+h(w)<0\right\}.$$

Then (cf. [CDM14, Section 3]) $\widetilde{\Omega}$ is smooth ($\mathcal{C}^{[q]}$ in the first case, $\mathcal{C}^\infty$ in the second), bounded and pseudoconvex. Therefore the $\overline{\partial}$-Neumann operator $\mathcal{N}_{\widetilde{\Omega}}$ is well defined. Note that there is no uniqueness for the numbers q and q_i satisfying $r={}^m/_q=\sum{}^1/_{q_i}$: r being fixed, we can choose q or the q_i as large as we want. In the case of domains whose rank of the Levi form is larger than $n-2$, the estimates for $P^\Omega_{\omega_0}$ are obtained in [CDM14] choosing the q_i large enough depending on the type of Ω.

Let us introduce two notation:

- If $u\in L^p(\Omega,\omega_0 d\lambda)$, $1\leq p<+\infty$, we denote by $I(u)$ the function, belonging to $L^p(\widetilde{\Omega})$, defined by $I(u)(z,w)=u(z)$ (the fact that $I(u)\in L^p(\widetilde{\Omega})$ follows Fubini's theorem). We extend this notation to forms $f=\sum f_i d\overline{z}_i$ in $L^p_{(0,1)}(\Omega,\omega_0 d\lambda)$ by $I(f)=\sum I(f_i)\,d\overline{z}_i$ (so that $I(f)\in L^p_{(0,1)}(\widetilde{\Omega})$ and $I(f)$ is $\overline{\partial}$-closed if f is so).
- If $v\in L^p(\widetilde{\Omega})$, $1\leq p<+\infty$, is holomorphic in w we denote by $R(v)$ the function, belonging to $L^p(\Omega,\omega_0 d\lambda)$ (by the mean value property applied to the subharmonic function $w\mapsto|v(z,w)|^p$), defined by $R(v)(z)=v(z,0)$.

If $r=0$, then we choose $\widetilde{\Omega}=\Omega$, and I and R are the identity operator.

Then:

Proposition 3.2. *For any function $u\in L^2(\Omega,\omega d\lambda)$, we have*

$$\eta P^\Omega_\omega(u)=P^\Omega_{\omega_0}(\eta u)+R\circ\left(\overline{\partial}^*\mathcal{N}_{\widetilde{\Omega}}\right)\circ I\left(P^\Omega_\omega(u)\overline{\partial}(\eta)\right).\tag{3.1}$$

Proof. By the preceding proposition, it suffices to note that the operator $R\circ\left(\overline{\partial}^*\mathcal{N}_{\widetilde{\Omega}}\right)\circ I$ is continuous from $L^2_{(0,1)}(\Omega,\omega_0 d\lambda)\cap\ker\overline{\partial}$ into $L^2(\Omega,\omega_0 d\lambda)$, solves the $\overline{\partial}$-equation and gives the solution which is orthogonal to holomorphic functions in that space. But if $f\in L^2_{(0,1)}(\Omega,\omega_0 d\lambda)\cap\ker\overline{\partial}$, then, by Fubini's theorem, $I(f)\in L^2_{(0,1)}(\widetilde{\Omega})\cap\ker\overline{\partial}$, and $\left(\overline{\partial}^*\mathcal{N}_{\widetilde{\Omega}}\right)\circ I(f)$ is the solution of $\overline{\partial}u=I(f)$ which is orthogonal to holomorphic functions in $L^2(\widetilde{\Omega})$ and satisfies

$$\left\|\left(\overline{\partial}^*\mathcal{N}_{\widetilde{\Omega}}\right)\circ I(f)\right\|_{L^2(\widetilde{\Omega})}\lesssim\|I(f)\|_{L^2_{(0,1)}(\widetilde{\Omega})}=C\,\|f\|_{L^2_{(0,1)}(\Omega,\omega_0 d\lambda)}$$

(recall that $\widetilde{\Omega}$ is pseudoconvex and that the volume of $\{h(w)<-\rho_0(z)\}$ is equal to $C\omega_0(z)$). As $I(f)$ is independent of the variable w, $\left(\overline{\partial}^*\mathcal{N}_{\widetilde{\Omega}}\right)\circ I(f)$ is holomorphic in w and

$$\overline{\partial}_z\left(\left(\overline{\partial}^*\mathcal{N}_{\widetilde{\Omega}}\right)\circ I(f)\right)(z,0)=f(z)$$

so $\overline{\partial}\left(R\circ\left(\overline{\partial}^{*}\mathcal{N}_{\widetilde{\Omega}}\right)\circ I(f)\right)=f$, and, by the mean value property (applied to the subharmonic function $w\mapsto\left|\left(\overline{\partial}^{*}\mathcal{N}_{\widetilde{\Omega}}\right)\circ I(f)(z,w)\right|^{2}$),

$$\left\|R\circ\left(\overline{\partial}^{*}\mathcal{N}_{\widetilde{\Omega}}\right)\circ I(f)\right\|_{L^{2}(\Omega,\omega_0 d\lambda)}\leq C\left\|\left(\overline{\partial}^{*}\mathcal{N}_{\widetilde{\Omega}}\right)\circ I(f)\right\|_{L^{2}(\widetilde{\Omega})}\lesssim\|f\|_{L^{2}_{(0,1)}(\Omega,\omega_0 d\lambda)}.$$

Moreover, if g is a holomorphic function in $L^{2}(\Omega,\omega_0 d\lambda)$,

$$\begin{aligned}\int_{\Omega}R\circ\left(\overline{\partial}^{*}\mathcal{N}_{\widetilde{\Omega}}\right)\circ I(f)\overline{g}\omega_0 d\lambda &= \int_{\Omega}\left(\overline{\partial}^{*}\mathcal{N}_{\widetilde{\Omega}}\right)\circ I(f)(z,0)\overline{g}(z)\omega_0(z)d\lambda(z)\\ &= C\int_{\Omega}\left(\int_{\{h(w)<-\rho_0(z)\}}\left(\overline{\partial}^{*}\mathcal{N}_{\widetilde{\Omega}}\right)\right.\\ &\qquad\left.\circ I(f)(z,w)d\lambda(w)\right)\overline{g}(z)d\lambda(z)\\ &= C\int_{\widetilde{\Omega}}\left(\overline{\partial}^{*}\mathcal{N}_{\widetilde{\Omega}}\right)\circ I(f)(z,w)\overline{g}(z)d\lambda(z,w)=0.\end{aligned}$$

□

An immediate density argument shows that:

Corollary. *Let $p\in\,]1,+\infty[$. Assume that the following properties are satisfied:*

- P^{Ω}_{ω} *and* $P^{\Omega}_{\omega_0}$ *continuously map* $L^{p}(\Omega,\omega d\lambda)$ *into itself;*
- $\overline{\partial}\mathcal{N}_{\widetilde{\Omega}}$ *continuously maps* $L^{p}(\widetilde{\Omega})$ *into itself.*

Then equation (3.1) *is valid for any function* $u\in L^{p}(\Omega,\omega d\lambda)$.

In the proofs of the theorems we need to use weighted Sobolev spaces $L^{p}_{s}(D,\nu d\lambda)$ defined for all $s\geq 0$ and some directional Sobolev spaces. For each definition we suppose D sufficiently regular.

It is well known that for $s\in[k,k+1]$, $k\in\mathbb{N}$, the fractional Sobolev space $L^{p}_{s}(D)$ is obtained using the complex interpolation method between $L^{p}_{k}(D)$ and $L^{p}_{k+1}(D)$ (see, for example, [Tri78]). By analogy, we extend the definition of the weighted Sobolev spaces $L^{p}_{k}(D,\nu d\lambda)$ to any index $s\geq 0$ using the complex interpolation method:

$$L^{p}_{s}(D,\nu d\lambda)=\left[L^{p}_{k}(D,\nu d\lambda),L^{p}_{k+1}(D,\nu d\lambda)\right]_{s-k},\text{ if } s\in[k,k+1].$$

Note that if ν_1 and ν_2 are two admissible weights such that $\nu_2=\eta\nu_1$ with η a strictly positive function in $\mathcal{C}^{[s]+1}\left(\overline{\Omega}\right)$, then the Banach spaces $L^{p}_{s}(D,\nu_1 d\lambda)$ and $L^{p}_{s}(D,\nu_2 d\lambda)$ are identical.

For all $s\geq 0$, we extend the definition of $L^{p}_{k,T}(D,\nu d\lambda)$ to $L^{p}_{s,T}(D,\nu d\lambda)=\left[L^{p}_{k,T}(D,\nu d\lambda),L^{p}_{k+1,T}(D,\nu d\lambda)\right]_{s-k}$, $k\leq s\leq k+1$, by complex interpolation between two consecutive integers. Clearly, the spaces $L^{2}_{s,T}(D,\nu d\lambda)$ are Hilbert spaces and $L^{p}_{s,T}(D,\nu d\lambda)$ are Banach spaces. When $\nu\equiv 1$ we denote this space $L^{p}_{s,T}(D)$.

Note that, $\widetilde{\rho}=\rho_0+h$ being the defining function of $\widetilde{\Omega}$, we have $T_{\widetilde{\rho}}=T_{\rho_0}+T_h$, with $T_h=q\,|w|^{2q-2}\sum\left(w_i\frac{\partial}{\partial w_i}-\overline{w_i}\frac{\partial}{\partial\overline{w_i}}\right)$ when $h(w)=|w|^{2q}$, $w\in\mathbb{C}^{m}$, and $T_h=\sum_{i=1}^{m}q_i\,|w_i|^{2q_i-2}\left(w_i\frac{\partial}{\partial w_i}-\overline{w_i}\frac{\partial}{\partial\overline{w_i}}\right)$ when $h(w)=\sum_{i=1}^{m}|w_i|^{2q_i}$, $w_i\in\mathbb{C}$.

Remark 3.1. The spaces $L^p_{s,T}(D,\nu d\lambda)$ depend on the choice of the vector field T (see Section 5 of [HM12]).

We now state some elementary properties of the operators I and R introduced before and related to these Sobolev spaces. It is convenient to introduce other spaces: for $1<p<+\infty$ and $s\geq 0$, let

$$L^p_s(\widetilde{\Omega})\cap\ker\overline{\partial}_w=\left\{u(z,w)\in L^p_s(\widetilde{\Omega})\text{ such that }\frac{\partial u}{\partial\overline{w_i}}\equiv 0,\ 1\leq i\leq m\right\}.$$

Lemma 3.1. *With the previous notation and for $1<p<+\infty$, we have:*

(1) *For all $s\geq 0$, I continuously maps $L^p_s(\Omega,\omega_0 d\lambda)$ into $L^p_s(\widetilde{\Omega})$.*
(2) *For all nonnegative integer k, R continuously maps $L^p_k(\widetilde{\Omega})\cap\ker\overline{\partial}_w$ into $L^p_k(\Omega,\omega_0 d\lambda)$.*
(3) *For all $s\geq 0$, if $q\geq[s]+2$, I continuously maps $L^p_{s,T_{\rho_0}}(\Omega,\omega_0 d\lambda)$ into $L^p_{s,T_{\widetilde{\rho}}}(\widetilde{\Omega})$.*
(4) *For all $s\geq 0$, if $q\geq[s]+2$, R continuously maps $L^p_s(\widetilde{\Omega})\cap\ker\overline{\partial}_w$ into $L^p_{s,T_{\rho_0}}(\Omega,\omega_0 d\lambda)$.*

Proof. As $D^\alpha_z I(h)=I(D^\alpha h)$ for any derivative D^α, Fubini's Theorem implies $\|D^\alpha_z I(h)\|^p_{L^p(\widetilde{\Omega})}=C\|D^\alpha h\|^p_{L^p(\Omega,\omega_0 d\lambda)}$ and (1) follows for $s\in\mathbb{N}$ and for all s by the interpolation theorem.

The second point of the lemma is also very simple. If $u\in L^p_k(\widetilde{\Omega})\cap\ker\overline{\partial}_w$, then, for all derivative D^α, $D^\alpha(Ru)(z)=D^\alpha_z u(z,0)$, and $w\mapsto|D^\alpha_z u(z,w)|^p$ is subharmonic. Therefore the mean value property gives

$$C\left|D^\alpha(Ru)(z)\right|^p\omega_0(z)\leq\int_{\{h(w)<-\rho_0(z)\}}\left|D^\alpha_z u(z,w)\right|^p d\lambda(w).$$

Integrating this inequality over Ω gives the result.

As $T^l_{\widetilde{\rho}}(I(h))=I\left(T^l_{\rho_0}(h)\right)$, Fubini's Theorem gives (3) when s is an integer. Therefore (3) follows by interpolation.

To see (4), let us denote by M_0u the mean with respect to the variable w of a function u in $L^p(\widetilde{\Omega})$:

$$M_0u(z)=\frac{1}{C\omega_0(z)}\int_{\{h(w)<-\rho_0(z)\}}u(z,w)d\lambda(w).$$

As T_{ρ_0} is tangent to ρ_0, we have $T_{\rho_0}(\omega_0)\equiv T_{\rho_0}(\rho_0)\equiv 0$, and, for all integer l we get

$$T^l_{\rho_0}M_0u(z)=\frac{1}{C\omega_0(z)}\int_{\{h(w)<-\rho_0(z)\}}T^l_{\rho_0}u(z,w)d\lambda(w).$$

Then, by Hölder's inequality we have

$$C\left|T^l_{\rho_0}M_0u(z)\right|^p\omega_0(z)\leq\int_{\{h(w)<-\rho_0(z)\}}\left|T^l_{\rho_0}u(z,w)\right|^p d\lambda$$

and, integrating this inequality over Ω, we get that M_0 continuously maps $L^p_k(\widetilde{\Omega})$ into $L^p_{k,T_{\rho_0}}(\Omega,\omega_0 d\lambda)$. Therefore, by the interpolation theorem, M_0 continuously maps $L^p_s(\widetilde{\Omega})$ into $L^p_{s,T_{\rho_0}}(\Omega,\omega_0 d\lambda)$ for all $s\geq 0$.

This proves (4) of the lemma because, by the mean value property for holomorphic functions, $M_0u=Ru$ when $u\in L^p_s(\widetilde{\Omega})\cap\ker\overline{\partial}_w$. $\square$

4. Proof of Theorems 2.1 and 2.2

For $r > 0$, we choose $h(w) = |w|^{2q}$ with $q \in \mathbb{R}_+$, $q \geq 1$. Then $\partial\Omega$ is only $\mathcal{C}^{[q]}$-regular and we cannot use classical L^2-Sobolev estimates with gain for the $\overline{\partial}$-Neumann problem of $\widetilde{\Omega}$. Thus, we first show that the estimates we need are valid using the results of [Har07] and [Str97].

Recall that ρ_0 is chosen so that there exists $t \in]0,1]$ such that $\rho_1 = -(-\rho_0)^t$ is strictly plurisubharmonic in Ω. Moreover, the proof of [DF77, Theorem 1] shows that ρ_0 can be chosen so that

$$i\partial\overline{\partial}\rho_1 \geq c_1 (-\rho_0)^t \, i\partial\overline{\partial} |z|^2, \text{ in } \Omega,$$

where c_1 is a positive constant depending on Ω.

Lemma 4.1. *Assume that $q \geq 2$ and $qt \geq 1$. Then there exist two positive constants c and β and a plurisubharmonic function $\lambda \in \mathcal{C}(\overline{\widetilde{\Omega}}) \cap \mathcal{C}^2(\widetilde{\Omega})$ such that:*

(1) $0 \leq \lambda \leq 1$.
(2) *In a neighborhood of $\partial\widetilde{\Omega}$ in $\widetilde{\Omega}$, $i\partial\overline{\partial}\lambda \geq c\delta_{\partial\widetilde{\Omega}}^{-\beta} i\partial\overline{\partial}\left(|z|^2 + |w|^2\right)$, where $\delta_{\partial\widetilde{\Omega}}$ denotes the distance to the boundary of $\widetilde{\Omega}$.*

Proof. It is enough (see [Str97, Section 4]) to prove that there exist positive constants C, c and β such that, for all $0 < \delta \leq \delta_0$, δ_0 small, there exists a plurisubharmonic function $\lambda_\delta \in \mathcal{C}(\overline{\Omega})$ such that

$$\begin{gathered} 0 \leq \lambda_\delta \leq C, \text{ and} \\ i\partial\overline{\partial}\lambda_\delta \geq c\delta^{-\beta} i\partial\overline{\partial}\left(|z|^2 + |w|^2\right), \text{ on the set } -\delta \leq \rho_0(z) + |w|^{2q} < 0. \end{gathered}$$

Since Ω is a smooth pseudoconvex domain of finite type, by the main result of [Cat87] (and [Str97, Section 4]) there exist two positive constants c_0 and β_0 such that there exists a plurisubharmonic function $\lambda_\Omega \in \mathcal{C}^\infty(\overline{\Omega})$ such that $0 \leq \lambda_\Omega \leq 1$ and $i\partial\overline{\partial}\lambda_\Omega \geq c_0 (-\rho_0)^{-\beta_0} i\partial\overline{\partial} |z|^2$ in a neighborhood U_0 of $\partial\Omega$ in $\overline{\Omega}$.

For K and β two positive constants to be chosen later, let us define

$$\begin{aligned} \lambda_\delta &= \lambda_\Omega(z) + \exp\left(\frac{\rho_1(z) + |w|^{2qt}}{\delta^t}\right)\left(K - \exp\left(-\frac{|w|^2}{\delta^\beta}\right)\right) \\ &= \lambda_\Omega(z) + \lambda_\delta^1(z,w)\left(K - \lambda_\delta^2(z,w)\right). \end{aligned}$$

Clearly, in $\overline{\widetilde{\Omega}}$, $0 \leq \lambda_\delta \leq K+1$. Let us evaluate $i\partial\overline{\partial}\lambda_\delta$ in $\widetilde{\Omega}$.

$$\begin{aligned}\left(K-\lambda_\delta^2\right) i\partial\overline{\partial}\lambda_\delta^1 &= \lambda_\delta^1\left(K-\lambda_\delta^2\right)\left[\frac{i\partial\overline{\partial}\rho_1}{\delta^t}+\frac{i\partial\overline{\partial}\left|w\right|^{2qt}}{\delta^t}\right.\\ &\qquad\left.+\frac{\partial\left(\rho_1+\left|w\right|^{2qt}\right)\wedge\overline{\partial}\left(\rho_1+\left|w\right|^{2qt}\right)}{\delta^{2t}}\right]\\ &\geq (K-1)\lambda_\delta^1\left[\frac{\left|w\right|^{2qt-2}i\partial\overline{\partial}\left|w\right|^2}{\delta^t}\right.\\ &\qquad\left.+\frac{\partial\left(\rho_1+\left|w\right|^{2qt}\right)\wedge\overline{\partial}\left(\rho_1+\left|w\right|^{2qt}\right)}{\delta^{2t}}\right]\end{aligned}$$

and

$$-\lambda_\delta^1 i\partial\overline{\partial}\lambda_\delta^2 \geq \lambda_\delta^1\lambda_\delta^2\left(\frac{1}{\delta^\beta}-\frac{\left|w\right|^2}{\delta^{2\beta}}\right)i\partial\overline{\partial}\left|w\right|^2$$

and

$$\begin{aligned}2\left|\partial\lambda_\delta^1\wedge\overline{\partial}\lambda_\delta^2\right| &= 2\lambda_\delta^1\lambda_\delta^2\left|\frac{\partial\left(\rho_1+\left|w\right|^{2qt}\right)}{\delta^t}\wedge\frac{\overline{\partial}\left|w\right|^2}{\delta^\beta}\right|\\ &\leq \lambda_\delta^1\lambda_\delta^2\left[(K-2)\frac{\partial\left(\rho_1+\left|w\right|^{2qt}\right)\wedge\overline{\partial}\left(\rho_1+\left|w\right|^{2qt}\right)}{\delta^{2t}}\right.\\ &\qquad\left.+\frac{4}{K-2}\frac{\left|w\right|^2}{\delta^{2\beta}}i\partial\overline{\partial}\left|w\right|^2\right].\end{aligned}$$

If $z\notin U_0$, then there exists a positive constant c_2 such that $\lambda_\delta^1\left(K-\lambda_\delta^2\right)\frac{i\partial\overline{\partial}\rho_1}{\delta^t}\geq c_2\lambda_\delta^1\left(K-\lambda_\delta^2\right)\frac{i\partial\overline{\partial}|z|^2}{\delta^t}$, and if $z\in U_0$

$$i\partial\overline{\partial}\lambda_\Omega+\lambda_\delta^1\left(K-\lambda_\delta^2\right)\frac{i\partial\overline{\partial}\rho_1}{\delta^t}\geq\left[c_0\left(-\rho_0\right)^{-\beta_0}+\lambda_\delta^1\left(K-\lambda_\delta^2\right)\frac{\left(-\rho_0\right)^t}{\delta^t}\right]i\partial\overline{\partial}\left|z\right|^2,$$

so that $i\partial\overline{\partial}\lambda_\Omega+\lambda_\delta^1\left(K-\lambda_\delta^2\right)\frac{i\partial\overline{\partial}\rho_1}{\delta^t}\geq 0$ in $\widetilde{\Omega}$ and there exist two positive constants c_3 and $\beta_1=\frac{\beta_0 t}{2}$ such that, if $-\delta\leq\rho_0(z)+\left|w\right|^{2q}<0$ (so that $\lambda_\delta^1\geq e^{-1}$),

$$i\partial\overline{\partial}\lambda_\Omega+\lambda_\delta^1\left(K-\lambda_\delta^2\right)\frac{i\partial\overline{\partial}\rho_1}{\delta^t}\geq\frac{c_3}{\delta^{\beta_1}}i\partial\overline{\partial}\left|z\right|^2.$$

Thus, assuming $K\geq 10$, if $\left|w\right|^2<\frac{\delta^\beta}{2}$, then $i\partial\overline{\partial}\lambda_\delta\geq 0$ in $\widetilde{\Omega}$, and, if, in addition, $-\delta\leq\rho_0(z)+\left|w\right|^{2q}<0$, there exists a constant c_4 such that

$$i\partial\overline{\partial}\lambda_\delta\geq\frac{c_3}{\delta^{\beta_1}}i\partial\overline{\partial}\left|z\right|^2+\frac{c_4}{\delta^\beta}i\partial\overline{\partial}\left|w\right|^2.$$

To conclude the proof assume $|w|^2 \geq \frac{\delta^\beta}{2}$ and write $|w|^2 = \tau\delta^\beta$, with $1/2 \leq \tau \leq L < +\infty$. Then

$$i\partial\overline{\partial}\lambda_\delta \geq i\partial\overline{\partial}\lambda_\Omega + \lambda_\delta^1 \left(K - \lambda_\delta^2\right) \frac{i\partial\overline{\partial}\rho_1}{\delta^t} + \frac{\lambda_\delta^1}{\delta^\beta} \left\{ (K-1) \frac{\left(\tau\delta^\beta\right)^{qt}}{\delta^t} \right.$$
$$\left. - \tau e^{-\tau} \left(1 + \frac{4}{K-2}\right) \right\} i\partial\overline{\partial} |w|^2$$

and the proof is finished choosing $\beta < 1/2q$ and $K \geq \max\left(10, 2^{qt+1}+1\right)$. ☐

This lemma and the results of [Str97] and [Har07] give the estimates we need for the $\overline{\partial}$-Neumann problem on $\widetilde{\Omega}$:

Lemma 4.2. *Let k be a positive integer. Assume $q \in \mathbb{R}_+$, $q \geq k+2$ and $qt \geq 1$. Then there exists $\varepsilon > 0$ such that the $\overline{\partial}$-Neumann problem on $\widetilde{\Omega}$ satisfies the following estimates: for $s \in [0,k]$,*

1. *$\overline{\partial}^* \mathcal{N}_{\widetilde{\Omega}}$ continuously maps the space of $(0,1)$-forms having coefficients in $L^2_s(\widetilde{\Omega})$ into $L^2_{s+\varepsilon}(\widetilde{\Omega})$.*
2. *The Bergman projection $P_{\widetilde{\Omega}}$ continuously maps the space $L^2_s(\widetilde{\Omega})$ into itself.*

If $T = aT_\rho + L$ is the vector field given in Theorem 2.1, then (writing $\rho = \varphi\rho_0$) we have $T = a\varphi T_{\rho_0} + (a\rho_0 T_\varphi + L) = a\varphi T_{\rho_0} + L'$ with $\varphi > 0$ on $\overline{\Omega}$ and $L' = L'_1 + \overline{L'_2}$ where L'_1 and L'_2 are $(1,0)$-type vector fields tangential to $\partial\Omega$. Moreover, writing $a = a' + b$ where a' is nowhere vanishing on $\overline{\Omega}$ and b is identically 0 in a neighborhood of $\partial\Omega$, we get $T = a'\varphi T_{\rho_0} + L''$ with $L'' = L''_1 + \overline{L''_2}$ where L''_1 and L''_2 are $(1,0)$-type vector fields tangential to $\partial\Omega$ and $a'\varphi$ is nowhere vanishing on $\overline{\Omega}$.

We now prove the following reformulation of Theorem 2.1:

Theorem 4.1. *Let Ω be as in Theorem 2.1. Let k be a nonnegative integer. Let ρ_0, ω_0 and ω be as at the end of section 2 with $\eta \in \mathcal{C}^{k+1}(\overline{\Omega})$ and $r \in \mathbb{R}_+$. Let $\varphi \in \mathcal{C}^\infty(\overline{\Omega})$ be a function which is nowhere vanishing on $\overline{\Omega}$ and let $T = \varphi T_{\rho_0} + L$ with $L = L_1 + \overline{L_2}$ where L_1 and L_2 are $\mathcal{C}^\infty(\overline{\Omega})$ vector fields of type $(1,0)$ tangential to $\partial\Omega$. Then for $s \in [0,k]$, the weighted Bergman projection P^Ω_ω continuously maps the directional weighted Sobolev space $L^2_{s,T}(\Omega, \omega_0 d\lambda)$ into $L^2_s(\Omega, \omega_0 d\lambda)$.*

Proof. The case $r = 0$ is a direct application of Proposition 3.2 and [HMS14, Theorem 1.1] so we assume $r > 0$. By results of [CDM14], $\widetilde{\Omega}$ is a $\mathcal{C}^{[q]}$-smoothly bounded pseudoconvex domain in $\mathbb{C}^{n+m}$. First we note that the estimate of the theorem for $P^\Omega_{\omega_0}$ is a consequence of a theorem of A.-K. Herbig, J. D. McNeal and E. Straube:

Lemma 4.3. *The Bergman projection $P^\Omega_{\omega_0}$ continuously maps the directional space $L^2_{s,T}(\Omega, \omega_0 d\lambda)$ into $L^2_s(\Omega, \omega_0 d\lambda)$.*

Proof of the lemma. According to [CDM14], Section 4, we have

$$P^\Omega_{\omega_0} = R \circ P^{\widetilde{\Omega}} \circ I,$$

where $P^{\widetilde{\Omega}}$ is the standard Bergman projection of $\widetilde{\Omega}$.

Lemma 4.4. *There exists a vector field $W = \sum a_i \frac{\partial}{\partial w_i} + b_i \frac{\partial}{\partial \overline{w_i}}$ with coefficients a_i and b_i in $\mathcal{C}^{[q]-1}(\overline{\widetilde{\Omega}})$ such that $T + W$ is $\mathcal{C}^{[q]-1}$-smooth in $\overline{\widetilde{\Omega}}$, tangential and complex transversal to $\partial\widetilde{\Omega}$.*

Proof of Lemma 4.4. This a very simple calculus. $T+W = \varphi T_{\rho_0} + L + W = \varphi T_{\widetilde{\rho}} - \varphi T_{|w|^{2q}} + L + W$, where $\widetilde{\rho}$ denotes the defining function of $\widetilde{\Omega}$. As $\varphi T_{\widetilde{\rho}}$ is tangential and complex transversal to $\partial\widetilde{\Omega}$ it is enough to see that the coefficients of W can be chosen so that the $(1,0)$ and $(0,1)$ parts of $-\varphi T_{|w|^{2q}} + L + W$ are both tangential to $\partial\widetilde{\Omega}$. For example, the $(1,0)$ part of this vector field is

$$-q\varphi\,|w|^{2q-2}\sum w_i\frac{\partial}{\partial w_i} + L_1 + \sum a_i\frac{\partial}{\partial w_i},$$

and it is tangent to $\partial\widetilde{\Omega}$ if $q^2\varphi\,|w|^{4q-2} - L_1\rho_0 \equiv \sum q a_i\,|w|^{2q-2}\,\overline{w}_i$ on $\partial\widetilde{\Omega}$. As L_1 is tangential to $\partial\Omega$, $L_1\rho_0$ vanishes at $\partial\Omega$ and there exists a function $\psi_1 \in \mathcal{C}^\infty(\overline{\Omega})$ such that $-L_1\rho_0 = \psi_1(-\rho_0)$. If $(z,w) \in \partial\widetilde{\Omega}$, then $-\rho_0(z) = |w|^{2q}$, and it suffices to choose

$$a_i = \frac{1}{q}w_i\left[q^2\varphi\,|w|^{2q-2} + \psi_1\right].$$

Similarly, the $(0,1)$ part of $-\varphi T_{|w|^{2q}} + L + W$ is tangent to $\partial\widetilde{\Omega}$ choosing $b_i = \frac{1}{q}\overline{w}_i\left[q^2\varphi\,|w|^{2q-2} + \psi_2\right]$. □

Let us now finish the proof of Lemma 4.3. By Lemma 4.2 and Theorem 1.1 of [HMS14], for any nonnegative integer k, if q is chosen sufficiently large (depending on k), $P^{\widetilde{\Omega}}$ continuously maps $L^2_{k,T+W}(\widetilde{\Omega})$ into $L^2_k(\widetilde{\Omega})$. As $(T+W)^l\,(I(u)) = T^l(u)$, for each $u \in L^2_{k,T}(\Omega, \omega_0 d\lambda)$, we have $I(u) \in L^2_{k,T+W}(\widetilde{\Omega})$, and, for $s = k$, the lemma follows (2) of Lemma 3.1. The general case $s \geq 0$ is therefore obtained by interpolation. □

Now we use the formula of Proposition 3.2 to prove Theorem 4.1, by induction, for $s \in \{0,\dots,k\}$, the general case $s \in [0,k]$ being then a consequence of the interpolation theorem. Let us assume the theorem is true for $s-1$, $1 < s \leq k$ and let us prove it for s. Let N be an integer whose inverse is smaller than the ε of Lemma 4.2. Let $u \in L^2_{s,T}(\Omega,\omega_0 d\lambda)$. To prove that $P^\Omega_\omega(u) \in L^2_s(\Omega,\omega_0 d\lambda)$, let us prove, by induction over $l \in \{0,1,\dots,N\}$, that $P^\Omega_\omega(u) \in L^2_{s-1+l/N}(\Omega,\omega_0 d\lambda)$. Assume $P^\Omega_\omega(u) \in L^2_{s-1+l/N}(\Omega,\omega_0 d\lambda)$, $l \leq N-1$. As $\eta \in \mathcal{C}^{k+1}(\overline{\Omega})$, by (1) of Lemma 3.1, $I\left(P^\Omega_\omega(u)\overline{\partial}(\eta)\right) \in L^2_{s-1+l/N}(\widetilde{\Omega})$. By Lemma 4.2,

$$\left(\overline{\partial}^*\mathcal{N}_{\widetilde{\Omega}}\right)\circ I\left(P^\Omega_\omega(u)\overline{\partial}(\eta)\right) \in L^2_{s-1+(l+1)/N}(\widetilde{\Omega}).$$

By (4) of Lemma 3.1,

$$R\circ\left(\overline{\partial}^*\mathcal{N}_{\widetilde{\Omega}}\right)\circ I\left(P^\Omega_\omega(u)\overline{\partial}(\eta)\right) \in L^2_{s-1+(l+1)/N,T_{\rho_0}}(\Omega,\omega_0 d\lambda).$$

By Lemma 4.3 $P^\Omega_{\omega_0}(\eta u) \in L^2_s(\Omega,\omega_0 d\lambda)$, thus, as $\eta^{-1} \in \mathcal{C}^{k+1}(\overline{\Omega})$, Proposition 3.2 gives

$$P^\Omega_\omega(u) \in L^2_{s-1+(l+1)/N,T_{\rho_0}}(\Omega,\omega_0 d\lambda),$$

and, as $P^\Omega_{\omega_0}\circ P^\Omega_\omega = P^\Omega_\omega$, Lemma 4.3 implies $P^\Omega_\omega(u) \in L^2_{s-1+(l+1)/N}(\Omega,\omega_0 d\lambda)$ finishing the proof. □

Proof of the corollary of Theorem 2.1. It is enough to see that, if l_r is a positive integer such that $2l_r \geq r$, then, for any integer $k \geq l_r + 1$, for $u \in L^2_{k,T}(\Omega,\omega_0 d\lambda)$ we have $P^\Omega_\omega(u) \in L^2_{k-l_r}(\Omega)$. But this is a consequence of the theorem and of Theorem

1.1 of [CK03] (see also [Det81]): $P_\omega^\Omega(u) \in L_k^2(\Omega, \omega_0 d\lambda) \subset L_k^2\left(\Omega, \delta_{\partial\Omega}^{2l_r} d\lambda\right)$, $\delta_{\partial\Omega}$ being the distance to the boundary of Ω, and a harmonic function in $L_k^2\left(\Omega, \delta_{\partial\Omega}^{2l_r} d\lambda\right)$ is in $L_{k-l_r}^2(\Omega)$. □

Remark.

(1) The proof of Theorem 4.1 shows that the operator

$$A_\omega = \left(\mathrm{Id} - P_\omega^\Omega\right) \circ \left(R \circ \overline{\partial}\mathcal{N}_{\widetilde{\Omega}} \circ I\right)$$

which solves the equation $\overline{\partial} v = f$ for f a $\overline{\partial}$-close $(0,1)$-form in $L^2(\Omega, \omega d\lambda)$ and gives the solution orthogonal to holomorphic functions, satisfies the following "weak" L^2-Sobolev estimates:
There exists $\vartheta > 0$ (depending on the type of Ω and on ω) such that, for all $s \geq 0$,

$$\|A_\omega(f)\|_{L^2_{s+\vartheta, T_{\rho_0}}(\Omega, \omega d\lambda)} \lesssim \|f\|_{L^2_s(\Omega, \omega d\lambda)}.$$

(2) As noted in [HM12], $\bigcap_{k\in\mathbb{N}} L_{k,T}^2(\Omega, \omega_0 d\lambda)$ is, in general, strictly larger than $\mathcal{C}^\infty(\overline{\Omega})$.

(3) In Remark 4.1 (2) of [CDM14] we notice that, if Ω is a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$ (not assumed of finite type) admitting a defining function ρ_1 plurisubharmonic in Ω then, by a result of H. Boas and E. Straube ([BS91]) and the proof of Lemma 4.3, the weighted Bergman projection $P_{\omega_1}^\Omega$, $\omega_1 = (-\rho_1)^r$, $r \in \mathbb{Q}_+$, continuously maps the directional Sobolev space $L_{s,T}^2(\Omega, \omega_1 d\lambda)$ into $L_s^2(\Omega, \omega_1 d\lambda)$, $s \geq 0$. Moreover, the arguments of the proof of the above corollary show that $P_{\omega_1}^\Omega$ continuously maps

$$\bigcap_{k\in\mathbb{N}} L_{k,T}^2(\Omega, \omega_1 d\lambda)$$

into $\mathcal{C}^\infty(\overline{\Omega})$.
Similarly, $P_{\omega_0}^\Omega$ satisfies these estimates under a good hypothesis on the $\overline{\partial}$-Neuman problem on Ω (for example if Catlin's property (P) holds).
But, as there is no gain in the Sobolev scale for the $\overline{\partial}$-Neuman problem on $\widetilde{\Omega}$ under those hypotheses, if ρ is another defining function of such a domain, our method does not give the regularity of P_ν^Ω, $\nu = (-\rho)^r$.

(4) As indicated to us by the referee, if the $\overline{\partial}$-Neuman problem on $\widetilde{\Omega}$ admits a compactness estimate (and $\partial\widetilde{\Omega}$ is sufficiently regular), then, for all nonnegative integers k and all $\varepsilon > 0$ there exists a positive constant $\widetilde{C}_{k,\varepsilon}$ such that

$$\begin{aligned}\left\|\left(\overline{\partial}^*\mathcal{N}_{\widetilde{\Omega}}\right) \circ I\left(P_\omega^\Omega(u)\overline{\partial}(\eta)\right)\right\|_{L_k^2(\widetilde{\Omega})} &\leq \varepsilon\left\|I\left(P_\omega^\Omega(u)\overline{\partial}(\eta)\right)\right\|_{L_k^2(\widetilde{\Omega})} \\ &\quad + \widetilde{C}_{k,\varepsilon}\left\|I\left(P_\omega^\Omega(u)\overline{\partial}(\eta)\right)\right\|_{L^2(\widetilde{\Omega})},\end{aligned}$$

and formula (3.1) combined with Lemma 4.3 gives

$$\left\|P_\omega^\Omega(u)\right\|_{L_k^2(\Omega, \omega_0 d\lambda)} \leq \varepsilon\left\|P_\omega^\Omega(u)\right\|_{L_k^2(\Omega, \omega_0 d\lambda)} + C_{k,\varepsilon}\|u\|_{L_{k,T}^2(\Omega, \omega_0 d\lambda)}.$$

Thus, if we can prove directly that $\left\|P_\omega^\Omega(u)\right\|_{L_k^2(\Omega, \omega_0 d\lambda)} < +\infty$ for all $u \in \mathcal{C}^\infty(\overline{\Omega})$, then Theorem 2.1 follows immediately. Without weights this kind of estimate is classically obtained using an elliptic regularization. It is a completely different approach of the problem to try to adapt such regularization to the context of weighted spaces and we will not discuss it here.

Proof of Theorem 2.2. The proof is very similar to the proof of Theorem 4.1. First, for each integer k, choosing q sufficiently large, Theorem 1.10 of [HMS14] implies that $P^{\widetilde{\Omega}}(I(\eta f h)) \in \mathcal{C}^k(\overline{\widetilde{\Omega}})$. Thus $P^{\Omega}_{\omega_0}(\eta f h) \in \mathcal{C}^\infty(\overline{\Omega})$. Then, by induction, the arguments used in the proof of Theorem 4.1 show that, for all nonnegative integers k, $P^{\Omega}_{\omega}(fh) \in L^2_k(\Omega, \omega d\lambda)$. Then, arguing as in the proof of the corollary of Theorem 2.1 we conclude that $P^{\Omega}_{\omega}(fh) \in L^2_{k-l_r}(\Omega)$ which completes the proof. $\square$

5. Proof of Theorem 2.3

The proof, based on the formula of Proposition 3.2 and on estimates for solutions of the $\overline{\partial}$-equation, is very similar to the one given in the previous section.

As in the preceding section we obtain the Sobolev estimates of Theorem 2.3 proving a stronger directional estimate:

Theorem 5.1. *Let Ω be as Theorem 2.3. Let ρ_0, ω_0 and ω be as at the end of section 2 with $r \in \mathbb{Q}_+$. Then:*

1. *Let k be a nonnegative integer. Assume $\eta \in \mathcal{C}^{k+1}(\overline{\Omega})$. Then, for $1 < p < +\infty$ and $s \in [0, k]$ the weighted Bergman projection P^{Ω}_{ω} continuously maps the directional weighted Sobolev space $L^p_{s,T_{\rho_0}}(\Omega, \omega_0 d\lambda)$ into $L^p_s(\Omega, \omega_0 d\lambda)$.*
2. *Let $\alpha \leq 1$. Assume $\eta \in \mathcal{C}^{[\alpha]+1}(\overline{\Omega})$. Then the weighted Bergman projection P^{Ω}_{ω} continuously maps the Lipschitz space $\Lambda_\alpha(\Omega)$ into itself.*

Choosing $h(w) = \sum |w_i|^{2q_i}$, $w_i \in \mathbb{C}$, $\sum 1/q_i = r$, we know ([CDM14, Theorem 3.2]) that the Levi form of the domain $\widetilde{\Omega}$ is locally diagonalizable at every point of $\partial\widetilde{\Omega}$. Thus we use the estimates for the $\overline{\partial}$-Neumann problem obtained by C. L. Fefferman, J. J. Kohn and M. Machedon in 1990 and by K. Koenig in 2004 for these domains:

Theorem 5.2 ([FKM90, Koe04])**.** *Let D be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$ of finite type whose Levi form is locally diagonalizable at every boundary point. Then there exists a positive integer N such that:*

1. *For every $\alpha \geq 0$, $\overline{\partial}^* \mathcal{N}_D$ continuously maps the Lipschitz space $\Lambda_\alpha(D)$ into $\Lambda_{\alpha+1/N}(D)$.*
2. *For $1 < p < +\infty$ and $s \geq 0$, $\overline{\partial}^* \mathcal{N}_D$ continuously maps the Sobolev space $L^p_s(D)$ into $L^p_{s+1/N}(D)$.*
3. *For $1 < p < +\infty$, $\overline{\partial}^* \mathcal{N}_D$ continuously maps $L^p(D)$ into $L^{p+1/N}(D)$.*
4. *For p sufficiently large $\overline{\partial}^* \mathcal{N}_D$ continuously maps $L^p(D)$ into $\Lambda_0(D)$.*

The first statement is explicitly stated in [FKM90], for N strictly larger than the type of D, for the $\overline{\partial}_b$-Neumann problem at the boundary, and exactly stated in [Koe04] (Corollary 6.3, p. 286). In [Koe04] it is also proved that $\overline{\partial}^* \mathcal{N}_D$ continuously maps the Sobolev space $L^p_s(D)$ into $L^p_{s+1/m-\varepsilon}(D)$, where m is the type of D and $\varepsilon > 0$. Therefore the third and fourth statements of the theorem follow the Sobolev embedding theorems (see, for example, [AF03]).

We need also directional Sobolev estimates for the standard Bergman projection $P^{\widetilde{\Omega}}$. Such estimates have been obtained for finite type domains in $\mathbb{C}^2$ by A. Bonami, D.-C. Chang and S. Grellier ([BCG96]) and by D.-C. Chang and B. Q. Li ([CL97]) in the case of decoupled domains of finite type in $\mathbb{C}^n$.

Following the proof of Lemma 3.4 of [CD06] but using the integral curve of the real normal to the boundary of D as in the proof of Theorem 4.2.1 of [BCG96], instead of a coordinate in a special coordinate system (also used in [MS94]), we easily write $\nabla^k P^D = \sum P_i^D T_d^i$ with "good" operators P_i^D and obtain the following estimate for P^D:

Theorem 5.3. *Let D be a smoothly bounded pseudoconvex domain of finite type in $\mathbb{C}^n$ whose Levi form is locally diagonalizable at every point of ∂D.*

If d is a defining function of D, let $T_d = \sum_i \frac{\partial d}{\partial \overline{z_i}} \frac{\partial}{\partial z_i} - \frac{\partial d}{\partial z_i} \frac{\partial}{\partial \overline{z_i}}$. Then, for $1 < p < +\infty$ and $s \geq 0$, the Bergman projection P^D of D continuously maps the space $L^p_{s,T_d}(D)$ into $L^p_s(D)$.

Proof of Theorem 5.1. Let us first prove the weighted L^p regularity of P^Ω_ω. Let $u \in L^p(\Omega, \omega d\lambda)$. Assume for the moment $p > 2$. Let N_p be an integer such that ${}^{p-2}/_{N_p} < {}^1/_N$ where N is the integer of Theorem 5.2 and let us prove, by induction over $l \in \{0, \dots, N_p\}$, that $P^\Omega_\omega(u) \in L^{2+l(p-2)/N_p}(\Omega, \omega d\lambda)$. Assume that $P^\Omega_\omega(u) \in L^{2+l(p-2)/N_p}(\Omega, \omega d\lambda)$ for $l < N_p$. Then Lemma 3.1 and Theorem 5.2 give $\overline{\partial}^* \mathcal{N}_{\widetilde{\Omega}} \circ I\left(P^\Omega_\omega(u)\overline{\partial}\eta\right) \in L^{p+(l+1)(p-2/N_p}(\widetilde{\Omega})$, and the second point of Lemma 3.1 gives the result. The L^p regularity of P^Ω_ω for $1 < p < 2$ is then obtained using the fact that P^Ω_ω is self-adjoint.

The Λ_α regularity is proved similarly. Suppose $u \in \Lambda_\alpha(\Omega)$. Then u belongs to all $L^p(\Omega, \omega d\lambda)$ spaces, $p < +\infty$, and the $L^p(\Omega, \omega d\lambda)$ regularity of P^Ω_ω, Lemma 3.1 and the last assertion of Theorem 5.2 show that $\overline{\partial}^* \mathcal{N}_{\widetilde{\Omega}} \circ I\left(P^\Omega_\omega(u)\overline{\partial}\eta\right) \in \Lambda_0\left(\widetilde{\Omega}\right)$; therefore $P^\Omega_\omega(u) \in \Lambda_0(\Omega)$. Then, using the first assertion of Theorem 5.2 it is easy to prove, by induction, that $P^\Omega_\omega(u) \in \Lambda_{l\alpha/N_\alpha}(\Omega)$, $l \in \{1, \dots, N_\alpha\}$, where N_α is a sufficiently large integer.

For the $L^p_s(\Omega, \omega d\lambda)$ regularity, we deduce from Theorem 5.3 the following extension of Lemma 4.3:

Lemma 5.1. *$P^\Omega_{\omega_0}$ continuously maps the space $L^p_{s,T_{\rho_0}}(\Omega, \omega_0 d\lambda)$ into $L^p_s(\Omega, \omega_0 d\lambda)$.*

As we already know that P^Ω_ω maps $L^p(\Omega, \omega_0 d\lambda)$ into itself, the proof of the $L^p_{s,T_{\rho_0}}(\Omega, \omega_0 d\lambda)$-$L^p_s(\Omega, \omega_0 d\lambda)$ regularity of P^Ω_ω is identical to the end of the proof of Theorem 4.1. □

Remark.

(1) Very recently Tran Vu Khanh & Andrew Raich ([KR14]) announced Sobolev L^p and Lipschitz estimates for the Bergman projection for all pseudoconvex domains of finite type in $\mathbb{C}^n$ (in fact for a more general class). Using this result, the method developed in [CDM14] shows that, for any pseudoconvex domains of finite type in $\mathbb{C}^n$ the weighted Bergman projection $P^\Omega_{\omega_0}$ satisfies the estimates stated in Theorem 2.3. But, without estimates with gain for the $\overline{\partial}$ problem on a general pseudoconvex domain of finite type, our method cannot give the same result for P^Ω_ω.

(2) To extend the result for convex domains of finite type stated in [CDM14] using our method it is necessary to prove weighted L^p-Sobolev and Lipschitz estimates with gain for solutions of the $\overline{\partial}$-equation. As this is certainly a long and technically complicated work we will not discuss it here.

References

[AF03] Robert A. Adams and John J. F. Fournier, *Sobolev spaces*, 2nd ed., Pure and Applied Mathematics (Amsterdam), vol. 140, Elsevier/Academic Press, Amsterdam, 2003. MR2424078 (2009e:46025)

[Bar92] David E. Barrett, *Behavior of the Bergman projection on the Diederich-Fornæss worm*, Acta Math. **168** (1992), no. 1-2, 1–10, DOI 10.1007/BF02392975. MR1149863 (93c:32033)

[BC00] Bo Berndtsson and Philippe Charpentier, *A Sobolev mapping property of the Bergman kernel*, Math. Z. **235** (2000), no. 1, 1–10, DOI 10.1007/s002090000099. MR1785069 (2002a:32039)

[BCG96] Aline Bonami, Der-Chen Chang, and Sandrine Grellier, *Commutation properties and Lipschitz estimates for the Bergman and Szegö projections*, Math. Z. **223** (1996), no. 2, 275–302, DOI 10.1007/PL00004561. MR1417432 (98a:32033)

[BG95] Aline Bonami and Sandrine Grellier, *Weighted Bergman projections in domains of finite type in* $\mathbf{C}^2$, Harmonic analysis and operator theory (Caracas, 1994), Contemp. Math., vol. 189, Amer. Math. Soc., Providence, RI, 1995, pp. 65–80, DOI 10.1090/conm/189/02256. MR1347006 (96f:32042)

[BS90] Harold P. Boas and Emil J. Straube, *Equivalence of regularity for the Bergman projection and the* $\overline{\partial}$*-Neumann operator*, Manuscripta Math. **67** (1990), no. 1, 25–33, DOI 10.1007/BF02568420. MR1037994 (90k:32057)

[BS91] Harold P. Boas and Emil J. Straube, *Sobolev estimates for the* $\overline{\partial}$*-Neumann operator on domains in* $\mathbf{C}^n$ *admitting a defining function that is plurisubharmonic on the boundary*, Math. Z. **206** (1991), no. 1, 81–88, DOI 10.1007/BF02571327. MR1086815 (92b:32027)

[Cat87] David Catlin, *Subelliptic estimates for the* $\overline{\partial}$*-Neumann problem on pseudoconvex domains*, Ann. of Math. (2) **126** (1987), no. 1, 131–191, DOI 10.2307/1971347. MR898054 (88i:32025)

[CD06] Philippe Charpentier and Yves Dupain, *Estimates for the Bergman and Szegö projections for pseudoconvex domains of finite type with locally diagonalizable Levi form*, Publ. Mat. **50** (2006), no. 2, 413–446, DOI 10.5565/PUBLMAT_50206_08. MR2273668 (2007h:32003)

[CDM14] P. Charpentier, Y. Dupain, and M. Mounkaila, *Estimates for weighted Bergman projections on pseudo-convex domains of finite type in* $\mathbb{C}^n$, Complex Var. Elliptic Equ. **59** (2014), no. 8, 1070–1095, DOI 10.1080/17476933.2013.805411. MR3197034

[Chr96] Michael Christ, *Global* C^∞ *irregularity of the* $\overline{\partial}$*-Neumann problem for worm domains*, J. Amer. Math. Soc. **9** (1996), no. 4, 1171–1185, DOI 10.1090/S0894-0347-96-00213-5. MR1370592 (96m:32014)

[CK03] Hong Rae Cho and Ern Gun Kwon, *Sobolev-type embedding theorems for harmonic and holomorphic Sobolev spaces*, J. Korean Math. Soc. **40** (2003), no. 3, 435–445, DOI 10.4134/JKMS.2003.40.3.435. MR1973911 (2004g:46049)

[CL97] Der-Chen Chang and Bao Qin Li, *Sobolev and Lipschitz estimates for weighted Bergman projections*, Nagoya Math. J. **147** (1997), 147–178. MR1475171 (98i:32035)

[Det81] Jacqueline Détraz, *Classes de Bergman de fonctions harmoniques* (French, with English summary), Bull. Soc. Math. France **109** (1981), no. 2, 259–268. MR623793 (83m:31006)

[DF77] Klas Diederich and John Erik Fornaess, *Pseudoconvex domains: bounded strictly plurisubharmonic exhaustion functions*, Invent. Math. **39** (1977), no. 2, 129–141. MR0437806 (55 #10728)

[FKM90] C. L. Fefferman, J. J. Kohn, and M. Machedon, *Hölder Estimates on CR Manifold with a diagonalizable Levi Form*, Advances in Math **84** (1990), no. 1, 1–90.

[Har07] Phillip S. Harrington, *Compact and subelliptic estimates for the* $\overline{\partial}$*-Neumann operator on* C^2 *pseudoconvex domains*, Math. Ann. **337** (2007), no. 2, 335–352, DOI 10.1007/s00208-006-0037-3. MR2262787 (2007g:35161)

[HM12] A.-K. Herbig and J. D. McNeal, *A smoothing property of the Bergman projection*, Math. Ann. **354** (2012), no. 2, 427–449, DOI 10.1007/s00208-011-0734-4. MR2965249

[HMS14] A.-K. Herbig, J. D. McNeal, and E. J. Straube, *Duality of holomorphic function spaces and smoothing properties of the Bergman projection*, Trans. Amer. Math. Soc. **366** (2014), no. 2, 647–665, DOI 10.1090/S0002-9947-2013-05827-8. MR3130312

[Koe04] Kenneth D. Koenig, *A parametrix for the $\overline{\partial}$-Neumann problem on pseudoconvex domains of finite type*, J. Funct. Anal. **216** (2004), no. 2, 243–302, DOI 10.1016/j.jfa.2004.06.004. MR2095685 (2005k:32051)

[Koh72] J. J. Kohn, *Boundary behavior of δ on weakly pseudo-convex manifolds of dimension two*, J. Differential Geometry **6** (1972), 523–542. Collection of articles dedicated to S. S. Chern and D. C. Spencer on their sixtieth birthdays. MR0322365 (48 #727)

[KR14] Tran Vu Khanh and Andrew Raich, *Local regularity properties for the Bergman projection on pseudoconvex domains of finite/infinite type*, arxiv.org/abs/1406.6532 (2014).

[MS94] J. D. McNeal and E. M. Stein, *Mapping properties of the Bergman projection on convex domains of finite type*, Duke Math. J. **73** (1994), no. 1, 177–199, DOI 10.1215/S0012-7094-94-07307-9. MR1257282 (94k:32037)

[PW90] Zbigniew Pasternak-Winiarski, *On the dependence of the reproducing kernel on the weight of integration*, J. Funct. Anal. **94** (1990), no. 1, 110–134, DOI 10.1016/0022-1236(90)90030-O. MR1077547 (91j:46035)

[Str97] Emil J. Straube, *Plurisubharmonic functions and subellipticity of the $\overline{\partial}$-Neumann problem on non-smooth domains*, Math. Res. Lett. **4** (1997), no. 4, 459–467, DOI 10.4310/MRL.1997.v4.n4.a2. MR1470417 (98m:32024)

[Tri78] Hans Triebel, *Interpolation theory, function spaces, differential operators*, North-Holland Mathematical Library, vol. 18, North-Holland Publishing Co., Amsterdam-New York, 1978. MR503903 (80i:46032b)

Institut de Mathématiques de Bordeaux, Université Bordeaux I, 351, Cours de la Libération, 33405, Talence, France
E-mail address: philippe.charpentier@math.u-bordeaux1.fr

Institut de Mathématiques de Bordeaux, Université Bordeaux I, 351, Cours de la Libération, 33405, Talence, France

Faculté des Sciences, Université Abdou Moumouni, B.P. 10662, Niamey, Niger
E-mail address: modi.mounkaila@yahoo.fr