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DISTANCE DEGENERATING HANDLE ADDITIONS

LIANG LIANG, FENGCHUN LEI, AND FENGLING LI

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ABSTRACT. Let $M = V \cup_S W$ be a Heegaard splitting of a 3-manifold M and let F be a component of ∂M lying in $\partial_- V$. A simple closed curve J in F is said to be distance degenerating if the distance of $M_J = V_J \cup_S W$ is less than the distance of $M = V \cup_S W$ where M_J is the 3-manifold obtained by attaching a 2-handle to M along J. In this paper, we will prove that for a strongly irreducible Heegaard splitting $M = V \cup_S W$, if V is simple or $M = V \cup_S W$ is locally complicated, then the diameter of the set of distance degenerating curves in F is bounded.

1. INTRODUCTION

Let F be a closed orientable surface. If the genus of F is at least 2, the curve complex of F, first defined by Harvey [5], is the complex whose vertices are the isotopy classes of essential simple closed curves in F, and $k+1$ vertices determine a k-simplex if they can be represented by pairwise disjoint curves. If F is a torus, the curve complex of F, defined by Masur and Minsky [11], is the complex whose vertices are the isotopy classes of essential simple closed curves in F, and $k+1$ vertices determine a k-simplex if they can be represented by a collection of curves any two of which intersect in only one point.

Denote the curve complex of F by $C(F)$. For any two vertices in $C(F)$, define the distance $d_{C(F)}(x,y)$ to be the minimal number of 1-simplices in a simplicial path joining x to y over all such possible paths. Let A and B be two sets of vertices of $C(F)$. The diameter of A, which is denoted by $diam_{C(F)}(A)$, is defined to be $\max\{d(x,y)|x,y \in A\}$. The distance between A and B, which is denoted by $d_{C(F)}(A,B)$, is defined to be $\min\{d(x,y)|x \in A, y \in B\}$.

Let M be a 3-manifold and let S be an embedded closed orientable surface in M. If S cuts M into two compression bodies V and W such that $S = \partial_+ V = \partial_+ W$, then $M = V \cup_S W$ is called a Heegaard splitting of M. The distance of $M = V \cup_S W$ is defined to be $d_{C(S)}(\mathscr{D}_V, \mathscr{D}_W)$ where $\mathscr{D}_V$ and $\mathscr{D}_W$ are sets of vertices in $C(S)$ represented by boundaries of essential disks in V and W, respectively.

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Hempel [3] introduced the concept of the distance of a Heegaard splitting to study 3-manifolds in 2001, and he showed [3] that there exist arbitrarily high distance Heegaard splittings for closed 3-manifolds (Ido-Jang-Kobayashi [1] recently showed that for any positive integers n and $g \geq 2$ there are closed 3-manifolds admitting a Heegaard splitting of genus g with distance exactly n), and a 3-manifold M admitting a distance at least 3 Heegaard splitting is hyperbolic. Since then, many properties of Heegaard splittings, as well as 3-manifolds, have been obtained. For example, Hartshorn [4] and Scharlemann [16] showed that there is no essential surface with small Euler characteristic in a 3-manifold admitting a high distance Heegaard splitting; Scharlemann-Tomova [17] showed that a high distance Heegaard splitting is a unique minimal Heegaard splitting up to isotopy; see [7], Lei-Yang [20], Yang-Lei [19] and many others showed that the amalgamation of Heegaard splittings is always efficient under some "high" distance conditions.

Let M be a 3-manifold with nonempty boundary and let $M = V \cup_S W$ be a Heegaard splitting of M. For a simple closed curve J on $\partial_- V$, M_J is defined to be the manifold obtained by adding a 2-handle to M along J and, if the component in which J lies is a torus, then filling in the resulting 2-sphere by a 3-ball (the operation is exactly a Dehn filling, and J is called a slope). When $\mathcal{J}$ is a collection of pairwise disjoint simple closed curves on $\partial_- V$, we can similarly define $M_{\mathcal{J}}$. $M_{\mathcal{J}}$ has a natural Heegaard splitting $M_{\mathcal{J}} = V_{\mathcal{J}} \cup_S W$.

For a ∂-reducible 3-manifold M, some sufficient conditions for $M_{\mathcal{J}}$ to be ∂-irreducible have been known; see, for example, [14], [6], [15], [8], [9]. Let M be a hyperbolic 3-manifold with at least one torus boundary component T_0, and r a slope on T_0. A slope r is said to be exceptional if the resulting manifold M_r is either reducible, boundary-reducible, annular, toroidal, or a small Seifert fiber space. It follows from the solution of the Geometrization Conjecture due to G. Perelman that M_r is hyperbolic if and only if the slope r is not exceptional. Thurston's hyperbolic Dehn surgery theorem shows that there are only finitely many exceptional slopes on T_0. On the other side, there are known examples of hyperbolic 3-manifolds with boundary of genus greater than one, such that infinitely many distinct handle additions yield nonhyperbolic 3-manifolds. However, Scharlemann-Wu [18] showed that this phenomenon is limited in some sense.

Let $V \cup_S W$ be a Heegaard splitting of a 3-manifold M with nonempty boundary, let J be a simple closed curve on $\partial_- V$, and let $M_J = V_J \cup_S W$. Let m and n be the Heegaard distances of $V \cup_S W$ and $V_J \cup_S W$, respectively. Clearly, $m \geq n$. If $m = n$, then the handle addition along J is called distance nondegenerating handle addition and J is called a nondegenerating curve. Otherwise, J is called a degenerating curve.

Let $M = V \cup_S W$ be an irreducible Heegaard splitting and let F be a component of ∂M. The main result of the paper is that if V has only one separating or nonseparating essential disk up to isotopy or $M = V \cup_S W$ is locally complicated, then the diameter of the set of degenerating curves is bounded by a constant, which is dependent only on the genus of S.

The article is organized as follows. In section 2, we review some necessary preliminaries. The statement and proof of the main result is given in section 3. In section 4, we will use a similar method to give a sufficient condition for H_{J_1,J_2} to have incompressible boundary, where H is a handlebody of genus at least 3 and J_1, J_2 are disjoint simple closed curves in ∂H.

2. Preliminaries

Let M be a compact orientable 3-manifold. A Heegaard splitting $M = V \cup_S W$ is said to be reducible if there are essential disks D in V and E in W such that $\partial D = \partial E$; otherwise, it is irreducible. A Heegaard splitting $M = V \cup_S W$ is said to be weakly reducible if there are essential disks D in V and E in W such that $\partial D \cap \partial E = \emptyset$; otherwise it is strongly irreducible (see [2]).

Let F be a properly embedded surface in M. F is said to be compressible if either F is a 2-sphere which bounds a 3-ball or there is an essential simple closed curve on F which bounds a disk in M; otherwise, F is said to be incompressible. A 3-manifold M is said to be ∂-reducible if ∂M is compressible in M. Otherwise, M is called ∂-irreducible. Let J be a simple closed curve in ∂M. J is said to be disk-busting if $\partial M - J$ is incompressible in M. A curve J' in $\partial M - J$ is coplanar with J if J' cuts a planar surface from $\partial M - J$.

Let F be a compact surface of genus at least 1 with nonempty boundary. Define the arc and curve complex $AC(F)$ as follows: Each vertex of $AC(F)$ is the isotopy class of an essential simple closed curve or an essential properly embedded arc in F, and a set of vertices form a simplex of $AC(F)$ if these vertices are represented by pairwise disjoint arcs or curves in F. For any two disjoint vertices, we place an edge between them. All the vertices and edges form a 1-skeleton of $AC(F)$, denoted by $AC^1(F)$. And for each edge, we assign it length 1. Thus for any two vertices α and β in $AC^1(F)$, the distance $d_{AC(F)}(\alpha, \beta)$ is defined to be the minimal length of paths in $AC^1(F)$ connecting α and β. A subsurface F' of F is called essential if $\partial F'$ consists of essential curves in F. Fix a compact essential subsurface $F' \subset F$. By the definition of projections to subsurfaces in [11], there is a natural map $\kappa_{F'}$ from vertices of $C(F)$ to finite subsets of vertices of $AC(F)$ defined as follows: For every vertex $[\gamma]$ in $C(F)$, take a curve γ in the isotopy class such that $|\gamma \cap F'|$ is minimal. If $\gamma \cap F' = \emptyset$, then $\kappa_{F'}([\gamma]) = \emptyset$. If $\gamma \cap F' \neq \emptyset$, then $\kappa_{F'}([\gamma])$ is the union of the isotopy classes of essential components of $\gamma \cap F'$. Furthermore, there is a natural map $\sigma_{F'}$ from vertices of $AC(F')$ to finite subsets of vertices of $C(F')$: For every vertex β in $AC(F')$, $\sigma_{F'}(\beta)$ is the union of the isotopy classes of essential boundary components of the regular neighborhood of $\beta \cup \partial F'$. It is obvious that if β is the isotopy class of a simple closed curve in F', then $\sigma_{F'}(\beta) = \beta$. Then we have a map $\pi_{F'} = \sigma_{F'} \circ \kappa_{F'}$ from vertices of $C(F)$ to finite subsets of vertices of $C(F')$. For any two vertices α, β in $C(F)$, define $d_{F'}(\alpha, \beta) = diam(\pi_{F'}(\alpha) \cup \pi_{F'}(\beta))$. For any $\alpha \in C(F)$, α is said to cut F' if $\pi_{F'}(\alpha) \neq \emptyset$. If $d_{C(F)}(\alpha, \beta) = 1$, then $d_{F'}(\alpha, \beta) \leq 2$.

The following so-called Bounded Geodesic Image Theorem, due to Masur-Minsky, will be used in our discussion.

Lemma 2.1 ([11]). *Let F' be an essential subsurface of F, and let γ be a geodesic segment in $C(F)$, such that $\pi_{F'}(v) \neq \emptyset$ for every vertex v of γ. Then there is a constant $\mathcal{M}$ depending only on F so that $diam_{C(F')}(\pi_{F'}(\gamma)) \leq \mathcal{M}$.*

The following Disk Image Theorem was proved by Li [10] and Masur-Schleimer [12] independently.

Lemma 2.2. *Let M be a compact orientable and irreducible 3-manifold. S is a boundary component of M. Suppose $\partial M - S$ is incompressible. Let $\mathscr{D}$ be the disk*

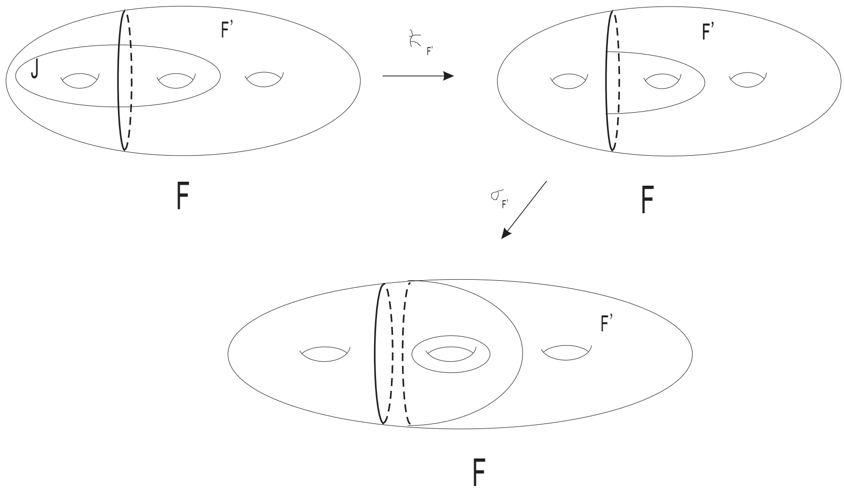

FIGURE 1. Composition of $\sigma_{F'}$ and $\kappa_{F'}$

complex of S, and $F \subset S$ an essential subsurface. Assume each component of ∂F is disk-busting. Then either

(1) M is an I-bundle over some compact surface and F is a horizontal boundary of the I-bundle and the vertical boundary is a single annulus; or

(2) the image of this complex, $\kappa_F(\mathscr{D})$, lies in a ball of radius 3 in AC(F). In particular, $\kappa_F(\mathscr{D})$ has diameter 6 in AC(F). Moreover, $\pi_F(\mathscr{D})$ has distance at most 12 in C(F).

Let $M = V \cup_S W$ be a strongly irreducible Heegaard splitting and let F be a component of $\partial_- V$. Let A be a separating essential disk in V such that one component of $V - A$ is homeomorphic to $F \times I$. Denote the other component by V' which is also a compression body. Denote the component of $S - \partial A$ lying in $\partial_+ V'$ by S_A. Let $\pi_{S_A} = \sigma_{S_A} \circ \kappa_{S_A}$ be defined as above and let f'_{S_A} be an embedding from S_A to $\partial_+ V$. Then f'_{S_A} induces a surjective map from vertices of $C(S)$ to vertices of $C(\partial_+ V)$ which we also denote by f'_{S_A}. Let $f_{S_A} = f'_{S_A} \circ \pi_{S_A}$. Then f_{S_A} is a map from vertices of $C(S)$ to finite subsets of vertices of $C(\partial_+ V)$. Let $\mathscr{D}_W$ be the set of vertices in $C(S)$ represented by boundaries of essential disks in W and let $\mathscr{D}_{V'}$ be the set of vertices in $C(\partial_+ V')$ represented by boundaries of essential disks in V'.

The Heegaard splitting $M = V \cup_S W$ is called *locally complicated* if there exists a separating essential disk A in V, which is described as above such that $d_{C(\partial_+ V')}(f_{S_A}(\mathscr{D}_W), \mathscr{D}_{V'})$ is larger than a constant depending only on the genus of S.

Remark 2.3. Since $M = V \cup_S W$ is strongly irreducible, $S - \partial A$ is incompressible. If W is homeomorphic to $S_A \times I$, then V has either only one separating or only one nonseparating essential disk up to isotopy. Otherwise, we can always find an essential disk A' in V which is disjoint from A. A simple argument of the genus of S_A shows that there is an essential arc α in S_A such that $\alpha \cap A' = \emptyset$; then $\alpha \times I$ is an essential disk in W disjoint from A' in S. This implies that $M = V \cup_S W$ is weakly reducible, a contradiction. If W is not homeomorphic to $S_A \times I$, by Lemma 2.2, $diam_{C(\partial_+ V')}(f_{S_A}(\mathscr{D}_W))$ is bounded. By Theorem 5.2 in [13], the set of simple closed curves J on $\partial_+ V'$ with distance $d_{C(\partial_+ V')}(J, \mathscr{D}_{V'}) \geq \mathcal{M} + 1$ is generic with respect to a Borel measure where $\mathcal{M}$ is a constant only depending on S. From this point of view, most strongly irreducible Heegaard splittings are locally complicated.

3. Proof of main result

Let V be a compression body and let $\mathscr{D}$ denote the set of vertices in the curve complex $C(\partial_+ V)$ represented by boundaries of essential disks in V. A compression body V is called simple if V has either only one separating or only one nonseparating essential disk up to isotopy; see Figure 2.

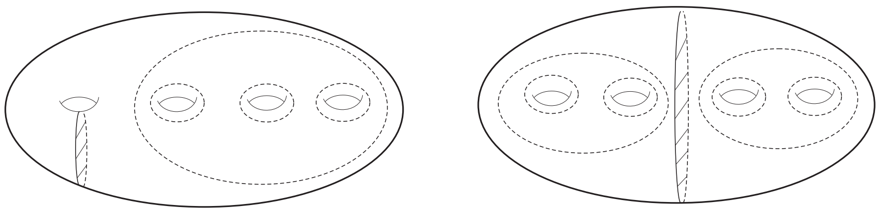

Figure 2. Simple compression bodies

Lemma 3.1. *If $\partial_- V$ is connected and $g(\partial_+ V) - g(\partial_- V) = 1$, then V has only one nonseparating essential disk D up to isotopy and for any essential disk E in V, $E \cap D = \emptyset$ under isotopy.*

Proof. Let D be a nonseparating essential disk in V. First, we consider nonseparating compressing disks in V.

Claim 1. V has only one nonseparating compressing disk D up to isotopy.

Assume that there is another nonseparating compressing disk E which is not isotopic to D. If $E \cap D = \emptyset$ under isotopy, then we can compress $\partial_+ V$ along D to get a manifold $M_1 = S \times I$ where $S = \partial_- V$ is an orientable closed surface with $g(S) \geq 1$. E is nonseparating and not isotopic to D, so ∂E is not coplanar with ∂D. Thus E is a compressing disk for $S \times \{1\}$ in M_1, a contradiction.

So every nonseparating compressing disk of V which is not isotopic to D has intersections with D. We can choose one such that it has minimal intersections with D under isotopy among nonseparating compressing disks which intersect D. We also denote it by E. Then an innermost loop argument implies that $E \cap D$ consists of arcs. Choose an outermost arc γ in D which cuts a disk D_0 from D such that $D_0 \cap E = \gamma$. Let $D'_i = D_i \cup_\gamma D_0$ where D_i is obtained by cutting E along γ, for $i = 1, 2$. Both D'_1 and D'_2 are essential in V, otherwise we can isotope E to reduce $|E \cap D|$. Since E can be recovered by a band sum of D'_1 and D'_2 and E is nonseparating, at least one of D'_1 and D'_2 is nonseparating. Assume D'_1 is nonseparating. As $|D'_1 \cap D| < |E \cap D|$, by assumption, D'_1 is isotopic to D. As $D'_1 \cap E = \emptyset$, $E \cap D = \emptyset$ up to isotopy, a contradiction.

Now we consider the relationship between a separating compressing disk and D. We have the following claim.

Claim 2. Any separating compressing disk in V is a band sum of D and a copy of D.

Assume the claim fails. Then there is a separating compressing disk which is not a band sum of D and a copy of D. Denote it by D'. First, we choose a separating compressing disk D_1 in V which is a band sum of D and a copy of D. Then D_1 cuts V into a solid torus H with D as a meridian disk and a 3-manifold homeomorphic to $\partial_- V \times I$. Let $M_1 = \overline{V - H}$.

If $D' \cap D_1 = \emptyset$ up to isotopy, then D' lies in H or M_1. Since M_1 is an I-bundle of $\partial_- V$, D' lies in H and is isotopic to D_1, a contradiction.

So $D' \cap D_1 \neq \emptyset$ up to isotopy. Then we can choose one from all separating disks none of which is a band sum of D and a copy of D such that it has minimal intersections with D_1 up to isotopy. Also denote it by D'. We can ∂-compress D' along D_1 to get two essential disks D'_1 and D'_2. Then each of D'_1 and D'_2 is isotopic to D or D_1. Since D' is a band sum of D'_1 and D'_2, in each case, we can prove that D' is isotopic to D or a band sum of D and a copy of D, a contradiction. So the claim holds.

So for any compressing disk E in V, E is isotopic to D, or E is a band sum of D and a copy of D. So $E \cap D = \emptyset$. □

Corollary 3.2. *If V is a simple compression body, then $diam_{C(\partial_+ V)}(\mathscr{D}) \leq 2$.*

Proof. When V has only one separating essential disk up to isotopy, then $\mathscr{D} = \{\partial D\}$ and V has only one essential disk under isotopy. So we only need to consider the case that V has only one nonseparating essential disk. By Lemma 3.1, the corollary holds. □

Let F be a closed orientable surface with $g(F) \geq 2$ and J, J' essential curves in F. Let A and B be sets of vertices in $C(F)$. Then we have the following lemma:

Lemma 3.3. *If $diam_{C(F)}(A)$ and $diam_{C(F)}(B)$ are bounded, then we have:*

$$d_{C(F)}(A, J) \leq diam_{C(F)}(A) + d_{C(F)}(A, B) + diam_{C(F)}(B) + d_{C(F)}(B, J)$$

and

$$d_{C(F)}(A, J) \leq diam_{C(F)}(A) + d_{C(F)}(A, J') + d_{C(F)}(J, J').$$

Proof. Let $a' \in A$ and $b, b' \in B$ such that $d_{C(F)}(A, B) = d_{C(F)}(a', b)$, $d_{C(F)}(B, J) = d_{C(F)}(b', J)$. So

$$\begin{aligned} d_{C(F)}(A, J) &= d_{C(F)}(a, J) \\ &\leq d_{C(F)}(a, a') + d_{C(F)}(a', b) + d_{C(F)}(b, b') + d_{C(F)}(b', J). \end{aligned}$$

Thus

$$d_{C(F)}(A, J) \leq diam_{C(F)}(A) + d_{C(F)}(A, B) + diam_{C(F)}(B) + d_{C(F)}(B, J).$$

Let $B = \{J'\}$. Then we have

$$d_{C(F)}(A, J) \leq diam_{C(F)}(A) + d_{C(F)}(A, J') + d_{C(F)}(J, J').$$ □

Let A be a separating essential disk in V such that one component of $\overline{V - A}$ is homeomorphic to $F \times I$. Let S_F be a component of $\partial_+ V - A$ which lies in $\partial_+(F \times I)$. Let f'_A be a natural homeomorphism from $S_F \cup A$ to F and let f_{S_F} be defined as above. Then $f_A = f'_A \circ f_{S_F}$ is a map from vertices of $C(\partial_+ V)$ to finite subsets of vertices of $C(F)$.

The following is our main theorem:

Theorem 3.4. *Let $M = V \cup_S W$ be a strongly irreducible Heegaard splitting and let F be a component of ∂M. If V is simple or $M = V \cup_S W$ is locally complicated, then the diameter of the set of distance degenerating curves in F is bounded by a constant depending only on the genus of S.*

Proof. Assume F is a component of $\partial_- V$. Let $\mathcal{J}$ be the set of distance degenerating curves in F. First we consider the case $g(F) \geq 2$. Let $J \in \mathcal{J}$ and let k be the distance of $M_J = V_J \cup_S W$. Then $k < n$ where n is the distance of $M = V \cup_S W$. Thus there are essential disks D and E in V_J and W such that $d_{C(S)}(\partial D, \partial E) = k$. Let $a_0 = \partial E, a_1, \ldots, a_k = \partial D$ be a geodesic segment between ∂D and ∂E in $C(S)$. There are three subcases:

(1) V has only one separating essential disk up to isotopy.

In this case, V is a boundary sum of two trivial compression bodies. Let A be the essential separating disk in V and $\partial_- V = F \cup F'$. Assume A cuts V into $V_F = F \times I$ and $V_{F'} = F' \times I$.

Isotope D such that $|D \cap A|$ is minimal. Since V is irreducible, an innermost loop argument implies that there is no closed curve component in $D \cap A$. For any integer i where $0 \leq i \leq k-1$, $A \cap a_i \neq \emptyset$. Otherwise, the distance of $M = V \cup_S W$ is at most $i+1$. So $n \leq i+1 \leq k$, a contradiction.

Assume $D \cap A = \emptyset$. Then D lies in V_F. ∂D is not parallel to ∂A. Otherwise, the distance of $M = V \cup_S W$ is less than n, a contradiction. So $f_A(\partial D)$ is a simple closed curve in F. Since a_i cuts S_A for $i = 0, 1, \ldots, k$, by Lemma 2.1, $d_{C(F)}(f_A(\partial D), f_A(\partial E)) \leq \mathcal{M}$.

Let V'_F be the compression body obtained by attaching a 2-handle to $F \times [0,1]$ along J. If J is separating in F, then V'_F is a compression body with only one separating essential disk in V'_F. So $f_A(\partial D)$ is isotopic to J. Since E is an essential disk in W and $d_{C(F)}(f_A(\partial D), f_A(\partial E)) \leq \mathcal{M}$,

$$d_{C(F)}(f_A(\mathscr{D}), J) \leq d_{C(F)}(f_A(\partial E), f_A(\partial D)) \leq \mathcal{M}.$$

If J is nonseparating in F, then V'_F is a compression body with $\partial_- V'_F$ connected and $g(\partial_+ V'_F) - g(\partial_- V'_F) = 1$.

So by Lemma 3.1, $d_{C(F)}(f_A(\partial D), J) \leq 1$. And by Lemma 3.3, we have

$$\begin{aligned} d_{C(F)}(f_A(\mathscr{D}), J) &\leq d_{C(F)}(f_A(\partial E), J) \\ &\leq diam_{C(F)}(f_A(\partial E)) + d_{C(F)}(f_A(\partial E), f_A(\partial D)) \\ &\quad + d_{C(F)}(f_A(\partial D), J) \\ &\leq \mathcal{M} + 3. \end{aligned}$$

Now consider the case $D \cap A \neq \emptyset$. Then $D \cap A$ consists of arcs and $f_A(\partial D)$ is a set of simple closed curves in F with $diam_{C(F)}(f_A(\partial D)) \leq 2$. Choose an outermost arc α of $D \cap A$ in D which cuts a disk D_α from D such that $D_\alpha \cap A = \alpha$. Assume α cuts A into D'_1 and D'_2. Let $D_i = D_\alpha \cup_\alpha D'_i$ where $i = 1, 2$. Since $V_{F'} = F' \times I$, D_α must lie in V_F. Otherwise, we can prove that $\partial_+ V_{F'}$ is compressible in $V_{F'}$, a contradiction. Both D_1 and D_2 are essential disks in V'_F. Otherwise, we can isotope D to reduce $|D \cap A|$. So $f_A(\partial D_i)$ is a simple closed curve in F for $i = 1, 2$ and both of $f_A(\partial D_1)$ and $f_A(\partial D_2)$ lie in $f_A(\partial D)$. By the same argument as above, $d_{C(F)}(f_A(\partial D_i), J) \leq 1$ where $i = 1, 2$. So $d_{C(F)}(f_A(\partial D), J) \leq d_{C(F)}(f_A(\partial D_1), J) \leq 1$.

So by Lemma 3.3,

$$\begin{aligned} d_{C(F)}(f_A(\mathscr{D}), J) &\leq d_{C(F)}(f_A(\partial E), J) \\ &\leq diam_{C(F)}(f_A(\partial E)) + d_{C(F)}(f_A(\partial E), f_A(\partial D)) \\ &+ diam_{C(F)}(f_A)(\partial D) + d_{C(F)}(f_A(\partial D), J) \\ &\leq \mathcal{M} + 5. \end{aligned}$$

So for any two simple closed curves J_1 and J_2 in $\mathcal{J}$,

$$d_{C(F)}(f_A(\mathscr{D}), J_i) \leq \mathcal{M} + 5,$$

where $i = 1, 2$. By Lemma 3.3,

$$\begin{aligned} d_{C(F)}(J_1, J_2) &\leq 2(\mathcal{M} + 5) + diam(f_A(\mathscr{D})) \\ &\leq 2\mathcal{M} + 22. \end{aligned}$$

So $diam_{C(F)}(\mathcal{J})$ is bounded and the theorem holds.

(2) V has only one nonseparating essential disk.

In this case, let A be the nonseparating essential disk in V. Then $V' = \overline{V - A}$ is homeomorphic to $F \times I$. Isotope D such that $|D \cap A|$ is minimal. With the same method, we can prove that $a_i \cap A \neq \emptyset$ where $0 \leq i \leq k - 1$.

If $D \cap A = \emptyset$, just like case (1), we can prove that D is an essential disk in V'_J and $d_{C(F)}(f_A(\mathscr{D}), J) \leq \mathcal{M} + 3$.

If $D \cap A \neq \emptyset$, then we can choose an outermost arc α of $D \cap A$ from D which cuts a disk D_α from D such that $D_\alpha \cap A = \alpha$. Assume α cuts A into D'_1 and D'_2. Let $D_i = D_\alpha \cup_\alpha D'_i$ where $i = 1, 2$. By Lemma 2.6 in [21], both D_1 and D_2 are essential disks in V'_J. With the same argument as in case (1), we can prove that $d_{C(F)}(f_A(\mathscr{D}), J) \leq \mathcal{M} + 5$.

So for any two simple closed curves J_1 and J_2 in $\mathcal{J}$, we have

$$d_{C(F)}(f_A(\mathscr{D}), J_i) \leq \mathcal{M} + 5,$$

where $i = 1, 2$.

So by Lemma 3.3,

$$\begin{aligned} d_{C(F)}(J_1, J_2) &\leq 2(\mathcal{M} + 5) + diam(f_A(\mathscr{D})) \\ &\leq 2\mathcal{M} + 22. \end{aligned}$$

Thus $diam_{C(F)}(\mathcal{J})$ is bounded and the theorem holds.

(3) $M = V \cup_S W$ is locally complicated.

Let A be the separating essential disk in V described as in the definition of locally complicated. Then a component V_F of $V - A$ is homeomorphic to $F \times I$. Denote the other component of $V - A$ by V'. Let $S' = \partial_+ V' \cap S$ and let $f_{S'}$ be the map from vertices of $C(S)$ to $C(S')$ defined as in the definition of locally complicated. Since $M = V \cup_S W$ is locally complicated, $d_{C(\partial_+ V')}(f_{S'}(\mathscr{D}), \mathscr{D}_{V'}) \geq \mathcal{M} + 1$ where $\mathscr{D}_{V'}$ is the set of vertices in $C(\partial_+ V')$ represented by boundaries of essential disks in V'. If $D \cap A = \emptyset$, we return to case (1) and the theorem holds. So we only need to consider the case $D \cap A \neq \emptyset$. Isotope D such that $|D \cap A|$ is minimal. Choose an outermost arc α of $D \cap A$ from D which cuts a disk D_α from D such that $D_\alpha \cap A = \alpha$. Assume α cuts A into two disks D'_1 and D'_2. Let $D_i = D_\alpha \cup_\alpha D'_i$ where $i = 1, 2$. If D_α lies in V', then both D_1 and D_2 are essential disks in V'.

Otherwise, assuming $D_\alpha \subset V'$, we can isotope D to reduce $|D \cap A|$, a contradiction. So both ∂D_1 and ∂D_2 are essential in $\partial_+ V'$ and lie in $f_{S'}(\partial D)$. With the same argument as in case (1), we can prove that $a_i \cap A \neq \emptyset$ where $0 \le i \le k-1$. So by Lemma 2.1,

$$d_{C(\partial_+ V')}(f_{S'}(\mathscr{D}), \mathscr{D}_{V'}) \le d_{C(\partial_+ V')}(f_{S'}(\partial E), f_{S'}(\partial D)) \le \mathcal{M},$$

which contradicts the assumption that $M = V \cup_S W$ is locally complicated. So D_α must lie in V_F and return to case (1). Thus the theorem holds.

When F is a torus, with the same method, we can prove that the theorem holds. This completes the proof of the theorem. □

4. ∂-irreducible handle additions to a handlebody

Now we consider the ∂-irreducible handle additions to a handlebody. Let H be a handlebody with genus greater than or equal to 3. Let J_1, J_2 be two disjoint simple closed curves in ∂H. Denote the component of $\partial H - J_1$ which J_2 lies in by F. Let $\mathscr{D}$ be the set of vertices in $C(\partial H)$ represented by the boundaries of compressing disks for ∂H. Then we have the following theorem:

Theorem 4.1. *Let H be a handlebody with genus at least 3. J_1, J_2 are disjoint simple closed curves on ∂H. If $\partial H - J_1$ is incompressible in H and $d_{C(F)}(J_2, \pi_F(\mathscr{D})) \ge 8$, then $\partial H_{J_1,J_2}$ is incompressible.*

Proof. Let V be the compression body obtained by attaching a 2-handle to $\partial H \times I$ along J_1 and let V' be the compression body obtained by attaching 2-handles to $\partial H \times I$ along J_1, J_2. Then $H_{J_1} = W \cup_S V$ and $H_{J_1,J_2} = W \cup_S V'$ are Heegaard splittings of H_{J_1} and H_{J_1,J_2} where $W = H$ and $S = \partial H$. Let D_{J_1} be the compressing disk in V with ∂D_{J_1} isotopic to J_1 in ∂H.

Claim. $H_{J_1} = W \cup_S V$ is a strongly irreducible Heegaard splitting.

Otherwise, the Heegaard splitting is weakly reducible. So we can find essential disks D, E in W and V such that $\partial D \cap \partial E = \emptyset$. If J_1 is separating, then ∂E is isotopic to J_1 and D is a compressing disk for $\partial H - J_1$, a contradiction. Assume J_1 is nonseparating. If E is nonseparating, then by Lemma 3.1, ∂E is isotopic to J_1. So D is a compressing disk for $\partial H - J_1$, a contradiction. If E is separating, by Lemma 3.1 ∂E cuts a once-punctured torus T from ∂H with J_1 lying in T. Since $\partial H - J_1$ is incompressible in H, $J_1 \cap D \neq \emptyset$. So ∂D lies in T and ∂E also bounds an essential disk in H which is a compressing disk for $\partial H - J_1$, a contradiction. So the claim holds.

Assume the theorem is false. Then $\partial H_{J_1,J_2}$ is compressible and the Heegaard splitting $H_{J_1,J_2} = W \cup_S V'$ is weakly reducible. We can find essential disks D and E in W and V' such that $\partial D \cap \partial E = \emptyset$. Isotope E such that $|E \cap D_{J_1}|$ is minimal. An innermost loop argument implies that $E \cap D_{J_1}$ consists of arcs. There are two cases:

Case 1. J_1 is nonseparating in ∂H.

Let V'' be the manifold obtained by cutting V' along D_{J_1}. There are two subcases:

(1) $E \cap D_{J_1} = \emptyset$. Then E lies in V'' and $\pi_F(\partial E)$ is a simple closed curve in F. ∂E is not coplanar to J_1 in ∂H. Otherwise ∂E bounds an essential disk in V which contradicts that the Heegaard splitting $H_{J_1} = W \cup_S V$ is strongly irreducible. So

E is essential in V''. If J_2 is separating in ∂H, then V'' is a compression body with only one separating essential disk. So ∂E is isotopic to J_2. If J_2 is nonseparating in ∂H, then V'' is a compression body with only one nonseparating essential disk. By Lemma 3.1, $\partial E \cap J_2 = \emptyset$. So $d_{C(F)}(\pi_F(\partial E), J_2) \leq 1$. So by Lemma 3.2,

$$\begin{aligned} d_{C(F)}(\pi_F(\partial D), J_2) &\leq diam_{C(F)}(\pi_F(\partial D)) + d_{C(F)}(\pi_F(\partial D), \pi_F(\partial E)) \\ &\quad + d_{C(F)}(\pi_F(\partial E), J_2) \\ &\leq 5. \end{aligned}$$

Thus

$$d_{C(F)}(\pi_F(\mathscr{D}), J_2) \leq d_{C(F)}(\pi_F(\partial D), J_2) \leq 5,$$

a contradiction.

(2) $E \cap D_{J_1} \neq \emptyset$. We can choose an arc γ from $E \cap D_{J_1}$ which is outermost in E. So γ cuts a disk E' from E such that $E' \cap D_{J_1} = \gamma$. Let $E_i = E' \cup_\gamma D_i$, where D_i is obtained by cutting D_{J_1} along γ, $i = 1, 2$. By Lemma 2.6 in [21], both D_1 and D_2 are essential in V''. So both ∂D_1 and ∂D_2 lie in $\pi_F(\partial E)$. By the same argument as above, whether J_2 is separating or not,

$$d_{C(F)}(\pi_F(\partial E), J_2) \leq 1.$$

So

$$\begin{aligned} d_{C(F)}(\pi_F(\partial D), J_2) &\leq diam_{C(F)}(\partial D) + d_{C(F)}(\pi_F(\partial D), \pi_F(\partial E)) \\ &\quad + diam_{C(F)}(\pi_F(\partial E)) + d_{C(F)}(\pi_F(\partial E), J_2) \\ &\leq 2 + 2 + 2 + d_{C(F)}((\partial D_1, J_2) \\ &\leq 7. \end{aligned}$$

So

$$d_{C(F)}(\pi_F(\mathscr{D}), J_2) \leq d_{C(F)}(\pi_F(\partial D), J_2) \leq 7,$$

a contradiction. Thus the theorem holds.

(2) J_1 is separating in ∂H_n.

In this case, D_{J_1} cuts V' into two compression bodies V_1' and V_2'. Assume J_2 lies in $\partial_- V_1'$. Let V'' be the compression body obtained by attaching a 2-handle to V_1' along J_2. Let γ be an outermost arc of $E \cap D_{J_1}$ in E which cuts a disk E_0 from E such that $E_0 \cap D_{J_1} = \gamma$. Assume γ cuts D_{J_1} into D_1 and D_2. Let $E_i = E_0 \cup_\gamma D_i$ where $i = 1, 2$. Then E_0 lies in V_1' and both E_1 and E_2 are essential disks in V_1'. Otherwise, both E_1 and E_2 lie in V_2' or are parallel to the boundary of V_1'. In the first case, since V_2' is a trivial compression body, E_i is parallel to the boundary of V_2'. So in both cases, we can isotope E to reduce $|E \cap D_{J_1}|$, a contradiction. Since both E_1 and E_2 are essential disks in V_1', both ∂E_1 and ∂E_2 lie in $\pi_F(\partial E)$. By Lemma 3.1, $d_{C(F)}(\partial E_1, J_2) \leq 1$. Thus $d_{C(F)}(\pi_F(\partial E), J_2) \leq 1$. Thus by Lemma 3.2,

$$\begin{aligned} d_{C(F)}(\pi_F(\partial D), J_2) &\leq diam_{C(F)}(\pi_F(\partial D)) + d_{C(F)}(\pi_F(\partial D), \pi_F(\partial E)) \\ &\quad + diam_{C(F)}(\pi_F(\partial E)) + d_{C(F)}(\pi_F(\partial E_1), J_2) \\ &\leq 7. \end{aligned}$$

So $d_{C(F)}(\pi_F(\mathscr{D}), J_2) \leq d_{C(F)}(\pi_F(\partial D), J_2) \leq 7$, a contradiction. This completes the proof of the theorem. □

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School of Mathematical Science, Dalian University of Technology, Dalian 116024, People's Republic of China

Current address: School of Mathematics, Liaoning Normal University, Dalian 116029, People's Republic of China

E-mail address: `liang_liang@aliyun.com`

School of Mathematical Science, Dalian University of Technology, Dalian 116024, People's Republic of China

E-mail address: `fclei@dlut.edu.cn`

School of Mathematical Science, Dalian University of Technology, Dalian 116024, People's Republic of China

E-mail address: `dutlfl@163.com`