PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 144, Number 1, January 2016, Pages 411–422
http://dx.doi.org/10.1090/proc/12707
Article electronically published on July 24, 2015

CYCLIC GROUP ACTIONS AND EMBEDDED SPHERES IN 4-MANIFOLDS

M. J. D. HAMILTON

(Communicated by Martin Scharlemann)

Abstract. In this note we derive an upper bound on the number of 2-spheres in the fixed point set of a smooth and homologically trivial cyclic group action of prime order on a simply-connected 4-manifold. This improves the a priori bound which is given by one half of the Euler characteristic of the 4-manifold. The result also shows that in some cases the 4-manifold does not admit such actions of a certain order at all or that any such action has to be pseudofree.

1. Introduction

Actions of finite groups, in particular cyclic groups $\mathbb{Z}_p$ of prime order p, on simply-connected 4-manifolds have been studied in numerous places in the literature. An interesting subclass includes those actions which act trivially on homology. In the topological setting, Edmonds has shown [9, Theorem 6.4] that every closed, simply-connected, topological 4-manifold admits for every $p > 3$ a (non-trivial) homologically trivial action which is *locally linear*. However, it is an open question from the Kirby list if such actions exist in the *smooth* setting for 4-manifolds like the $K3$ surface (it is known that there is no such action of $\mathbb{Z}_2$ [21, 25] on $K3$ and no such action of $\mathbb{Z}_p$ which is holomorphic [6, 23] or symplectic [7]).

The actions in the theorem of Edmonds can be assumed to be pseudofree, i.e. the fixed point set consists of isolated points. In general, if the action is homologically trivial, the fixed point set will consist of isolated points and disjoint embedded 2-spheres. We recall this fact in Proposition 2.3. If m is the number of points and n the number of spheres, then $m + 2n$ is equal to the Euler characteristic $\chi(M)$ of the 4-manifold. This implies an a priori upper bound on the number of spheres:

$$n \leq \frac{\chi(M)}{2}.$$

A natural question is whether all cases of possible values for n can occur. We will show that this upper bound can indeed be improved, for example, by a factor of roughly $\frac{1}{2}$ if the 4-manifold M and the action are smooth, M is smoothly minimal and the Seiberg-Witten invariants of M are non-vanishing. More precisely we will show the following: We say that a 4-manifold M satisfies property $(*)$ if every smoothly embedded 2-sphere in M that represents a non-zero rational homology class has negative self-intersection. For example, a 4-manifold M with $b_2^+(M) > 1$

Received by the editors July 29, 2014 and, in revised form, December 4, 2014.
2010 *Mathematics Subject Classification.* Primary 57M60, 57S17, 57N13; Secondary 57R57.
Key words and phrases. 4-manifold, group action, fixed point set, G-signature theorem.

©2015 American Mathematical Society

and non-vanishing Seiberg-Witten invariants satisfies property $(*)$. Then we have (cf. Corollary 4.1):

Corollary. *Let the group $\mathbb{Z}_p$ act homologically trivially and smoothly on a simply-connected, smooth 4-manifold M that satisfies property $(*)$. Then*

$$n \leq \frac{p\chi(M) - c_1^2(M)}{3(p-1)}.$$

If in addition M is smoothly minimal, then

$$n \leq \frac{p\chi(M) - c_1^2(M)}{2(2p-1)}.$$

Independently of $c_1^2(M)$ we have in these cases the bounds

$$n < \frac{\chi(M)}{3}\left(1 + \frac{2}{p-1}\right)$$

and

$$n < \frac{\chi(M)}{4}\left(1 + \frac{3}{2p-1}\right),$$

respectively.

The proof uses the G-signature theorem together with an estimate on the signature defects at the fixed points. Even though the proof is elementary, it seems worthwhile to record this fact together with a number of corollaries, in particular in the situation that the theorem implies $n < 0$ (no action possible) or $0 \leq n < 1$ (every action is pseudofree).

This result has applications especially for smooth, homologically trivial $\mathbb{Z}_2$- and $\mathbb{Z}_3$-actions on general, smooth, simply-connected 4-manifolds as well as for $\mathbb{Z}_p$-actions on possible examples of exotic smooth 4-manifolds homeomorphic to $S^2 \times S^2$ or $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$. Three consequences of the main theorem are Corollary 5.5, Corollary 5.7 and Corollary 6.2 that lead to implications in particular for smooth involutions. The first corollary is related to a special case of the problem from the Kirby list and implies that a simply-connected, *non-spin* 4-manifold with *positive signature* that satisfies property $(*)$ does not admit homologically trivial, *smooth* involutions. The same is true according to the second corollary if the signature is equal to -1 and the manifold is in addition smoothly minimal. Both results are a partial extension of a theorem of Ruberman for spin 4-manifolds to the non-spin case and contrast a theorem of Edmonds, who has shown that every smooth, simply-connected, non-spin 4-manifold admits a homologically trivial, *locally linear* involution. The third corollary implies that a homologically trivial, *smooth* involution on a simply-connected 4-manifold that satisfies property $(*)$ and has vanishing signature (this is, by Ruberman's theorem, the case if the 4-manifold is spin, for example) is necessarily pseudofree, i.e. the fixed point set consists of a collection of isolated points.

Convention. All 4-manifolds in the following will be closed, oriented and connected and have $b_2(M) > 0$. All spheres embedded in 4-manifolds will be 2-dimensional. All group actions will be non-trivial and orientation-preserving.

2. Spheres in the fixed point set and the G-signature theorem

Let M denote a *simply-connected*, topological 4-manifold with a *locally linear* action of a cyclic group $G = \mathbb{Z}_p$, with $p \geq 2$ a prime. The group action is generated by a locally linear homeomorphism $\tau\colon M \to M$ of order p, such that τ is not equal to the identity. There is an induced action of G on $H^2(M;\mathbb{Z})$ preserving the intersection form. According to [11, 17] this action decomposes over the integers into t copies of the trivial action of rank 1, c copies of the cyclotomic action of rank $p-1$ and r copies of the regular action of rank p, where t, c, r are certain non-negative integers. As a consequence, the second Betti number of M is equal to

$$b_2(M) = t + c(p-1) + rp.$$

In particular we have:

Lemma 2.1. *If $p > b_2(M) + 1$, then G acts trivially on homology.*

Let F denote the fixed point set of the locally linear homeomorphism τ. Since G is of prime order, the set F is the fixed point set of every group element in G different from the identity. The fixed point set F is a closed topological submanifold of M [5, p. 171]. The action is locally linear and hence given by an orthogonal action in a neighbourhood of a fixed point. Since the action preserves orientation, the fixed point set F has even codimension [26]. It consists of a disjoint union of finitely many isolated points and finitely many closed surfaces. If p is odd, then every surface in the fixed point set is orientable [5, p. 175].

The next lemma follows from [11, Proposition 2.5]:

Lemma 2.2. *Suppose that the fixed point set F has more than one component. Then every surface component of F represents a non-zero class in $H_2(M;\mathbb{Z}_p)$.*

If the action is not free, then according to [11, Proposition 2.4] the $\mathbb{Z}_p$-Betti numbers of the fixed point F satisfy

$$b_1(F;\mathbb{Z}_p) = c,$$
$$b_0(F;\mathbb{Z}_p) + b_2(F;\mathbb{Z}_p) = t + 2.$$

Let $\chi(M) = b_2(M) + 2$ denote the Euler characteristic of M. If G acts trivially on homology, then $\chi(F) = \chi(M)$ by the Lefschetz fixed point theorem. Hence the action is not free and we get:

Proposition 2.3. *Suppose that G acts trivially on the homology of M. Then F consists of a disjoint union of m isolated points and n spheres, with $m+2n = \chi(M)$. Since $b_2(M) > 0$, after a choice of orientation, every sphere in F represents a non-zero class in $H_2(M;\mathbb{Z})$.*

From now on we assume that the action of G is *trivial on homology.* We want to improve the upper bound $\frac{1}{2}\chi(M)$ on the number n of spheres. We can use the G-signature theorem [4], which is valid not only for smooth, but also for locally linear actions in dimension 4; cf. [28] and a remark in [9, p. 164] (all of our applications will be for smooth actions). Let $S_1, \ldots, S_n$ denote the spherical components of the fixed point set F, and let P be the set of isolated fixed points. Note that the signature satisfies

$$\operatorname{sign}(M/G) = \operatorname{sign}(M),$$

since the action of G is trivial on homology. The G-signature theorem implies [13, pp. 14–17]:

$$(p-1)\mathrm{sign}(M) = \sum_{x\in P} \mathrm{def}_x + \frac{p^2-1}{3}\sum_{i=1}^{n}[S_i]^2.$$

Here $[S_i]^2$ denotes the self-intersection number of the sphere S_i. The numbers def_x are equal, in Hirzebruch's notation, to $\mathrm{def}(p;q,1)$ for certain integers q coprime to p and depending on x. We have

$$\mathrm{def}(p;q,1) = -\frac{2}{3}(q,p) = -4p\sum_{k=0}^{p-1}\left(\left(\frac{k}{p}\right)\right)\left(\left(\frac{qk}{p}\right)\right).$$

In this equation (q,p) denotes the Dedekind symbol, while $((\cdot))\colon \mathbb{R}\to\mathbb{R}$ is a certain function introduced by Rademacher and given by

$$\begin{aligned} ((z)) &= z-[z]-\frac{1}{2}, \quad \text{if } z \text{ is not an integer,}\\ ((z)) &= 0, \quad \text{if } z \text{ is an integer.}\end{aligned}$$

Here $[z]$ denotes the greatest integer less than or equal to z. We want to prove the following estimate:

Lemma 2.4. *For all prime numbers p and integers q coprime to p we have*

$$|\mathrm{def}(p;q,1)| \le |\mathrm{def}(p;1,1)| = \frac{1}{3}(p-1)(p-2).$$

Proof. We have by Cauchy-Schwarz

$$\begin{aligned}\left|\sum_{k=0}^{p-1}\left(\left(\frac{k}{p}\right)\right)\left(\left(\frac{qk}{p}\right)\right)\right| &\le \left(\sum_{k=1}^{p-1}\left(\left(\frac{k}{p}\right)\right)^2\right)^{\frac{1}{2}}\cdot\left(\sum_{k=1}^{p-1}\left(\left(\frac{qk}{p}\right)\right)^2\right)^{\frac{1}{2}}\\ &= \sum_{k=1}^{p-1}\left(\left(\frac{k}{p}\right)\right)^2,\end{aligned}$$

because q generates $\mathbb{Z}_p$ and $((0)) = 0$. Since $0 < \frac{k}{p} < 1$ for all $k = 1,\ldots,p-1$ we have

$$\begin{aligned}\sum_{k=1}^{p-1}\left(\left(\frac{k}{p}\right)\right)^2 &= \sum_{k=1}^{p-1}\left(\frac{k}{p}-\frac{1}{2}\right)^2\\ &= \sum_{k=1}^{p-1}\left(\frac{k^2}{p^2}-\frac{k}{p}+\frac{1}{4}\right)\\ &= \frac{1}{6p^2}(p-1)p(2p-1) - \frac{1}{2p}(p-1)p + \frac{p-1}{4}\\ &= \frac{1}{6p}(2p^2-3p+1) - \frac{1}{2p}(p^2-p) + \frac{p-1}{4}\\ &= \frac{1}{12p}(4p^2-6p+2-6p^2+6p+3p^2-3p)\\ &= \frac{1}{12p}(p^2-3p+2)\\ &= \frac{1}{12p}(p-1)(p-2).\end{aligned}$$

This implies the claim. The number $\operatorname{def}(p;1,1)$ has also been calculated in equation (28) in [13]. □

We can now prove the main theorem. We use the standard notation

$$c_1^2(M) = 2\chi(M) + 3\operatorname{sign}(M)$$

for every 4-manifold M. We abbreviate the following conditions on the action and the manifold by simply saying that "$\mathbb{Z}_p$ acts homologically trivially on a simply-connected 4-manifold M":

> The group $\mathbb{Z}_p$, with $p \geq 2$ prime, acts locally linearly and homologically trivially on a simply-connected, topological 4-manifold M.

We consider in the following only actions of this kind.

Theorem 2.5. *Let $\mathbb{Z}_p$ act homologically trivially on a simply-connected 4-manifold M. Suppose that all spheres S in the fixed point set of the action satisfy an a priori bound $[S]^2 \leq s < 0$ for some integer s. Then the number n of spheres in the fixed point set satisfies the upper bound*

$$n \leq \frac{p\chi(M) - c_1^2(M)}{p(2-s) - (4+s)}.$$

For all possible values of $c_1^2(M)$ we have the bound

$$n < \frac{\chi(M)}{2-s}\left(1 + \frac{6}{p(2-s) - (4+s)}\right).$$

Proof. By Proposition 2.3 the number of isolated fixed points in F is $\chi(M) - 2n$. By the G-signature theorem and Lemma 2.4 we have

$$(p-1)\operatorname{sign}(M) \leq \frac{1}{3}(p-1)(p-2)(\chi(M) - 2n) + \frac{1}{3}sn(p^2-1).$$

This implies the first claim (note that the denominator is positive under our assumption $s < 0$). The second claim follows from the estimate $\operatorname{sign}(M) > -\chi(M)$, which is true for all oriented 4-manifolds with $b_1(M) = 0$. □

3. Smoothly embedded spheres

Definition 3.1. We say that a smooth 4-manifold M satisfies **property** $(*)$ if the following holds:

> Every smoothly embedded sphere S in M that represents a non-zero homology class $[S] \in H_2(M;\mathbb{Q})$ has negative self-intersection number.

We are interested in under which conditions a 4-manifold M satisfies property $(*)$. The following is clear:

Proposition 3.2. *Let M be a smooth 4-manifold. Assume that $b_2^+(M) = 0$. Then M satisfies property $(*)$.*

The next theorem is well known; cf. [15, Proposition 1]. The statement also follows from the adjunction inequality [12, 16].

Proposition 3.3. *Let M be a smooth 4-manifold. Assume that $b_2^+(M) > 1$ and the Seiberg-Witten invariants of M do not vanish identically. Then M satisfies property $(*)$.*

We did not find in the literature a similarly general theorem in the case of 4-manifolds M with $b_2^+(M) = 1$. To describe what we can show in this case, recall that a *rational surface* is a smooth 4-manifold diffeomorphic to $S^2 \times S^2, \mathbb{CP}^2$ or $\mathbb{CP}^2 \# n\overline{\mathbb{CP}}^2$ with $n \geq 1$, while a *ruled surface* is an oriented S^2-bundle over a Riemann surface Σ_g of genus $g \geq 0$ (there exist up to diffeomorphism two such ruled surfaces for every genus g; the ruled surface is called *irrational* if $g \geq 1$). We can then prove the following:

Proposition 3.4. *Let M be a smooth 4-manifold. Assume that $b_2^+(M) = 1$, $b_2^-(M) \leq 9$, $b_1(M) = 0$ and M is not diffeomorphic to a rational surface. If M admits a symplectic form, then M satisfies property $(*)$.*

Remark 3.5. In this situation, the assumption $b_2^-(M) \leq 9$ is equivalent to $K^2 \geq 0$, where K denotes the canonical class of the symplectic form, because $K^2 = 2\chi(M) + 3\mathrm{sign}(M)$.

For the proof recall the following theorem of Liu [20, Theorem B] (slightly adapted to make the statement more precise):

Theorem 3.6 (Liu). *Let M be a symplectic 4-manifold with $b_2^+(M) = 1$. If $K \cdot \omega < 0$, then M must be either rational or (a blow-up of) an irrational ruled 4-manifold.*

We also need an adjunction inequality of Li and Liu [19, p. 467]:

Theorem 3.7 (Li-Liu). *Suppose M is a symplectic 4-manifold with $b_2^+(M) = 1$ and ω is the symplectic form. Let C be a smooth, connected, embedded surface with non-negative self-intersection. If $[C] \cdot \omega > 0$, then the genus of C satisfies $2g(C) - 2 \geq K \cdot [C] + [C]^2$.*

We have the following general light cone lemma; compare with [19, Lemma 2.6]:

Lemma 3.8. *Let M be a 4-manifold with $b_2^+(M) = 1$. The forward cone is one of the two connected components of $\{a \in H^2(M; \mathbb{R}) \mid a^2 > 0\}$. Then the following holds for all elements $a, b \in H^2(M; \mathbb{R})$:*

(a) *If a is in the forward cone and b in the closure of the forward cone with $b \neq 0$, then $a \cdot b > 0$.*
(b) *If a and b are in the closure of the forward cone, then $a \cdot b \geq 0$.*
(c) *If a is in the forward cone and b satisfies $b^2 \geq 0$ and $a \cdot b \geq 0$, then b is in the closure of the forward cone.*

Proof. With respect to a suitable basis of the vector space $H^2(M; \mathbb{R})$ we have $a \cdot b = a_0 b_0 - \sum a_i b_i$ where the elements a in the forward cone satisfy $a_0 > 0$. Then (a) and (b) follow by applying the Cauchy-Schwarz inequality:

$$\sum a_i b_i \leq \sqrt{\sum a_i^2} \sqrt{\sum b_i^2}.$$

For (c) assume by contradiction $b_0 < 0$. Then the vector $-b$ is in the closure of the forward cone, so (a) implies $a \cdot (-b) > 0$ and hence $a \cdot b < 0$, a contradiction. $\square$

We can now prove Proposition 3.4:

Proof. Let the forward cone be defined by the class of ω. Our assumptions together with Theorem 3.6 and Lemma 3.8 imply that the canonical class K is in the closure of the forward cone. Suppose that the class $[S]$ of a sphere S satisfies $[S] \neq 0$ and $[S]^2 \geq 0$. Choose the orientation on S such that $[S]$ is in the closure of the forward cone. By Lemma 3.8, $[S] \cdot \omega > 0$. Then Theorem 3.7 applies and shows that $-2 \geq K \cdot [S] + [S]^2$. However, Lemma 3.8 implies that $K \cdot [S] \geq 0$. This is a contradiction. □

We conjecture the following:

Conjecture 3.9. *Let M be a smooth 4-manifold. Assume that $b_2^+(M) = 1$, $b_2^-(M) \leq 9$, $H_1(M;\mathbb{Z}) = 0$ and M has non-trivial small perturbation Seiberg-Witten invariants. Then M satisfies property $(*)$.*

For a definition of the small perturbation Seiberg-Witten invariants see [27].

4. The main corollary for smooth actions

Recall that an oriented 4-manifold is called *(smoothly) minimal* if it does not contain smoothly embedded spheres of self-intersection -1.

Corollary 4.1. *Let the group $\mathbb{Z}_p$ act homologically trivially and smoothly on a simply-connected, smooth 4-manifold M that satisfies property $(*)$. Then*

$$n \leq \frac{p\chi(M) - c_1^2(M)}{3(p-1)}.$$

If in addition M is smoothly minimal, then

$$n \leq \frac{p\chi(M) - c_1^2(M)}{2(2p-1)}.$$

Independently of $c_1^2(M)$ we have in these cases the bounds

$$n < \frac{\chi(M)}{3}\left(1 + \frac{2}{p-1}\right)$$

and

$$n < \frac{\chi(M)}{4}\left(1 + \frac{3}{2p-1}\right),$$

respectively.

Proof. If the action is smooth, then every sphere in F is smoothly embedded [5, p. 309]. The first claim follows with Theorem 2.5, since $[S]^2 \leq -1$ for every embedded sphere S representing a non-zero homology class if M satisfies property $(*)$. If M is smoothly minimal, spheres of self-intersection -1 do not exist in M, hence $[S]^2 \leq -2$. □

This improves the a priori bound $n \leq \frac{1}{2}\chi(M)$ by a factor of approximately $\frac{2}{3}$ and $\frac{1}{2}$, at least for large p.

Example 4.2. Let $M = E(k)_{a,b}$ be a simply-connected, minimal elliptic surface with multiple fibres of coprime indices a, b. Assume that either $k \geq 2$, or $k = 1$ and both $a, b \neq 1$. Then M is smoothly minimal, symplectic and irrational and thus satisfies property $(*)$. We have $c_1^2(M) = 0$ and $\chi(M) = 12k$. Therefore

$$n \leq 3k\left(1 + \frac{1}{2p-1}\right).$$

This rules out some of the possible $\mathbb{Z}_3$-actions on elliptic surfaces in [18].

5. The case $n < 0$: Non-existence of actions

Since the integer n has to be non-negative if an action exists, we get:

Proposition 5.1. *Let the group $\mathbb{Z}_p$ act homologically trivially on a simply-connected 4-manifold M. Suppose that all spheres S in the fixed point set of the action satisfy an a priori bound $[S]^2 \leq s < 0$ for some integer s. Then*

$$p\chi(M) \geq c_1^2(M).$$

Corollary 5.2. *Let the group $\mathbb{Z}_p$ act homologically trivially and smoothly on a simply-connected, smooth 4-manifold M that satisfies property $(*)$. If $p = 2$, then $\mathrm{sign}(M) \leq 0$. If $p = 3$, then $c_1^2(M) \leq 3\chi(M)$.*

Remark 5.3. Ruberman [25] has shown that if $\mathbb{Z}_2$ acts homologically trivially and *locally linearly* on a simply-connected *spin* 4-manifold, then $\mathrm{sign}(M) = 0$. The first part of Corollary 5.2 is a partial extension of this result to *smooth* $\mathbb{Z}_2$-actions on *non-spin* 4-manifolds. Regarding the second statement, it is not known if there exist simply-connected, smooth 4-manifolds with non-trivial Seiberg-Witten invariants and $c_1^2(M) > 3\chi(M)$ (for more on this question see [12, Section 10.3]). Note that any simply-connected 4-manifold satisfies a priori $c_1^2(M) < 5\chi(M)$.

A non-singular, odd, integral, bilinear form Q on a finitely generated free abelian group V is said to have *characteristic signature* if there exists an indivisible characteristic element $v \in V$ such that $v \cdot v = \mathrm{sign}(Q)$. The intersection forms of *smooth*, simply-connected, non-spin 4-manifolds are direct sums of copies of the forms $(+1)$ and (-1) (this is clear in the indefinite case and follows in the definite case by Donaldson's theorem [8]) and hence are always characteristic. The next theorem of Edmonds then follows from [10, Corollary 11]:

Theorem 5.4 (Edmonds). *Every smooth, simply-connected, non-spin 4-manifold M admits a homologically trivial, locally linear involution whose fixed point set consists of a single sphere S with $[S]^2 = \mathrm{sign}(M)$ and a collection of isolated points.*

By contrast, the following corollary is implied by Corollary 5.2:

Corollary 5.5. *Let M be a smooth, simply-connected 4-manifold M that satisfies property $(*)$ and has positive signature. Then M does not admit a homologically trivial, smooth involution.*

This corollary is relevant only if M is non-spin because of Ruberman's theorem.

Example 5.6. Let M be a simply-connected, complex algebraic surface of general type and positive signature with $b_2^+(M) > 1$ (see e.g. [24] and the references therein for the construction of such surfaces). Then M satisfies property $(*)$ by Proposition 3.3 and the non-triviality of the Seiberg-Witten invariants for surfaces of general type [29]. Hence M does not admit a homologically trivial, smooth $\mathbb{Z}_2$-action. However, if M is non-spin (for example, if M is a blow-up of a spin surface of general type), then it admits a homologically trivial, locally linear $\mathbb{Z}_2$-action by Theorem 5.4 of Edmonds.

We can also prove the following:

Corollary 5.7. *Let M be a simply-connected, smooth, minimal 4-manifold with $\mathrm{sign}(M) = -1$ that satisfies property $(*)$. Then M does not admit a homologically trivial, smooth involution.*

Proof. Since $\mathrm{def}_x = 0$ for isolated fixed points of involutions, the G-signature theorem implies for such an action

$$\mathrm{sign}(M) = \sum_{i=1}^{n} [S_i]^2.$$

This cannot be satisfied, because $\mathrm{sign}(M) = -1$ and $[S_i]^2 \leq -2$ under our assumptions. □

Remark 5.8. Note that such a manifold is always non-spin according to Rohlin's theorem. The proof of Corollary 5.7 also gives a further explanation for the result in Corollary 5.5.

6. The case $0 \leq n < 1$: Action is pseudofree

We can also study the case $0 \leq n < 1$. This will elucidate the situation close to or on the boundary of the allowed regions given by Proposition 5.1 and Corollary 5.2.

Proposition 6.1. *Let the group $\mathbb{Z}_p$ act homologically trivially on a simply-connected 4-manifold M. Suppose that all spheres S in the fixed point set of the action satisfy an a priori bound $[S]^2 \leq s < 0$ for some integer s and that M satisfies*

$$p\chi(M) - c_1^2(M) < p(2 - s) - (4 + s).$$

Then $n = 0$, hence the fixed point set consists only of isolated points, i.e. the action is pseudofree.

The following is an application to involutions on 4-manifolds with $\mathrm{sign}(M) = 0$:

Corollary 6.2. *Let the group $\mathbb{Z}_2$ act homologically trivially and smoothly on a simply-connected, smooth 4-manifold M that satisfies property $(*)$. Assume that* $\mathrm{sign}(M) = 0$. *Then the action is pseudofree. In particular, every smooth, homologically trivial involution on a simply-connected, smooth, spin 4-manifold that satisfies property $(*)$ is pseudofree.*

Proof. We have $c_1^2(M) = 2\chi(M) + 3\mathrm{sign}(M)$. We can take $s = -1$ in Proposition 6.1 and the inequality is $0 < 3$, which is true. The second part follows from Ruberman's theorem [25] since under these assumptions $\mathrm{sign}(M) = 0$. □

Remark 6.3. Atiyah-Bott [3, Proposition 8.46] have shown that all components of the fixed point set have the same dimension, so that the fixed point set consists *either* of isolated fixed points *or* of a collection of embedded surfaces, if $\mathbb{Z}_2$ acts smoothly and orientation-preservingly on a simply-connected spin 4-manifold (there are generalizations to the locally linear and general case by Edmonds [11, Corollary 3.3] and Ruberman [25]). Under our additional assumptions that the involution is homologically trivial and M satisfies property $(*)$ the second case of a fixed point set of dimension 2 does not occur.

We can prove a similar statement for $\mathbb{Z}_3$-actions on 4-manifolds close to or on the Bogomolov-Miyaoka-Yau line $c_1^2(M) = 3\chi(M)$:

Corollary 6.4. *Let the group $\mathbb{Z}_3$ act homologically trivially and smoothly on a simply-connected, smooth 4-manifold M that satisfies property $(*)$. Assume that either $c_1^2(M) = 3\chi(M) - l$ with $0 \leq l \leq 4$, or M is minimal and $c_1^2(M) = 3\chi(M) - l$ with $0 \leq l \leq 8$. Then the action is pseudofree.*

Proof. The proof is similar to the proof of Corollary 6.2. For Proposition 6.1 to work, l has to be less than 6 in the first case and less than 10 in the second case. □

Remark 6.5. Note that

$$l = 3\chi(M) - c_1^2(M) = \chi(M) - 3\mathrm{sign}(M) = 2 - 2b_2^+(M) + 4b_2^-(M)$$

is always an *even* number. If $b_1(M) = 0$, the Seiberg-Witten invariants can be non-zero or M can have a symplectic form only if $b_2^+(M)$ is odd. Then l is divisible by 4. Hence if we want to apply Proposition 3.3 and Proposition 3.4, then $l \in \{0, 4\}$ in the first case and $l \in \{0, 4, 8\}$ in the second case of Corollary 6.4.

Example 6.6. Let M be a smooth, minimal 4-manifold homeomorphic, but not diffeomorphic, to the manifold $\mathbb{CP}^2\#2\overline{\mathbb{CP}}^2$; cf. [2]. Suppose that M admits a symplectic form ω (such an example for M is constructed in that paper). Then M satisfies property $(*)$ according to Proposition 3.4. Hence there does not exist a smooth, homologically trivial involution on M and every smooth, homologically trivial $\mathbb{Z}_3$-action is pseudofree.

7. Actions on exotic $S^2 \times S^2$ and $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$

Lemma 7.1. *Let $\mathbb{Z}_p$, with $p \geq 3$ prime, act on M, where M is a 4-manifold homeomorphic to $S^2 \times S^2$ or $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$. Then the action is homologically trivial.*

Proof. This follows as in [14, Proposition 5.8] (it follows from Lemma 2.1 in all cases except $p = 3$). □

Corollary 7.2. *Let $\mathbb{Z}_p$ act smoothly on M, where M is a smooth, minimal 4-manifold homeomorphic, but not diffeomorphic, to $S^2 \times S^2$ or $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$ and satisfying property $(*)$. If $p = 2$, assume in addition that the action is homologically trivial. Then the action is pseudofree.*

Proof. We have $\chi(M) = 4$ and $c_1^2(M) = 8$. Hence the inequality in Proposition 6.1 with $s = -2$ is

$$4p - 8 < 4p - 2.$$

Since this is true, the claim follows. □

Note that every smooth 4-manifold homeomorphic to $S^2 \times S^2$ is minimal because its intersection form is even. It is not known if there exist exotic 4-manifolds homeomorphic, but not diffeomorphic, to $S^2 \times S^2$ or $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$. The smallest (in terms of Euler characteristic) known, simply-connected 4-manifold that admits exotic copies is $\mathbb{CP}^2\#2\overline{\mathbb{CP}}^2$, mentioned above in Example 6.6. However, if the trend for $\mathbb{CP}^2\#n\overline{\mathbb{CP}}^2$ with $n \geq 2$ generalizes to even smaller 4-manifolds, it is quite likely that exotic copies of $S^2 \times S^2$ or $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$ exist, at least some of which could be symplectic, so that Corollary 7.2 applies to them.

Remark 7.3. All statements in this paper remain true (except possibly Theorem 5.4 of Edmonds) if the assumption that M is simply-connected is replaced by $H_1(M; \mathbb{Z}) = 0$. This follows from [22, Corollary 3.3, Proposition 3.5], since in this situation Proposition 2.3 above remains true. The results of Section 7 then apply to smooth 4-manifolds with the integral cohomology of $S^2 \times S^2$ and $\mathbb{CP}^2\#\overline{\mathbb{CP}}$ (for example, the symplectic cohomology $S^2 \times S^2$ constructed in [1]).

Acknowledgements

The author would like to thank Dieter Kotschick for helpful comments regarding reference [15] and an anonymous referee for valuable suggestions to improve the quality of the paper.

References

[1] Anar Akhmedov, *Small exotic 4-manifolds*, Algebr. Geom. Topol. **8** (2008), no. 3, 1781–1794, DOI 10.2140/agt.2008.8.1781. MR2448872 (2009g:57040)

[2] Anar Akhmedov and B. Doug Park, *Exotic smooth structures on small 4-manifolds with odd signatures*, Invent. Math. **181** (2010), no. 3, 577–603, DOI 10.1007/s00222-010-0254-y. MR2660453 (2012b:57048)

[3] M. F. Atiyah and R. Bott, *A Lefschetz fixed point formula for elliptic complexes. II. Applications*, Ann. of Math. (2) **88** (1968), 451–491. MR0232406 (38 #731)

[4] M. F. Atiyah and I. M. Singer, *The index of elliptic operators. III*, Ann. of Math. (2) **87** (1968), 546–604. MR0236952 (38 #5245)

[5] Glen E. Bredon, *Introduction to compact transformation groups*, Academic Press, New York-London, 1972. Pure and Applied Mathematics, Vol. 46. MR0413144 (54 #1265)

[6] Dan Burns Jr. and Michael Rapoport, *On the Torelli problem for kählerian $K-3$ surfaces*, Ann. Sci. École Norm. Sup. (4) **8** (1975), no. 2, 235–273. MR0447635 (56 #5945)

[7] Weimin Chen and Slawomir Kwasik, *Symplectic symmetries of 4-manifolds*, Topology **46** (2007), no. 2, 103–128, DOI 10.1016/j.top.2006.12.003. MR2313067 (2008h:53147)

[8] S. K. Donaldson, *An application of gauge theory to four-dimensional topology*, J. Differential Geom. **18** (1983), no. 2, 279–315. MR710056 (85c:57015)

[9] Allan L. Edmonds, *Construction of group actions on four-manifolds*, Trans. Amer. Math. Soc. **299** (1987), no. 1, 155–170, DOI 10.2307/2000487. MR869405 (88d:57013)

[10] Allan L. Edmonds, *Involutions on odd four-manifolds*, Topology Appl. **30** (1988), no. 1, 43–49, DOI 10.1016/0166-8641(88)90079-X. MR964061 (89m:57041)

[11] Allan L. Edmonds, *Aspects of group actions on four-manifolds*, Topology Appl. **31** (1989), no. 2, 109–124, DOI 10.1016/0166-8641(89)90075-8. MR994404 (90h:57050)

[12] Robert E. Gompf and András I. Stipsicz, *4-manifolds and Kirby calculus*, Graduate Studies in Mathematics, vol. 20, American Mathematical Society, Providence, RI, 1999. MR1707327 (2000h:57038)

[13] F. Hirzebruch, *The signature theorem: reminiscences and recreation*, Prospects in mathematics (Proc. Sympos., Princeton Univ., Princeton, NJ, 1970), Ann. of Math. Studies, No. 70, Princeton Univ. Press, Princeton, NJ, 1971, pp. 3–31. MR0368023 (51 #4265)

[14] Michael Klemm, *Finite group actions on smooth 4-manifolds with indefinite intersection form*, ProQuest LLC, Ann Arbor, MI, 1995. Thesis (Ph.D.)–McMaster University (Canada). MR2695049

[15] D. Kotschick, *Orientations and geometrisations of compact complex surfaces*, Bull. London Math. Soc. **29** (1997), no. 2, 145–149, DOI 10.1112/S0024609396002287. MR1425990 (97k:32047)

[16] P. B. Kronheimer and T. S. Mrowka, *The genus of embedded surfaces in the projective plane*, Math. Res. Lett. **1** (1994), no. 6, 797–808, DOI 10.4310/MRL.1994.v1.n6.a14. MR1306022 (96a:57073)

[17] Sławomir Kwasik and Reinhard Schultz, *Homological properties of periodic homeomorphisms of 4-manifolds*, Duke Math. J. **58** (1989), no. 1, 241–250, DOI 10.1215/S0012-7094-89-05812-2. MR1016421 (91d:57023)

[18] Hongxia Li, *Cyclic group actions on elliptic surfaces $E(2n)$*, J. Math. Comput. Sci. **2** (2012), no. 6, 1759–1765. MR3004191

[19] T. J. Li and A. Liu, *Symplectic structure on ruled surfaces and a generalized adjunction formula*, Math. Res. Lett. **2** (1995), no. 4, 453–471, DOI 10.4310/MRL.1995.v2.n4.a6. MR1355707 (96m:57052)

[20] Ai-Ko Liu, *Some new applications of general wall crossing formula, Gompf's conjecture and its applications*, Math. Res. Lett. **3** (1996), no. 5, 569–585, DOI 10.4310/MRL.1996.v3.n5.a1. MR1418572 (97k:57038)

[21] Takao Matumoto, *Homologically trivial smooth involutions on $K3$ surfaces*, Aspects of low-dimensional manifolds, Adv. Stud. Pure Math., vol. 20, Kinokuniya, Tokyo, 1992, pp. 365–376. MR1208316 (94e:57022)

[22] M. McCooey, *Symmetry groups of non-simply connected four-manifolds*, preprint: arXiv:0707.3835v2.

[23] C. A. M. Peters, *On automorphisms of compact Kähler surfaces*, Journées de Géometrie Algébrique d'Angers, Juillet 1979/Algebraic Geometry, Angers, 1979, Sijthoff & Noordhoff, Alphen aan den Rijn—Germantown, Md., 1980, pp. 249–267. MR605346 (82c:32029)

[24] Ulf Persson, Chris Peters, and Gang Xiao, *Geography of spin surfaces*, Topology **35** (1996), no. 4, 845–862, DOI 10.1016/0040-9383(95)00046-1. MR1404912 (98h:14046)

[25] Daniel Ruberman, *Involutions on spin 4-manifolds*, Proc. Amer. Math. Soc. **123** (1995), no. 2, 593–596, DOI 10.2307/2160919. MR1231042 (95c:57023)

[26] P. A. Smith, *Transformations of finite period. IV. Dimensional parity*, Ann. of Math. (2) **46** (1945), 357–364. MR0013304 (7,136e)

[27] Zoltán Szabó, *Exotic 4-manifolds with $b_2^+ = 1$*, Math. Res. Lett. **3** (1996), no. 6, 731–741, DOI 10.4310/MRL.1996.v3.n6.a2. MR1426531 (97j:57049)

[28] C. T. C. Wall, *Surgery on compact manifolds*, 2nd ed., Edited and with a foreword by A. A. Ranicki, Mathematical Surveys and Monographs, vol. 69, American Mathematical Society, Providence, RI, 1999. MR1687388 (2000a:57089)

[29] Edward Witten, *Monopoles and four-manifolds*, Math. Res. Lett. **1** (1994), no. 6, 769–796, DOI 10.4310/MRL.1994.v1.n6.a13. MR1306021 (96d:57035)

Institute for Geometry and Topology, University of Stuttgart, Pfaffenwaldring 57, 70569 Stuttgart, Germany

E-mail address: `mark.hamilton@math.lmu.de`