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THE HAUSDORFF DIMENSION ESTIMATION FOR AN ERGODIC HYPERBOLIC MEASURE OF C^1-DIFFEOMORPHISM

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Abstract. This paper provides the Hausdorff dimension estimation for an ergodic hyperbolic measure of C^1-diffeomorphism on an m-dimensional compact Riemannian manifold with the assumption that its Oseledet's splitting is a dominated splitting.

1. Introduction

This paper is interested in the Hausdorff dimension of an ergodic hyperbolic invariant measure of C^1-diffeomorphism. We relate it to the entropy and Lyapunov exponents of the map which has dominated Oseledet's splitting. First we introduce some notation. Let M be an m-dimensional compact Riemannian manifold, and $\dim_H Z$ be the Hausdorff dimension of the subset $Z \subseteq M$ (see page 112 in [12]). Hausdorff dimension measures the size of a set, and is useful for distinguishing between sets of Lebesgue measure zero. Given a measure μ on $\Lambda \subseteq M$, *the Hausdorff dimension of* μ is defined by

$$\dim_H \mu = \inf\{\dim_H Z : \ Z \subseteq \Lambda \text{ and } \mu(Z) = 1\}.$$

The lower and upper pointwise dimensions of μ at the point $x \in \Lambda$ are defined by

$$\underline{d}_\mu(x) = \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \quad \text{and} \quad \overline{d}_\mu(x) = \limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}$$

where $B(x,r)$ denotes the ball of radius r centered at x. We first recall two basic properties relating these quantities with the Hausdorff dimension of subsets of M ([11, Theorems 7.1 and 7.2])

(i) if $\underline{d}_\mu(x) \geq \alpha$ for μ almost every $x \in \Lambda$, then $\dim_H \mu \geq \alpha$;
(ii) if $\underline{d}_\mu(x) \leq \alpha$ for every $x \in Z \subseteq \Lambda$, then $\dim_H Z \leq \alpha$.

The quantity $\dim_H \mu$ can indeed be defined in terms of a local quantity. Namely, [4]

$$\dim_H \mu = \text{ess sup}\{\underline{d}_\mu(x) : \ x \in \Lambda\},$$

here the essential supremum taken with respect to μ. In particular, if there exists a number d such that

$$\lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} = d$$

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for μ almost every $x \in \Lambda$, then $\dim_H \mu = d$. This criterion was established by Young in [12]. The limit when it exists is called *the pointwise dimension* of μ at x.

Let $f : M \to M$ be a C^1-diffeomorphism with a compact f-invariant locally maximal hyperbolic set $\Lambda \subseteq M$ and μ be an f-invariant probability on Λ. For each $x \in \Lambda$ and $v \in T_xM$ we consider the Lyapunov exponent

$$\lambda_\mu(x, v) = \limsup_{n\to\infty} \frac{1}{n} \log \|Df^n(x)(v)\|.$$

We denote $\lambda_\mu(f, x)$ as *the Lyapunov exponent* of x. By the work of Brin and Katok in [5] for μ almost every $x \in \Lambda$ there exists the limit

$$h_\mu(f, x) = \lim_{\varepsilon\to 0} \lim_{n\to\infty} -\frac{1}{n} \log \mu(B(x, \varepsilon, n, m)),$$

where $B(x, \varepsilon, n, m) := \{y \in \Lambda : \ d(f^i x, f^i y) < \varepsilon \text{ for } i = -m+1, \cdots, -1, 0, 1, \cdots, n-1\}$. The number $h_\mu(f, x)$ is called *the local entropy* of μ at the point x.

Let μ be a compactly supported finite measure invariant under a $C^{1+\alpha}$-diffeomorphism f. We say μ is *hyperbolic* if the Lyapunov exponents of μ*.a.e.* $x \in M$ are all non-zero. If μ is hyperbolic, Ledrappier and Young [9], and Barreira, Pesin and Schmeling [3] have proved that the pointwise dimension exists almost everywhere. In the two dimensional case, Young [12] showed that if μ is an ergodic invariant hyperbolic measure, then

$$\dim_H \mu = h_\mu(f)\left(\frac{1}{\lambda_u(\mu)} - \frac{1}{\lambda_s(\mu)}\right),$$

where $h_\mu(f)$ is the Kolmogorov-Sinai entropy of f with respect to μ, and $\lambda_u(\mu)$ and $\lambda_s(\mu)$ are respectively the positive and negative values of the Lyapunov exponent of μ. Let Λ be a compact f-invariant locally maximal hyperbolic set on which f is conformal, and let μ be an f-invariant probability measure on Λ. Barreira and Wolf [4] have proved that for μ almost every $x \in \Lambda$

$$\underline{d}_\mu(x) = \overline{d}_\mu(x) = h_\mu(f, x)\ \left(\frac{1}{\lambda_u(f, x)} - \frac{1}{\lambda_s(f, x)}\right).$$

This paper is mainly motivated by the existence of the pointwise dimension of the invariant measure μ for C^1-diffeomorphism. In this paper, we investigate the pointwise dimension of the ergodic hyperbolic measure μ for C^1-diffeomorphism. In fact, it may not exist with the assumption that the Oseledet's splitting of the hyperbolic measure is dominated. But we give the relations between the Hausdorff dimension of the measure and other invariants of the system, such as the local entropy and Lyapunov exponents. In [2], Ban, Cao and Hu introduced the concept of average conformal repeller. In this paper, a hyperbolic set Λ will be called *average conformal* if it has two unique Lyapunov exponents, one positive and one negative. That is, for any ergodic invariant measure μ, the Lyapunov exponents are $\lambda_1(\mu) = \lambda_2(\mu) = ... = \lambda_k(\mu) > 0$, $\lambda_{k+1}(\mu) = \lambda_{k+2}(\mu) = ... = \lambda_m(\mu) < 0$ with $0 < k < m$. The pointwise dimension does exist for the average conformal hyperbolic set (see Corollaries 1 and 2). Here if $k = m$, such as an average conformal repeller, please see Theorem A in [8]. For more information about dimension of ergodic measure on an average conformal repeller, see also [7]. In our paper, we always assume that a C^1-diffeomorphism on an m-dimensional compact Riemannian manifold M satisfying its Oseledet's splitting is a dominated splitting. Our main result is as follows.

Theorem 1.1. *Let $f : M \to M$ be a C^1-diffeomorphism of an m-dimensional compact Riemannian manifold M. Suppose μ is an ergodic hyperbolic probability measure on M and its Oseledet's splitting $E^u \oplus E^s$ is a dominated splitting. Assume that $\lambda_1(\mu) > \cdots > \lambda_k(\mu) > 0 > \lambda_{k+1}(\mu) > \cdots > \lambda_p(\mu)$, where $0 < k < p$, $0 < p \leq m$. Then*

$$h_\mu(f)(\frac{1}{\lambda_1(\mu)} - \frac{1}{\lambda_p(\mu)}) \leq \dim_H \mu \leq h_\mu(f)(\frac{1}{\lambda_k(\mu)} - \frac{1}{\lambda_{k+1}(\mu)}).$$

Instead of Pesin theory, we just use several definitions and some related properties. Particularly, we find "fake invariant foliations" $\mathcal{F}^u$, $\mathcal{F}^s$ to make our proof work in general. Moreover, if the positive Lyapunov exponents and the negative Lyapunov exponents stay the same respectively, the following result is a direct corollary of the previous theorem.

Corollary 1. *Let $f : M \to M$ be a C^1-diffeomorphism of an m-dimensional compact Riemannian manifold M. Suppose μ is an ergodic hyperbolic probability measure on M and its Oseledet's splitting $E^u \oplus E^s$ is a dominated splitting. Assume that $\lambda_u(\mu) := \lambda_1(\mu) = \cdots = \lambda_k(\mu) > 0$ and $\lambda_s(\mu) := \lambda_{k+1}(\mu) = \cdots = \lambda_m(\mu) < 0$, where $0 < k < m$. Then*

$$\dim_H \mu = h_\mu(f)(\frac{1}{\lambda_u(\mu)} - \frac{1}{\lambda_s(\mu)}).$$

In particular, for an average conformal hyperbolic set $\Lambda \subseteq M$ of C^1-diffeomorphism $f : M \to M$, we obtain:

Corollary 2. *Let $f : M \to M$ be a C^1-diffeomorphism of an m-dimensional compact Riemannian manifold M. Suppose $\Lambda \subseteq M$ is an average conformal hyperbolic set of f. Then for every invariant measure μ supported on Λ,*

$$\dim_H \mu = \text{ess sup}\{h_\mu(f,x)(\frac{1}{\lambda_u(f,x)} - \frac{1}{\lambda_s(f,x)})\}.$$

The remainder of this paper is organized as follows. Section 2 gives some notation and preliminaries. Section 3 provides the proof of the main result.

2. Preliminaries

Let M be an m-dimensional compact Riemannian manifold, and f be a C^1-diffeomorphism from M to itself. We denote

$$\|Df(x)\| = \sup_{u \in T_x M} \frac{|Df(x)u|}{|u|}, \quad m(Df(x)) = \inf_{0 \neq u \in T_x M} \frac{|Df(x)u|}{|u|}.$$

Let $\Lambda \subseteq M$ be an f-invariant compact subset. A Df-invariant splitting $T_\Lambda M = E \oplus F$ of the tangent bundle over Λ is *dominated* if there exists $N \geq 1$ such that given any $x \in \Lambda$, any unitary vectors $v \in E(x)$ and $w \in F(x)$, then

$$\|D_x f^N(v)\| \leq \frac{1}{2}\|D_x f^N(w)\|.$$

Given a hyperbolic measure μ, then its hyperbolic Oseledet's splitting, defined at $\mu.a.e.x$, is the Df-invariant splitting given by

$$E^u(x) = \bigoplus_{\lambda_\mu(f,x)>0} E(\lambda_\mu(f,x)) \quad \text{and} \quad E^s(x) = \bigoplus_{\lambda_\mu(f,x)<0} E(\lambda_\mu(f,x))$$

where $E(\lambda_\mu(f,x))$ is the Lyapunov space corresponding to the Lyapunov exponent $\lambda_\mu(f,x)$ at x.

Lemma 2.1 (Lemma 8.4 in [1]). *Let f be a C^1-diffeomorphism, μ be an ergodic invariant probability measure, and $E \subseteq T_{\operatorname{supp}\mu}M$ be a Df-invariant continuous subbundle defined over the supported set* $\operatorname{supp}\mu$ *of μ. Let $\lambda_E^{\max}$ be the upper Lyapunov exponent in E of the measure μ.*

Then, for any $\varepsilon > 0$, there exists an integer $N_1(\varepsilon)$ such that, for μ-almost every point $x \in M$ and any $N \geq N_1(\varepsilon)$, the Birkhoff averages

$$\frac{1}{kN}\sum_{l=0}^{k-1}\log\|Df^N|_{E(f^{lN}(x))}\|$$

converge towards a number contained in $[\lambda_E^{\max}, \lambda_E^{\max}+\varepsilon)$, where k goes to $+\infty$.

The following lemma is analogous.

Lemma 2.2. *Let f be a C^1-diffeomorphism, μ be an ergodic invariant probability measure, and $E \subseteq T_{\operatorname{supp}\mu}M$ be a Df-invariant continuous subbundle defined over suppμ. Let $\lambda_E^{\min}$ be the lower Lyapunov exponent in E of the measure μ.*

Then, for any $\varepsilon > 0$, there exists an integer $N_2(\varepsilon)$ such that, for μ-almost every point $x \in M$ and any $N \geq N_2(\varepsilon)$, the Birkhoff averages

$$\frac{1}{kN}\sum_{l=0}^{k-1}\log m(Df^N|_{E(f^{lN}(x))})$$

converge towards a number contained in $(\lambda_E^{\min}-\varepsilon, \lambda_E^{\min}]$, where k goes to $+\infty$.

Proof. It is slight modifications of the proof of Lemma 2.1. We omit it here. □

Lemma 2.3 (Proposition 3.1 in [6]). *Let $f: M \to M$ be a C^1-diffeomorphism of an m-dimensional compact Riemannian manifold M and $\Lambda \subseteq M$ be an f-invariant compact set such that the tangent space over Λ admits a dominated splitting $T_\Lambda M = E^1 \oplus E^2$. For every $\zeta > 0$, there exist constants $\rho > r_0 > 0$ such that the neighborhood $B(x,\rho)$ of every $x \in \Lambda$ admits foliation $\mathcal{F}_x^1$, $\mathcal{F}_x^2$ with the following properties, for each $* \in \{1,2\}$:*

1. *For each $y \in B(x,\rho)$, the leaf $\mathcal{F}_x^*(x)$ is C^1 and $T_y\mathcal{F}_x^*(y)$ lies in a cone of width ζ about $E^*(y)$.*
2. *For each $y \in B(x,r_0)$, $f(\mathcal{F}_x^*(y,r_0)) \subseteq \mathcal{F}_{fx}^*(fy)$ and $f^{-1}(\mathcal{F}_x^*(y,r_0)) \subseteq \mathcal{F}_{f^{-1}x}^*(f^{-1}y)$.*

Remark 1. We call these $\mathcal{F}^1$, $\mathcal{F}^2$ "fake invariant foliations".

3. The proof of Theorem 1.1

Proof. Since Oseledet's splitting $E^u \oplus E^s$ is a dominated splitting and the splitting $E^u \oplus E^s$ extends to a dominated splitting over suppμ, $T_{\operatorname{supp}\mu}M = E^u \oplus E^s$ is a dominated splitting. Therefore the angles between $E^u(x)$ and $E^s(x)$ are uniformly bounded from zero for every $x \in$ suppμ.

Since the measure μ is ergodic, Oseledet's Theorem and Brin-Katok's Theorem for local entropy in [5] say that for μ.a.e. $x \in M$

$$\lambda_k(f,x) := \lim_{n\to\infty}\frac{1}{n}\log m(Df^n|_{E^u(x)}) = \lambda_k(\mu), \tag{3.1}$$

$$\lambda_{k+1}(f,x) := \lim_{n\to\infty} \frac{1}{n} \log \|Df^n|_{E^s(x)}\| = \lambda_{k+1}(\mu), \tag{3.2}$$

$$h_\mu(f,x) := \lim_{\varepsilon\to 0} \lim_{n,m\to\infty} \frac{-\log \mu(B(x,\varepsilon,n,m))}{n+m} = h_\mu(f). \tag{3.3}$$

Given any $\eta > 0$, there exists a subset $\Lambda_\eta \subseteq M$ such that $\mu(\Lambda_\eta) > 1-\eta$ and the convergence of (3.1), (3.2) and (3.3) in Λ_η is uniform. For any $r > 0$, $\exists\ \varepsilon_0(r) > 0$, such that $\forall\ 0 < \varepsilon < \varepsilon_0(r)$,

$$h_\mu(f) - r < \lim_{n,m\to\infty} \frac{-\log \mu(B(x,\varepsilon,n,m))}{n+m} < h_\mu(f) + r$$

for all $x \in \Lambda_\eta$. Given small δ with $0 < \delta < \varepsilon_0(r)$, $\lambda_k(\mu) - \delta > 0$ and $\lambda_{k+1}(\mu) + \delta < 0$, for any $x \in \Lambda_\eta$, $\exists\ N = N_3(\delta)$ such that for any $n \geq N$

$$m(Df^n|_{E^u(x)}) > e^{n(\lambda_k(\mu)-\delta)} > 1 \quad \text{and} \quad \|Df^n|_{E^s(x)}\| < e^{n(\lambda_{k+1}(\mu)+\delta)} < 1.$$

By Lemma 2.1, for μ almost every point $x \in M$ and any $N \geq N_1(\delta)$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{l=0}^{n-1} \log \|Df^N|_{E^u(f^{lN}x)}\| < (\lambda_1(\mu) + \delta)N, \tag{3.4}$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{l=0}^{n-1} \log \|Df^{-N}|_{E^s(f^{-lN}x)}\| < (-\lambda_p(\mu) + \delta)N. \tag{3.5}$$

By Lemma 2.2, for μ almost every point $x \in M$ and any $N \geq N_2(\delta)$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{l=0}^{n-1} \log m(Df^N|_{E^u(f^{lN}x)}) > (\lambda_k(\mu) - \delta)N, \tag{3.6}$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{l=0}^{n-1} \log m(Df^{-N}|_{E^s(f^{-lN}x)}) > (-\lambda_{k+1}(\mu) - \delta)N. \tag{3.7}$$

Fix $L_0 > \max\{N_1(\delta),\ N_2(\delta),\ N_3(\delta)\}$ with $(-\lambda_k(\mu)+\delta)L_0 + 2\delta < 0$ and $(\lambda_{k+1}(\mu) + \delta)L_0 + 2\delta < 0$. Then let $\zeta > 0$ and $r_* > 0$ be sufficiently small so that

$$e^{-\delta} \leq \frac{\|Df^{L_0}(x)u\|}{\|Df^{L_0}(y)v\|} \leq e^\delta \quad \text{and} \quad e^{-\delta} \leq \frac{\|Df^{-L_0}(x)u\|}{\|Df^{-L_0}(y)v\|} \leq e^\delta \tag{3.8}$$

whenever $d(x,y) \leq r_*$ and $\angle(u,v) \leq 3\zeta$. By Lemma 2.3, the relation (3.8) and taking $\varepsilon < \min\{r_*, r_0, \varepsilon_0(r)\}$ small enough,

$$e^{-\delta} \leq \frac{\|Df^{-L_0}|_{T_z\mathcal{F}^u_{f^{iL_0}x}(z)}\|}{\|Df^{-L_0}|_{T_{f^{iL_0}x}\mathcal{F}^u_{f^{iL_0}x}(f^{iL_0}x)}\|} \leq e^\delta \text{ for every } z \in \mathcal{F}^u_{f^{iL_0}x}(f^{iL_0}x,\varepsilon) \text{ and } i \in \mathbb{Z}, \tag{3.9}$$

$$e^{-\delta} \leq \frac{\|Df^{L_0}|_{T_z\mathcal{F}^s_{f^{iL_0}x}(z)}\|}{\|Df^{L_0}|_{T_{f^{iL_0}x}\mathcal{F}^s_{f^{iL_0}x}(f^{iL_0}x)}\|} \leq e^\delta \text{ for every } z \in \mathcal{F}^s_{f^{iL_0}x}(f^{iL_0}x,\varepsilon) \text{ and } i \in \mathbb{Z}. \tag{3.10}$$

Let $\Lambda'_\eta = \{x \in \Lambda_\eta :\ (3.4),\ (3.5),\ (3.6),\ (3.7) \text{ hold with } N = L_0\}$. It is easy to see $\mu(\Lambda_\eta) = \mu(\Lambda'_\eta)$. Given any $x \in \Lambda'_\eta$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{l=0}^{n-1} \log \|Df^{L_0}|_{E^u(f^{lL_0}x)}\| < (\lambda_1(\mu) + \delta)L_0,$$
$$\lim_{n\to\infty} \frac{1}{n} \sum_{l=0}^{n-1} \log \|Df^{-L_0}|_{E^s(f^{-lL_0}x)}\| < (-\lambda_p(\mu) + \delta)L_0.$$

Then $\exists\ N > 0,\ M > 0$ such that for any $n \geq N,\ m \geq M$ we have

$$\sum_{l=0}^{n-1} \log \|Df^{L_0}|_{E^u(f^{lL_0}x)}\| < (\lambda_1(\mu) + \delta)L_0 n,$$

$$\sum_{l=0}^{m-1} \log \|Df^{-L_0}|_{E^s(f^{-lL_0}x)}\| < (-\lambda_p(\mu) + \delta)L_0 m.$$

For each $n \geq N$, there exists $m = m(n)$ such that

$$e^{[(-\lambda_p(\mu)+\delta)L_0+2\delta]m} D < e^{[(\lambda_1(\mu)+\delta)L_0+2\delta]n} C \leq e^{[(-\lambda_p(\mu)+\delta)L_0+2\delta](m+1)} D \tag{3.11}$$

where $C = \sup\{Df^i|_{T_x\mathcal{F}^u_x} :\ i = 1, \cdots, L_0 - 1\}$, $D = \sup\{Df^{-i}|_{T_x\mathcal{F}^s_x} :\ i = 1, \cdots, L_0 - 1\}$. If $n \to \infty$, then $m \to \infty$. Therefore for large enough $n \geq N$, we have $m(n) \geq M$.

For each $x \in \Lambda'_\eta$ and $n \geq N$ large enough such that $m = m(n) \geq M$,

$$B(x, e^{-[(\lambda_1(\mu)+\delta)L_0+2\delta]n} C^{-1}\varepsilon) \subseteq B(x, \varepsilon, nL_0, mL_0).$$

Let $r_n = e^{-[(\lambda_1(\mu)+\delta)L_0+2\delta]n} C^{-1}\varepsilon$. Then $\lim\limits_{n\to\infty} r_n = 0$ and $\lim\limits_{n\to\infty} \frac{\log r_{n+1}}{\log r_n} = 1$. In fact, $\forall\ y \in B(x, r_n)$, for $i = 0, 1, \cdots, nL_0$, $\exists\ \xi \in B(x, r_n)$ such that

$$\begin{aligned}
&d_{\mathcal{F}^u}(f^i x, f^i y) \\
&\quad \leq \|Df^i|_{T_\xi\mathcal{F}^u_x(\xi)}\| d_{\mathcal{F}^u}(x, y) \\
&\quad \leq \Big(\prod_{l=0}^{p_i-1} \|Df^{L_0}|_{T_{f^{lL_0}\xi}\mathcal{F}^u_{f^{lL_0}x}(f^{lL_0}\xi)}\|\Big) \cdot \|Df^{q_i}|_{T_{f^{p_iL_0}\xi}\mathcal{F}^u_{f^{p_iL_0}x}(f^{p_iL_0}\xi)}\| d_{\mathcal{F}^u}(x, y) \\
&\qquad\qquad\qquad\qquad (\text{where } i = p_iL_0 + q_i,\ 0 \leq q_i < L_0) \\
&\quad \leq \Big(\prod_{l=0}^{p_i-1} \|Df^{L_0}|_{T_{f^{lL_0}x}\mathcal{F}^u_{f^{lL_0}x}(f^{lL_0}x)}\|\Big) e^{p_i\delta} C d_{\mathcal{F}^u}(x, y) \\
&\quad \leq \Big(\prod_{l=0}^{p_i-1} \|Df^{L_0}|_{E^u(f^{lL_0}x)}\|\Big) e^{2p_i\delta} C d_{\mathcal{F}^u}(x, y) \\
&\quad \leq \Big(\prod_{l=0}^{n-1} \|Df^{L_0}|_{E^u(f^{lL_0}x)}\|\Big) e^{2n\delta} C d_{\mathcal{F}^u}(x, y) \\
&\quad \leq e^{[(\lambda_1(\mu)+\delta)L_0+2\delta]n} C d_{\mathcal{F}^u}(x, y) \\
&\quad \leq \varepsilon.
\end{aligned}$$

And for $j = 0, 1, \cdots, mL_0$, $\exists\ \eta \in B(x, r_n)$ such that

$$\begin{aligned}
d_{\mathcal{F}^s}(f^{-j}x, f^{-j}y) &\le \|Df^{-j}|_{T_\eta \mathcal{F}^s_x(\eta)}\| d_{\mathcal{F}^s}(x,y) \\
&\le (\prod_{l=0}^{p_j-1} \|Df^{-L_0}|_{T_{f^{-lL_0}\eta}\mathcal{F}^s_{f^{-lL_0}x}(f^{-lL_0}\eta)}\|) \\
&\quad \times \|Df^{-q_j}|_{T_{f^{-p_jL_0}\eta}\mathcal{F}^s_{f^{-p_jL_0}x}(f^{-p_jL_0}\eta)}\| d_{\mathcal{F}^s}(x,y) \\
&\qquad\qquad (\text{where } j = p_jL_0 + q_j,\ 0 \le q_j < L_0) \\
&\le (\prod_{l=0}^{p_j-1} \|Df^{-L_0}|_{T_{f^{-lL_0}x}\mathcal{F}^s_{f^{-lL_0}x}(f^{-lL_0}x)}\|) e^{p_j\delta} D d_{\mathcal{F}^s}(x,y) \\
&\le (\prod_{l=0}^{p_j-1} \|Df^{-L_0}|_{E^s(f^{-lL_0}x)}\|) e^{2p_j\delta} D d_{\mathcal{F}^s}(x,y) \\
&\le (\prod_{l=0}^{m-1} \|Df^{-L_0}|_{E^s(f^{-lL_0}x)}\|) e^{2m\delta} D d_{\mathcal{F}^s}(x,y) \\
&\le e^{[(-\lambda_p(\mu)+\delta)L_0+2\delta]m} D d_{\mathcal{F}^s}(x,y) \\
&\le e^{[(\lambda_1(\mu)+\delta)L_0+2\delta]n} C d_{\mathcal{F}^s}(x,y) \quad (\text{by } (3.11)) \\
&\le \varepsilon.
\end{aligned}$$

Note that

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{nL_0} \log r_n &= \lim_{n\to\infty} \frac{-[(\lambda_1(\mu)+\delta)L_0 + 2\delta]n - \log C + \log \varepsilon}{nL_0} \\
&= -\lambda_1(\mu) - \delta - \frac{2\delta}{L_0},
\end{aligned}$$

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{mL_0} \log r_n &\ge \lim_{n\to\infty} \frac{-[(-\lambda_p(\mu)+\delta)L_0 + 2\delta](m+1) - \log D + \log \varepsilon}{mL_0} \\
&= \lambda_p(\mu) - \delta - \frac{2\delta}{L_0}.
\end{aligned}$$

Therefore

$$\liminf_{n\to\infty} -\frac{\log r_n}{(n+m)L_0} \frac{\log \mu(B(x, r_n))}{\log r_n} \ge \lim_{n\to\infty} \frac{-\log \mu(B(x, \varepsilon, nL_0, mL_0))}{(n+m)L_0}$$

for every $x \in \Lambda'_\eta$. So

$$\underline{d}_\mu(x) \ge (h_\mu(f) - r)\left(\frac{1}{\lambda_1(\mu) + \delta + \frac{2\delta}{L_0}} - \frac{1}{\lambda_p(\mu) - \delta - \frac{2\delta}{L_0}}\right)$$

for every $x \in \Lambda'_\eta$. Let $r \to 0$; then $\delta \to 0$. Thus the arbitrariness of r implies

$$\underline{d}_\mu(x) \ge h_\mu(f)\left(\frac{1}{\lambda_1(\mu)} - \frac{1}{\lambda_p(\mu)}\right)$$

for every $x \in \Lambda'_\eta$. Let $\eta \to 0$,

$$\underline{d}_\mu(x) \ge h_\mu(f)\left(\frac{1}{\lambda_1(\mu)} - \frac{1}{\lambda_p(\mu)}\right)$$

for μ*.a.e.* $x \in M$.

On the other hand, we need the following lemma.

Lemma 3.1 (Pliss [10]). *Given $a_* \le c_2 < c_1$ there exists $\theta = \dfrac{c_1 - c_2}{c_1 - a_*}$ such that, given any real numbers $a_1, \cdots, a_N$ with*

$$\sum_{i=1}^{N} a_i \le c_2 N \quad \text{and} \quad a_i \ge a_* \text{ for every } i,$$

there exist $l > N\theta$ and $1 \le n_1 < \cdots < n_l \le N$ such that

$$\sum_{i=n+1}^{n_j} a_i \le c_1(n_j - n) \quad \text{for all} \quad 0 \le n < n_j \text{ and } j = 1, \cdots, l.$$

By Lemma 2.2, for μ almost every point $x \in M$, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log m(Df^{L_0}|_{E^u(f^{iL_0}x)}) > (\lambda_k(\mu) - \delta)L_0.$$

It follows that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \log \|Df^{-L_0}|_{E^u(f^{iL_0}x)}\| < (-\lambda_k(\mu) + \delta)L_0.$$

Take $a_* = \min\{\log \|Df^{-L_0}|_{E^u(x)}\| : x \in M\}$ and note that $a_* \le (-\lambda_k(\mu)+\delta)L_0$. Let $(-\lambda_k(\mu) + \delta)L_0 < c_2 < c_1 = (-\lambda_k(\mu) + \delta)L_0 + \delta$. Applying Lemma 3.1 to $a_i = \log \|Df^{-L_0}|_{E^u(f^{iL_0}x)}\|$ and large values of N, we find an infinite sequence $1 \le n_1 < n_2 < \cdots < n_j < \cdots$ such that

$$\sum_{i=n+1}^{n_j} \log \|Df^{-L_0}|_{E^u(f^{iL_0}x)}\| \le [(-\lambda_k(\mu) + \delta)L_0 + \delta](n_j - n) \tag{3.12}$$

for every $0 \le n < n_j$ and $\forall j \ge 1$. From (3.12) and (3.10) one gets that

$$f^{(n-n_j)L_0}(\mathcal{F}^u_{f^{n_jL_0}x}(f^{n_jL_0}x, \varepsilon)) \subseteq \mathcal{F}^u_{f^{nL_0}x}(f^{nL_0}x, e^{(n_j-n)[(-\lambda_k(\mu)+\delta)L_0+3\delta]}\varepsilon) \tag{3.13}$$

for every $0 \le n < n_j$ and, in particular,

$$f^{-n_jL_0}(\mathcal{F}^u_{f^{n_jL_0}x}(f^{n_jL_0}x, \varepsilon)) \subseteq \mathcal{F}^u_x(x, e^{n_j[(-\lambda_k(\mu)+\delta)L_0+3\delta]}\varepsilon). \tag{3.14}$$

Consider f^{-L_0}; similarly there exists an infinite sequence $1 \le m_1 < m_2 < \cdots < m_j < \cdots$ such that

$$f^{m_jL_0}(\mathcal{F}^s_{f^{-m_jL_0}}(f^{-m_jL_0}x, \varepsilon)) \subseteq \mathcal{F}^s_x(x, e^{m_j[(\lambda_{k+1}(\mu)+\delta)L_0+3\delta]}\varepsilon). \tag{3.15}$$

For every $x \in \Lambda'_\eta$ and $n \in \mathbb{N}$, there exist n_i and m_j such that

$$B(x, \varepsilon, n_iL_0, m_jL_o) \subseteq B(x, R_n)$$

where $R_n = e^{n[(-\lambda_k(\mu)+\delta)L_0+3\delta]}\varepsilon$, and n_i, m_j satisfy

$$e^{n_i[(-\lambda_k(\mu)+\delta)L_0+3\delta]} \le \frac{R_n}{\varepsilon} < e^{(n_i-1)[(-\lambda_k(\mu)+\delta)L_0+3\delta]},$$

$$e^{m_j[(\lambda_{k+1}(\mu)+\delta)L_0+3\delta]} \le \frac{R_n}{\varepsilon} < e^{(m_j-1)[(\lambda_{k+1}(\mu)+\delta)L_0+3\delta]}.$$

It is easy to see $n_i \to +\infty$, $m_j \to +\infty$ if $n \to +\infty$. In fact, for any $y \in B(x, \varepsilon, n_iL_0, m_jL_0)$,

$$d_{\mathcal{F}^u}(f^{n_iL_0}x, f^{n_iL_0}y) < \varepsilon \quad \text{and} \quad d_{\mathcal{F}^s}(f^{-m_jL_0}x, f^{-m_jL_0}y) < \varepsilon.$$

By (3.14) and (3.15), we get

$$d_{\mathcal{F}^u}(x,y) < e^{n_i[(-\lambda_k(\mu)+\delta)L_0+3\delta]}\varepsilon \le R_n,$$

$$d_{\mathcal{F}^s}(x,y) < e^{m_j[(\lambda_{k+1}(\mu)+\delta)L_0+3\delta]}\varepsilon \le R_n.$$

We know that $\lim\limits_{n\to\infty} R_n = 0$, $\lim\limits_{n\to\infty} \frac{\log R_{n+1}}{\log R_n} = 1$ and

$$\lim_{n\to\infty} \frac{\log R_n}{n_i L_0} \le \lim_{n\to\infty} \frac{(n_i-1)[(-\lambda_k(\mu)+\delta)L_0+3\delta]+\log\varepsilon}{n_i L_0} = -\lambda_k(\mu)+\delta+\frac{3\delta}{L_0},$$

$$\lim_{n\to\infty} \frac{\log R_n}{m_j L_0} \le \lim_{n\to\infty} \frac{(m_j-1)[(\lambda_{k+1}(\mu)+\delta)L_0+3\delta]+\log\varepsilon}{m_j L_0} = \lambda_{k+1}(\mu)+\delta+\frac{3\delta}{L_0}.$$

Therefore

$$\limsup_{n\to\infty} \frac{(n_i+m_j)L_0}{-\log R_n} \; \frac{-\log\mu(B(x,\varepsilon,n_iL_0,m_jL_0))}{(n_i+m_j)L_0} \ge \limsup_{n\to\infty} \frac{\log\mu(B(x,R_n))}{\log R_n}.$$

Thus

$$\overline{d}_\mu(x) \le (h_\mu(f)+r)\Big(\frac{1}{\lambda_k(\mu)-\delta-\frac{3\delta}{L_0}} - \frac{1}{\lambda_{k+1}(\mu)+\delta+\frac{3\delta}{L_0}}\Big)$$

for every $x \in \Lambda'_\eta$. The arbitrariness of r, and $\delta \to 0$ if $r \to 0$, imply that

$$\overline{d}_\mu(x) \le h_\mu(f)\Big(\frac{1}{\lambda_k(\mu)} - \frac{1}{\lambda_{k+1}(\mu)}\Big)$$

for every $x \in \Lambda'_\eta$. Let $\eta \to 0$,

$$\overline{d}_\mu(x) \le h_\mu(f)\Big(\frac{1}{\lambda_k(\mu)} - \frac{1}{\lambda_{k+1}(\mu)}\Big)$$

for $\mu.a.e.$ $x \in M$. Thus

$$h_\mu(f)\Big(\frac{1}{\lambda_1(\mu)} - \frac{1}{\lambda_p(\mu)}\Big) \le \dim_H \mu \le h_\mu(f)\Big(\frac{1}{\lambda_k(\mu)} - \frac{1}{\lambda_{k+1}(\mu)}\Big).$$

□

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