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TENSOR PRODUCT SURFACES AND LINEAR SYZYGIES

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ABSTRACT. Let $U \subseteq H^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a,b))$ be a basepoint free four-dimensional vector space, with $a, b \geq 2$. The sections corresponding to U determine a regular map $\phi_U : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3$. We show that there can be at most one linear syzygy on the associated bigraded ideal $I_U \subseteq k[s,t;u,v]$. Existence of a linear syzygy, coupled with the assumption that U is basepoint free, implies the existence of an additional "special pair" of minimal first syzygies. Using results of Botbol, we show that these three syzygies are sufficient to determine the implicit equation of $\phi_U(\mathbb{P}^1 \times \mathbb{P}^1)$, and that $\mathrm{Sing}(\phi_U(\mathbb{P}^1 \times \mathbb{P}^1))$ contains a line.

1. Introduction

A tensor product surface is the image of a map $\mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3$. Such surfaces arise in geometric modeling, and it is often useful to find the implicit equation for the surface. Standard tools such as Gröbner bases and resultants tend to be slow, and the best current methods rely on Rees algebra techniques. The use of such methods was pioneered by the geometric modeling community (e.g. Sederberg-Chen [16], Sederberg-Goldman-Du [17], Sederberg-Saito-Qi-Klimaszewksi [18], Cox-Goldman-Zhang [9]). Further work on using Rees algebras in implicitization appears in Busé-Jouanolou [3], Busé-Chardin [4], Botbol [1], and Botbol-Dickenstein-Dohm [2]; see Cox [7] for a nice overview. A key tool is the approximation complex $\mathcal{Z}$, introduced by Herzog-Simis-Vasconcelos in [13], [14].

Definition 1.1. Let $I = \langle f_1, \ldots, f_n \rangle \subseteq R = k[x_1, \ldots, x_m]$, and let $K_i \subseteq \Lambda^i(R^n)$ be the kernel of the i^{th} Koszul differential on $\{f_1, \ldots, f_n\}$. The approximation complex $\mathcal{Z}$ is a complex of $S = R[y_1, \ldots, y_n]$ modules, with i^{th} term $\mathcal{Z}_i = S \otimes_R K_i$, and differential the Koszul differential on $\{y_1, \ldots, y_n\}$.

It follows from Definition 1.1 that $H_0(\mathcal{Z})$ is the symmetric algebra S_I on I, and that K_1 is $\mathrm{Syz}(I)$. For a fixed degree μ, the matrix representing the first differential d^1 in $\mathcal{Z}$ in degree μ is obtained by rewriting each syzygy on I

$$\sum_{i=1}^{n} g_i e_i \text{ with } \sum_{i=1}^{n} g_i f_i = 0$$

as $\sum_{i=1}^{n} g_i y_i$, but in terms of a choice of basis for R_μ, so that the entries of d^1_μ are elements of $k[y_1, \ldots, y_n]$. This generalizes to the bigraded setting. Let $R = k[s,t,u,v]$ be a bigraded ring over an algebraically closed field k, with s,t of degree

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$(1,0)$ and u, v of degree $(0,1)$. Note that the bidegree (a,b) graded piece $R_{a,b}$ of R corresponds exactly to the global sections $H^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a,b))$.

Definition 1.2. Suppose $U \subseteq R_{a,b}$ has basis $\{p_0, p_1, p_2, p_3\}$, such that the p_i have no common zeroes on $\mathbb{P}^1 \times \mathbb{P}^1$, and let I_U denote the ideal $\langle p_0, p_1, p_2, p_3\rangle \subset R$. Since the p_i have no common zeroes on $\mathbb{P}^1 \times \mathbb{P}^1$, they define a regular map $\phi_U : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3$, and we write X_U for $\phi_U(\mathbb{P}^1 \times \mathbb{P}^1) \subseteq \mathbb{P}^3$.

The assumption that U is basepoint free means that $\sqrt{I_U} = \langle s,t\rangle \cap \langle u,v\rangle$. In this setting, work of [2] gives conditions on μ so that the determinant of d^1_μ is a power of the implicit equation for X_U. Motivated by [8], in [15], Schenck-Seceleanu-Validashti show that for tensor product surfaces of bidegree $(2,1)$, the existence of a linear syzygy on I_U imposes very strong conditions on X_U. We show this is not specific to the bidegree $(2,1)$ case. Our main result is:

Theorem. *If $a, b \geq 2$ and U is basepoint free, then there is at most one linear first syzygy on I_U. A linear first syzygy gives rise to a special pair of additional first syzygies. These three syzygies determine the degree $(2a-1, b-1)$ component of the approximation complex $\mathcal{Z}$. By [1], the determinant of the resulting square matrix is a power of the implicit equation of X_U.*

Example 1.3. Suppose $(a,b) = (2,2)$, and
$$U = \mathrm{Span}\{t^2u^2 + s^2uv, t^2uv + s^2v^2, t^2v^2, s^2u^2\} \subseteq H^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2,2)),$$
which has a first syzygy of bidegree $(0,1)$. A computation shows that I_U has seven minimal first syzygies, in bidegrees
$$(0,1), (2,1), (2,1), (0,3), (2,2), (4,1), (6,0).$$
By Theorem 2.2, the three syzygies of bidegree $(0,1), (2,1), (2,1)$ are generated by the columns of
$$\begin{bmatrix} v & 0 & s^2u \\ -u & -t^2v & 0 \\ 0 & t^2u + s^2v & 0 \\ 0 & 0 & -t^2u - s^2v \end{bmatrix},$$
and the bidegree $(2a-1, b-1) = (3,1)$ component of the first differential in the approximation complex is
$$\begin{bmatrix} x_0 & 0 & 0 & 0 & x_2 & 0 & -x_3 & 0 \\ -x_1 & 0 & 0 & 0 & 0 & 0 & x_0 & 0 \\ 0 & x_0 & 0 & 0 & 0 & x_2 & 0 & -x_3 \\ 0 & -x_1 & 0 & 0 & 0 & 0 & 0 & x_0 \\ 0 & 0 & x_0 & 0 & -x_1 & 0 & 0 & 0 \\ 0 & 0 & -x_1 & 0 & x_2 & 0 & -x_3 & 0 \\ 0 & 0 & 0 & x_0 & 0 & -x_1 & 0 & 0 \\ 0 & 0 & 0 & -x_1 & 0 & x_2 & 0 & -x_3 \end{bmatrix}.$$
The determinant of this matrix is
$$(x_0^3x_2 + x_1^3x_3 - x_0^2x_1^2)^2.$$
By Corollary 2.3, this means the implicit equation defining X_U is $x_0^3x_2 + x_1^3x_3 - x_0^2x_1^2$, and ϕ_U is $2:1$ by Lemma 1.6. By Corollary 2.4, the codimension one singular locus of X_U contains $\mathbf{V}(x_0, x_1)$; in fact, in this case equality holds.

1.1. **Algebraic tools.** Two results from previous work will be especially useful; for additional background on approximation complexes and bigraded commutative algebra, see [15].

Lemma 1.4 ([15]). *If I_U has a linear first syzygy of bidegree $(0,1)$, then*

$$I_U = \langle pu, pv, p_2, p_3 \rangle,$$

where p is homogeneous of bidegree $(a, b-1)$.

A similar result holds if I_U has a first syzygy of degree $(1,0)$. The lemmas below (Lemmas 7.3 and 7.4 of Botbol [1]) also play a key role. Botbol notes that the Koszul cohomology module $(H_2)_{4a-1,3b-1}$ has dimension equal to the sum of the multiplicities at the basepoints, so if U is basepoint free, this module vanishes.

Lemma 1.5 ([1]). *Suppose $a \leq b$. If $\nu = (2a-1, b-1)$, then the determinant of the ν strand of the approximation complex is of degree $2ab - \dim(H_2)_{4a-1,3b-1}$.*

Lemma 1.6 ([1]). *If U has basepoints with multiplicities e_x, then*

$$\deg(\phi_U)\deg(F) = 2ab - \sum e_x, \text{ where } \langle F \rangle = I(X_U).$$

If U is basepoint free, the determinant of the ν strand is the determinant of $(d^1)_\nu$.

2. Proofs of main theorems

Theorem 2.1. *If $a, b \geq 2$ and U is basepoint free, then there can be at most one linear first syzygy on I_U.*

Proof. Suppose L is a linear syzygy of bidegree $(0,1)$ on I_U. By Lemma 1.4, we may assume

$$I_U = \langle pu, pv, p_2, p_3 \rangle = \langle p_0, p_1, p_2, p_3 \rangle,$$

where p is homogeneous of bidegree $(a, b-1)$. Suppose there is another minimal first linear syzygy of bidegree $(0,1)$,

$$\sum_{i=0}^{3} p_i \cdot (a_i u + b_i v) = 0.$$

Let

$$\begin{array}{rcl} \widetilde{p_2} & = & \sum a_i p_i, \\ \widetilde{p_3} & = & \sum b_i p_i, \end{array}$$

so $\widetilde{p_2}u + \widetilde{p_3}v = 0$. But the syzygy module on $[u, v]$ is generated by $[v, -u]$, so we must have $\widetilde{p_2} = qv, \widetilde{p_3} = -qu$ for some q of bidegree $(a, b-1)$. If in addition

$$D = \det \begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix} \text{ is nonzero, then}$$

$$I_U = \langle pu, pv, \widetilde{p_2}, \widetilde{p_3} \rangle = \langle pu, pv, qu, qv \rangle.$$

Example V.1.4.3 of [12] shows that curves $\mathbf{V}(f)$ of bidegree (a,b) and $\mathbf{V}(g)$ of bidegree (c,d) on $\mathbb{P}^1 \times \mathbb{P}^1$, sharing no common component, meet in $ad + bc$ points. If p and q share a common factor, then clearly I_U is not basepoint free; if they do not share a common factor, then $\mathbf{V}(p,q)$ consists of $2ab - 2a$ points; since $a, b \geq 2$, this again forces I_U to have basepoints. The same argument works if the additional syzygy is of bidegree $(1,0)$, save that in this case since q is of degree $(a-1, b)$, $\mathbf{V}(p,q)$ consists of $2ab - a - b + 1$ points, and again I_U is not basepoint free.

Next, suppose $D = 0$. If $a_2 = a_3 = b_2 = b_3 = 0$, then the second minimal first syzygy involves only pu and pv. If the syzygy is of bidegree $(0,1)$, then by Lemma 1.4, $(pu, pv) = (qv, qu)$. Thus

$$pu = qv \Longrightarrow p = fv, q = fu \Longrightarrow fv^2 = fu^2,$$

a contradiction. If the syzygy is of bidegree $(1,0)$, then $(pu, pv) = (qs, qt)$, and

$$pu = qs \Longrightarrow p = fs, q = fu \Longrightarrow fsv = fut,$$

again a contradiction.

Finally, if $D = 0$ and a_2, a_3, b_2, b_3 are not all zero, then $c \cdot [a_2, b_2] = [a_3, b_3]$ for some $c \neq 0$, so letting $\widetilde{p_2} = p_2 + cp_3$, we may assume the syzygy involves only $pu, pv, \widetilde{p_2}$. If the syzygy is of degree $(0,1)$, letting $l_i = a_i u + b_i v$ for $i = 0,1,2$, we have

$$pul_0 + pvl_1 + \widetilde{p_2} l_2 = 0.$$

Since $\langle l_2 \rangle$ is prime, either $l_2 | ul_0 + vl_1$ or $l_2 | p$. In the former case, $ul_0 + vl_1 = l_2 l_3$ for some $l_3 \in k[u,v]_1$, hence $pl_3 + \widetilde{p_2} = 0$. In particular $p | \widetilde{p_2}$, so $\mathbf{V}(p, p_3)$ contains $2ab - a$ points and I_U is not basepoint free.

In the latter case, $p' l_2 = p$ for some $p' \in R_{(a-2,b)}$, so $p' l_2 u l_0 + p' l_2 v l_1 + \widetilde{p_2} l_2 = 0$. Hence $p' u l_0 + p' v l_1 + \widetilde{p_2} = 0$, so p' is a common factor of p and $\widetilde{p_2}$ of bidegree $(a, b-2)$, so $\mathbf{V}(p', p_3)$ contains $2ab - 2a$ points and I_U is not basepoint free. A similar argument works if the additional syzygy is of bidegree $(1,0)$. $\square$

Theorem 2.2. *If U is basepoint free, $a, b \geq 2$, and there is a linear syzygy L of bidegree $(0,1)$ on I_U, then there are two additional first syzygies S_1, S_2 of bidegree $(a, b-1)$, such that*

$$\dim \langle L, S_1, S_2 \rangle_{(2a-1,b-1)} = 2ab.$$

Proof. By Lemma 1.4, we may assume $(p_0, p_1) = (pu, pv)$. Write $p_2 = g_2 v + f_2 u$. Then $f_2 p_0 + g_2 p_1 - pp_2 = 0$, so the kernel of $[pu, pv, p_2]$ contains the columns of the matrix

$$M = \begin{bmatrix} v & f_2 \\ -u & g_2 \\ 0 & -p \end{bmatrix}.$$

In fact, M is the syzygy matrix of $[pu, pv, p_2]$: the sequence $\{pu, p_2\}$ is not regular iff the two polynomials share a common factor. If $u | p_2$, then let $p_2' = p_2 + pv$; $u | p_2'$ or $p | p_2'$ imply I_U is not basepoint free. So the depth of the ideal of 2×2 minors of M is two and exactness follows from the Buchsbaum-Eisenbud criterion [10]. Writing $p_3 = f_3 u + g_3 v$, the syzygy module of I_U contains the columns of $N = \mathrm{Span}\{L, S_1, S_2\}$, where

$$N = \begin{bmatrix} v & f_2 & f_3 \\ -u & g_2 & g_3 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix}.$$

As the bottom 3×3 submatrix of N is upper triangular, $\{L, S_1, S_2\}$ span a free R-module. The linear syzygy L is of bidegree $(0,1)$, so in the degree ν strand of the approximation complex, it gives rise to

$$h^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2a-1, b-2)) = 2a(b-1)$$

columns of the matrix of the first differential d^1. The two syzygies S_1, S_2 of bidegree $(a, b-1)$ each give rise to

$$h^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a-1,0)) = a$$

columns of the matrix of d^1. That the columns are independent follows from the fact that $\{L, S_1, S_2\}$ span a free R-module. Hence, these syzygies yield $2ab$ columns of the degree ν component of the matrix of d^1. □

For Theorem 2.1 and Theorem 2.2 to hold, we need $a, b \geq 2$, even if U is basepoint free. If either a or b is at most one, there can be additional minimal linear syzygies. For example, if $(a,b) = (1,1)$, then there are four minimal linear first syzygies. However, it is easy to see that the theorems both hold if L is of bidegree $(1,0)$.

Corollary 2.3. *If $a, b \geq 2$, U is basepoint free, and I_U has a linear first syzygy, then the determinant of the degree $\nu = (2a-1, b-1)$ submatrix of the first differential in the approximation complex is determined by $\{L, S_1, S_2\}$.*

Proof. This follows from Lemma 1.5, Lemma 1.6, the remarks preceding those lemmas, and Theorem 2.2. □

Corollary 2.4. *If $a, b \geq 2$, U is basepoint free, and I_U has a linear first syzygy, then the singular locus of X_U contains a line.*

Proof. Let $I_U = \langle pu, pv, p_2, p_3\rangle$. By Corollary 2.3, the matrix representing the degree ν component d^1 has as its leftmost $2a(b-1)$ columns a block matrix P. For each monomial $m_c = s^{2a-1-c}t^c$ with $c \in \{0, \dots, 2a-1\}$, there is a $b \times b-1$ block B corresponding to elements $m_c \cdot \{v^{b-2}, \dots, u^{b-2}\} \cdot L$, with $L = vx_0 - ux_1$, hence

$$B = \begin{bmatrix} x_0 & 0 & \dots & \dots & 0 \\ -x_1 & x_0 & 0 & \vdots & 0 \\ \vdots & -x_1 & \ddots & \vdots & 0 \\ \vdots & 0 & x_0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 \\ 0 & 0 & 0 & -x_1 & x_0 \\ 0 & 0 & 0 & 0 & -x_1 \end{bmatrix} \text{ and } P = \begin{bmatrix} B & 0 & \dots & 0 \\ 0 & B & \ddots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & \dots & 0 & B \end{bmatrix}.$$

Computing the Laplace expansion of the determinant using the $2ab-2a$ minors of P shows the implicit equation for X_U takes the form

$$x_0^{2ab-2a} \cdot f_0 + x_0^{2ab-2a-1}x_1 \cdot f_1 + \cdots + x_1^{2ab-2a} \cdot f_{2ab-2a}.$$

So X_U is singular along $\mathbf{V}(x_0, x_1)$, with multiplicity at least $2ab - 2a$. □

Remark 2.5. The specific form of the implicit equation given above means that it suffices to find the f_i, and in turn speeds up the computation.

3. Application to the bidegree $(2,2)$ case

We close with some examples in the bidegree $(2,2)$ case; without loss of generality we assume I_U has a linear first syzygy of bidegree $(0,1)$, so $I_U = \langle pu, pv, p_2, p_3\rangle$.

Hence p is of bidegree $(2,1)$. There are three possible factorizations for p:

(1) p is irreducible.
(2) p is a product of an irreducible form of bidegree $(1,1)$, and a form of bidegree $(1,0)$. So $p = ql$, where $q = a_0su + a_1sv + a_2tu + a_3tv$ and $l = b_0s + b_1t$. The locus of such forms is the image of the map

$$\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1))) \times \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,0))) = \mathbb{P}^3 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^5,$$

$(a_0 : a_1 : a_2 : a_3) \times (b_0 : b_1) \mapsto (a_0b_0 : a_0b_1 + a_2b_0 : a_2b_1 : a_1b_0 : a_1b_1 + a_3b_0 : a_3b_1)$, which is a quartic hypersurface

$$Q = \mathbf{V}(x_2^2x_3^2 - x_1x_2x_3x_4 + x_0x_2x_4^2 + x_1^2x_3x_5 - 2x_0x_2x_3x_5 - x_0x_1x_4x_5 + x_0^2x_5^2).$$

Note that $\Sigma_{2,1} \subseteq \mathbf{V}(Q)$.
(3) p is a product of three linear forms, two of bidegree $(1,0)$ and one of bidegree $(0,1)$. Then identifying the coefficients of $p = a_0s^2u + a_1stu + a_2t^2u + a_3s^2v + a_4stv + a_5t^2v$ with a point of $\mathbb{P}^5$, such a decomposition corresponds to a point on the Segre variety $\Sigma_{2,1}$, whose ideal is defined by the 2×2 minors of

$$\begin{bmatrix} x_0 & x_1 & x_2 \\ x_3 & x_4 & x_5 \end{bmatrix}.$$

Examples of possible bigraded Betti tables for these three cases appear below, where p_2 and p_3 are chosen generically. When p_2 and p_3 are also nongeneric, there are many additional possible types of Betti table. It would be interesting to prove that the tables below are always the bigraded Betti tables for generic choices of p_2 and p_3, and to classify the bigraded resolutions which are possible in the $(2,2)$ case, and we are working on this. For brevity, we denote $R(a,b)$ by (a,b). In all three cases, X_U has degree $2ab = 8$, in contrast to Example 1.3.

Example 3.1. Suppose $p \notin \mathbf{V}(Q)$. Changing coordinates, we may assume p is the point $(1:0:0:0:0:1)$, which corresponds to $p = s^2u + t^2v$.

$$0 \leftarrow I_U \longleftarrow (-2,-2)^4 \longleftarrow \begin{matrix} (-2,-3) \\ \oplus \\ (-4,-3)^2 \\ \oplus \\ (-4,-4) \\ \oplus \\ (-3,-5)^2 \\ \oplus \\ (-6,-3) \\ \oplus \\ (-8,-2) \end{matrix} \longleftarrow \begin{matrix} (-4,-5)^3 \\ \oplus \\ (-6,-4)^2 \\ \oplus \\ (-8,-3)^2 \end{matrix} \longleftarrow \begin{matrix} (-6,-5) \\ \oplus \\ (-8,-4) \end{matrix} \longleftarrow 0$$

The reduced singular locus of X_U consists of curves of degrees 1, 2, and 3.

Example 3.2. Suppose $p \in \mathbf{V}(Q) \setminus \Sigma_{2,1}$. After a change of coordinates, we may assume p is the point $(1:2:1:1:1:0)$, which corresponds to $s^2u + 2stu + t^2u + s^2v + stv$.

$$0 \leftarrow I_U \longleftarrow (-2,-2)^4 \longleftarrow \begin{array}{c} (-2,-3) \\ \oplus \\ (-4,-3)^2 \\ \oplus \\ (-4,-4) \\ \oplus \\ (-3,-5)^2 \\ \oplus \\ (-6,-3) \\ \oplus \\ (-7,-2) \end{array} \longleftarrow \begin{array}{c} (-4,-5)^3 \\ \oplus \\ (-6,-4)^2 \\ \oplus \\ (-7,-3)^2 \end{array} \longleftarrow \begin{array}{c} (-6,-5) \\ \oplus \\ (-7,-4) \end{array} \longleftarrow 0$$

The reduced singular locus of X_U consists of curves of degrees 1, 1, and 4.

Example 3.3. Suppose $p \in \Sigma_{2,1}$. After a change of coordinates, we may assume p is the point $(1:1:1:1:1:1)$, which corresponds to $s^2u+stu+t^2u+s^2v+stv+t^2v$.

$$0 \leftarrow I_U \longleftarrow (-2,-2)^4 \longleftarrow \begin{array}{c} (-2,-3) \\ \oplus \\ (-4,-3)^2 \\ \oplus \\ (-4,-4) \\ \oplus \\ (-3,-5)^2 \\ \oplus \\ (-6,-2) \end{array} \longleftarrow \begin{array}{c} (-4,-5)^3 \\ \oplus \\ (-6,-4) \\ \oplus \\ (-6,-3) \end{array} \longleftarrow (-6,-5) \longleftarrow 0$$

The reduced singular locus of X_U consists of curves of degrees 1 and 4.

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`http://www.math.uiuc.edu/Macaulay2/`

and scripts to perform the computations are available at

`http://www.math.uiuc.edu/~schenck/Syzscript.`

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