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CONSTRUCTION OF NONAUTONOMOUS FORWARD ATTRACTORS

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Abstract. Autonomous systems depend only on the elapsed time, so their attractors and limit sets exist in current time. Similarly, the pullback limit defines a component set of a nonautonomous pullback attractor at each instant of current time. The forward limit defining a nonautonomous forward attractor is different as it is the limit to the asymptotically distant future. In particular, the limiting objects forward in time do not have the same dynamical meaning in current time as in the autonomous or pullback cases. Nevertheless, the pullback limit taken within a positively invariant family of compact subsets allows the component set of a forward attractor to be constructed at each instant of current time. Every forward attractor has such a positively invariant family of compact subsets, which ensures that the component sets of a forward attractor can be constructed in this way. It is, however, only a necessary condition and not sufficient for the constructed family of subsets to be a forward attractor. The analysis here is presented in the state space $\mathbb{R}^d$ to focus on the dynamical essentials rather than on functional analytical technicalities; in particular, those concerning asymptotic compactness properties.

1. Introduction

The global attractor A of an autonomous semi-dynamical system π on $\mathbb{R}^d$ with a compact positively invariant absorbing set B, i.e., a nonempty compact set of $\mathbb{R}^d$ with $\pi(t,B) \subset B$ for all $t \geq 0$, is given by

$$A = \bigcap_{t \geq 0} \pi(t,B). \tag{1.1}$$

A global attractor is asymptotically stable in the sense of Lyapunov, i.e., both Lyapunov stable and attracting; see Hale [7], LaSalle [14].

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An analogous expression holds for the component sets of a nonautonomous pullback attractor $\mathcal{A} = \{A(t) : t \in \mathbb{R}\}$ of a nonautonomous dynamical system ϕ (described here by a process or two-parameter semi-group),

$$A(t) = \bigcap_{t_0 \le t} \phi(t, t_0, B(t_0)), \qquad t \in \mathbb{R}. \tag{1.2}$$

Here $\mathcal{B} = \{B(t) : t \in \mathbb{R}\}$ is a ϕ-positively invariant family of nonempty compact subsets, i.e., with $\phi(t, t_0, B(t_0)) \subset B(t)$ for all $t \ge t_0$, which pullback absorbs bounded subsets of $\mathbb{R}^d$ under ϕ. See, e.g., Carvalho et al. [1], Cheban et al. [3], Chepyzhov & Vishik [4], Chueshov [5], Kloeden & Rasmussen [12], Pötzsche [15].

The sets A and $A(t)$ in (1.1) and (1.2) are nonempty compact subsets of $\mathbb{R}^d$ as the intersection of a nested compact subset. Generalisations hold for metric state spaces X instead of $\mathbb{R}^d$ with the absorbing sets consisting of closed and bounded subsets rather than compact subsets, provided π or ϕ satisfies in addition some kind of compactness or asymptotic compactness property [1, 4].

It is frequently repeated in the literature that there is no such expression for a nonautonomous forward attractor. It will be shown here the component subsets of nonautonomous forward attractors are also given by (1.2).

The situation is somewhat more complicated due to some peculiarities of forward attractors compared to pullback attractors [16], e.g., they need not be unique; see also [2]. To focus on the dynamics rather than technical details, the discussion below is given in terms of the state space $\mathbb{R}^d$. Some definitions and results for nonautonomous dynamical systems are first given in the next section.

2. Processes and nonautonomous pullback attractors

A fundamental difference between nonautonomous and autonomous dynamical systems is that a nonautonomous system depends on both the current time t and the starting time t_0, and not just on their difference $t - t_0$, the time elapsed since starting, as in an autonomous dynamical system. This has a profound effect on the nature of the nonautonomous attractors.

The 2-parameter semi-group formalism of a nonautonomous dynamical system, called a *process* by Dafermos [6] and LaSalle [14], will be used here. Define

$$\mathbb{R}^2_{\ge} := \{(t, t_0) \in \mathbb{R} \times \mathbb{R} \,:\, t \ge t_0\}\,.$$

Definition 2.1. A *process* on a state space $\mathbb{R}^d$ is a mapping $\phi : \mathbb{R}^2_{\ge} \times \mathbb{R}^d \to \mathbb{R}^d$ with the following properties:

i) *initial condition:* $\phi(t_0, t_0, x_0) = x_0$ for all $x_0 \in \mathbb{R}^d$ and $t_0 \in \mathbb{R}$,
ii) *2-parameter semi-group property:* $\phi(t_2, t_0, x_0) = \phi(t_2, t_1, \phi(t_1, t_0, x_0))$ for all (t_1, t_0), $(t_2, t_1) \in \mathbb{R}^2_{\ge}$ and $x_0 \in \mathbb{R}^d$,
iii) *continuity:* the mapping $(t, t_0, x_0) \mapsto \phi(t, t_0, x_0)$ is continuous.

The analogue of an invariant set for an autonomous semi-dynamical system is too restrictive for a process, i.e., a subset A of $\mathbb{R}^d$ such that $\phi(t, t_0, A) = A$ for all $(t, t_0) \in \mathbb{R}^2_{\ge}$. Such a subset contains steady state solutions if there are any, but excludes almost everything else such as the periodic solution. Instead a family $\mathcal{B} = \{B(t) : t \in \mathbb{R}\}$ of nonempty subsets of $\mathbb{R}^d$ must be used. Such a family $\mathcal{B}$ is said to be ϕ-invariant if $\phi(t, t_0, B(t_0)) = B(t)$ for all $(t, t_0) \in \mathbb{R}^2_{\ge}$ and ϕ-positively invariant if $\phi(t, t_0, B(t_0)) \subset B(t)$ for all $(t, t_0) \in \mathbb{R}^2_{\ge}$.

Similarly, nonautonomous attractors consist of ϕ-invariant families of nonempty compact subsets rather than a single invariant compact subset. There are two different types of nonautonomous attractors for processes, one with pullback convergence and one with forward convergence, with respect to the Hausdorff semi-distance between nonempty compact subsets of $\mathbb{R}^d$, which is denoted by $\text{dist}_{\mathbb{R}^d}(\cdot,\cdot)$.

Definition 2.2. Let ϕ be a process on a state space $\mathbb{R}^d$ with time set $\mathbb{R}$.

A ϕ-invariant family $\mathcal{A} = \{A(t), t \in \mathbb{R}\}$ of nonempty compact subsets of $\mathbb{R}^d$ is called a

i) *forward attractor* if it forward attracts all families $\mathcal{D} = \{D(t), t \in \mathbb{R}\}$ of nonempty bounded subsets of $\mathbb{R}^d$, i.e.,

$$\lim_{t\to\infty} \text{dist}_{\mathbb{R}^d}(\phi(t,t_0,D(t_0)), A(t)) = 0, \qquad (\text{fixed } t_0) \tag{2.1}$$

ii) *pullback attractor* if it pullback attracts all families $\mathcal{D} = \{D(t), t \in \mathbb{R}\}$ of nonempty bounded subsets of $\mathbb{R}^d$, i.e.,

$$\lim_{t_0\to-\infty} \text{dist}_{\mathbb{R}^d}(\phi(t,t_0,D(t_0)), A(t)) = 0, \qquad (\text{fixed } t). \tag{2.2}$$

Pullback attraction uses information of the system in the past, while forward attraction is concerned with the future behaviour. The two convergence concepts are independent. In general, a forward attractor need not be a pullback attractor, or vice versa; Cheban et al. [3], Kloeden & Rasmussen [12].

The existence of a pullback attractor is ensured by that of a pullback absorbing family (see, e.g., Carvalho et al. [1], Kloeden & Rasmussen [12]).

Theorem 2.3. *Suppose that a process ϕ on $\mathbb{R}^d$ has a ϕ-positively invariant pullback absorbing family $\mathcal{B} = \{B(t), t \in \mathbb{R}\}$ of nonempty compact subsets of $\mathbb{R}^d$, i.e., for each $t \in \mathbb{R}$ and every family $\mathcal{D} = \{D(t), t \in \mathbb{R}\}$ of nonempty bounded subsets of $\mathbb{R}^d$ there exists a $T_{t,\mathcal{D}} \in \mathbb{R}^+$ such that*

$$\phi(t,t_0,D(t_0)) \subseteq B(t) \qquad \textit{for all} \quad t_0 \le t - T_{t,\mathcal{D}}.$$

Then ϕ has a global pullback attractor $\mathcal{A} = \{A(t), t \in \mathbb{R}\}$ with component subsets determined by

$$A(t) = \bigcap_{t_0 \le t} \phi(t,t_0,B(t_0)) \qquad \textit{for each} \quad t \in \mathbb{R}. \tag{2.3}$$

Moreover, if $\mathcal{A}$ is uniformly bounded, i.e., if $\bigcup_{t\in\mathbb{R}} A(t)$ is bounded, then it is unique.

The uniqueness of the pullback attractor is assured if, for example, the pullback absorbing family $\mathcal{B}$ is uniformly bounded, i.e., if $\bigcup_{t\in\mathbb{R}} B(t)$ is bounded.

Theorem 2.3 is essentially a necessary and sufficient condition for the existence of a pullback attractor, since it can be shown that every pullback attractor has a positively invariant pullback absorbing family.

3. Nonautonomous forward attractors

The solutions of the nonautonomous scalar ODE $\dot{x} = 2tx$ converge in the pullback sense, but not in the forward sense to the zero solution $x(t) = 0$, whereas for the nonautonomous scalar ODE $\dot{x} = -2tx$ they converge in the forward sense, but not in the pullback sense. The pullback attractor of the first ODE has uniquely defined component subsets $A(t) = \{0\}$. These sets form a forward attractor in the second case, but it is not unique, since for any $r \in \mathbb{R}^+$, the sets $A_r(t) = re^{-t^2}[-1,1]$ are also the component sets of a forward attractor $\mathcal{A}_r$. Interestingly, for this second ODE every solution is bounded and complete (or entire), i.e., defined for all $t \in \mathbb{R}$. In fact, taken alone, each solution of the second ODE is globally attracting in the forward sense, i.e., forms a forward attractor with component subsets consisting of singleton points.

Some properties of forward attractors required later will be established.

Proposition 3.1. *A forward attractor $\mathcal{A} = \{A(t), t \in \mathbb{R}\}$ is Lyapunov asymptotic stable, i.e., both forward attracting* (2.1) *and Lyapunov stable: for each $t_0 \in \mathbb{R}$ and $\varepsilon > 0$ there exists $\delta = \delta(t_0, \varepsilon) > 0$ such that*

$$\operatorname{dist}_{\mathbb{R}^d}\left(\phi(t, t_0, x_0), A(t)\right) < \varepsilon \tag{3.1}$$

for all $t \geq t_0$ and $x_0 \in \mathbb{R}^d$ with $\operatorname{dist}_{\mathbb{R}^d}\left(x_0, A(t_0)\right) \leq \delta$.

This follows from the invariance of $\mathcal{A}$, the continuity with respect to the Hausdorff metric of the set-valued mapping $C \mapsto \phi(t, t_0, C)$ in compact sets uniformly on bounded time intervals $t \in [t_0, T]$ for each fixed t_0 (see Roxin [17]) and the forward attracting property (2.1).

Recall that uniformly bounded means that there exists an $R > 0$ such that $A(t) \subset B_R := \{x \in \mathbb{R}^d : \|x\| \leq R\}$ for each $t \in \mathbb{R}$ and define $B_1[A] := \{x \in \mathbb{R}^d : \operatorname{dist}_{\mathbb{R}^d}(x, A) \leq 1\}$ for any nonempty compact subset A of $\mathbb{R}^d$.

Proposition 3.2. *A uniformly bounded forward attractor $\mathcal{A} = \{A(t), t \in \mathbb{R}\}$ has a ϕ-positively invariant family $\mathcal{B} = \{B(t), t \in \mathbb{R}\}$ of nonempty compact subsets which is forward absorbing.*

Proof. It can be assumed without loss of generality that R is large enough so that $B_1[A(t)] \subset B_R$ for each $t \in \mathbb{R}$. By forward attraction of the set B_R, for each $t_0 \in \mathbb{R}$ there exists $T(t_0, 1) \geq 0$ such that

$$\operatorname{dist}_{\mathbb{R}^d}\left(\phi(t, t_0, B_R), A(t)\right) \leq 1, \qquad t \geq t_0 + T(t_0, 1).$$

In particular, this holds for $t = t_1 := t_0 + T(t_0, 1)$. Set $n_0 = t_0 = 0$ and define n_1 to be the first integer with $n_1 \geq n_0 + T(n_0, 1)$. Then

$$\phi(n_1, n_0, B_1[A(n_0)]) \subset \phi(n_1, n_0, B_R) \subset B_1[A(n_1)].$$

Repeat this construction for $n_k < n_{k+1}$ for $k = 0$, 1,2, $\cdots$ and define

$$B(t) := \phi\left(t, n_k, B_1[A(n_k)]\right), \qquad n_k \leq t < n_{k+1},\ k = 0, 1, 2, \cdots.$$

These sets are obviously nonempty and compact. In particular, $B(n_k) = B_1[A(n_k)]$ and $\phi(t, t_0, B(t_0)) \subset B(t)$ for all $t \geq t_0 \geq 0$.

The construction of $B(t)$ proceeds differently for $t < 0$. Define $r_0 := 1$. Then by continuity with respect to the Hausdorff metric (as used to prove Proposition 3.1) with $\varepsilon = 1$, there exists $r_{-1} := \delta(-1, r_0) = \delta(-1, 1) > 0$ such that

$$\phi(0, -1, B_{r_{-1}}[A(-1)] \cap B_R) \subset B_{r_0}[A(0)] = B_1[A(0)] = B_1[A(0)] \cap B_R.$$

Repeat this construction in the interval $[-k, -k+1]$ for $k = 1$,2, $\cdots$ and define

$$B(t) := \phi\left(t, -k, B_{r_{-k}}[A(-k)] \cap B_R\right), \qquad -k \leq t < -k+1,\ k = 1, 2, \cdots$$

with radius $r_{-k} = \delta(-k, 1) > 0$ according to Proposition 3.1, respectively. These sets are obviously nonempty and compact with $B(-k) = B_{r_{-k}}[A(-k)] \cap B_R$ and $\phi(t, t_0, B(t_0)) \subset B(t)$ for all $t_0 \leq t \leq 0$. Since $r_0 = 1$, the positive invariance property holds for all $t \in \mathbb{R}$.

The family $\mathcal{B} = \{B(t), t \in \mathbb{R}\}$ is by construction forward absorbing provided the time is taken large enough to reach one of the sets $B_1[A(n_k)]$ with $n_k \in \mathbb{N}$, which is possible by the forward attraction property. $\square$

3.1. **Construction of possible forward attractors.** It is possible to construct the component subsets of candidates for forward attractors with the same expression (1.2) as for a pullback attractor. This is based on the observation that a ϕ-positively invariant family of nonempty compact subsets contains a maximal ϕ-invariant family of nonempty compact subsets.

Theorem 3.3. *Suppose that a process ϕ on $\mathbb{R}^d$ has a ϕ-positively invariant family $\mathcal{B} = \{B(t), t \in \mathbb{R}\}$ of nonempty compact subsets of $\mathbb{R}^d$. Then ϕ has a maximal ϕ-invariant family $\mathcal{A} = \{A(t), t \in \mathbb{R}\}$ in $\mathcal{B}$ of nonempty compact subsets determined by*

$$A(t) = \bigcap_{t_0 \leq t} \phi\left(t, t_0, B(t_0)\right) \qquad \text{for each } \; t \in \mathbb{R}. \tag{3.2}$$

In view of Proposition 3.2 the component sets of all forward attractors can be constructed in this way. In particular, for the nonautonomous scalar ODE $\dot{x} = -2tx$ and any $r \in \mathbb{R}^+$ the family $\mathcal{B}_r = \{B_r(t), t \in \mathbb{R}\}$ of nonempty compact subsets $B_r(t) = re^{-t^2}[-1, 1]$ for $t \leq 0$ and $B_r(t) = r[-1, 1]$ for $t \geq 0$ is ϕ-positive invariant (and also forward absorbing). The corresponding subsets given by (3.2) are $A_r(t) = re^{-t^2}[-1, 1]$ for $t \in \mathbb{R}$.

Theorem 3.3 was proved in Theorem 5 of Kloeden & Marín-Rubio [9] by constructing complete solutions in $\mathcal{B}$. The expression (3.2) is clear, since the sets $\{\phi\left(t, t_0, B(t_0)\right), t_0 \leq t\}$ with t fixed are nonempty and compact by the continuity of ϕ and the assumed compactness of the subsets $B(t_0)$. They are also nested since by the 2-parameter semi-group property and the positive invariance,

$$\phi\left(t, t_0, B(t_0)\right) = \phi\left(t, t_1, \phi\left(t_1, t_0, B(t_0)\right)\right) \subset \phi\left(t, t_1, B(t_1)\right)$$

for (t, t_0), (t, t_1), $(t_1, t_0) \in \mathbb{R}^2_{\geq}$.

Note that this pullback construction is used only inside the ϕ-positively invariant family $\mathcal{B}$. It is equivalent to

$$\lim_{t_0 \to -\infty} \operatorname{dist}_{\mathbb{R}^d}\left(\phi(t, t_0, B(t_0)), A(t)\right) = 0, \quad (\text{fixed } t). \tag{3.3}$$

Theorem 3.3 does not imply that the subsets given by (3.2) form a pullback attractor, since nothing has been assumed so far about what is happening outside of the subsets $B(t)$ in $\mathcal{B}$. With the additional assumption that $\mathcal{B}$ is pullback absorbing, then Theorem 2.3 holds and the family $\mathcal{A} = \{A(t), t \in \mathbb{R}\}$ is a pullback attractor.

On the other hand, the assumption that the ϕ-positively invariant family $\mathcal{B} = \{B(t), t \in \mathbb{R}\}$ is forward absorbing, i.e., for each $t_0 \in \mathbb{R}$ and every family $\mathcal{D} = \{D(t), t \in \mathbb{R}\}$ of nonempty bounded subsets of $\mathbb{R}^d$ there exists a $T_{t_0, \mathcal{D}} \in \mathbb{R}^+$ such that

$$\phi\left(t, t_0, D(t_0)\right) \subseteq B(t) \qquad \text{for all} \quad t \geq t_0 + T_{t_0, \mathcal{D}},$$

need not ensure that forward convergence within $\mathcal{B}$

$$\lim_{t \to \infty} \operatorname{dist}_{\mathbb{R}^d}\left(\phi(t, t_0, B(t_0)), A(t)\right) = 0, \quad (\text{fixed } t_0) \tag{3.4}$$

or, more so, not for a general family $\mathcal{D} = \{D(t), t \in \mathbb{R}\}$ of nonempty bounded subsets of $\mathbb{R}^d$ instead of $\mathcal{B} = \{B(t), t \in \mathbb{R}\}$.

Similar difficulties occur for skew product flows, but the further assumptions on the nature of the nonautonomity through the driving system allow additional insights into the forward dynamics, e.g., Kloeden & Rasmussen [12].

3.2. **Forward asymptotic behaviour.** To avoid distracting complications, it will be supposed here that a process ϕ on $\mathbb{R}^d$ has a ϕ-positively invariant family $\mathcal{B} = \{B(t), t \in \mathbb{R}\}$ of nonempty compact subsets of $\mathbb{R}^d$, so by Theorem 3.3 it has a ϕ-invariant family $\mathcal{A} = \{A(t), t \in \mathbb{R}\}$ of nonempty compact subsets of $\mathbb{R}^d$ with $A(t) \subset B(t)$ for each $t \in \mathbb{R}$. Moreover, it will be supposed also that $\mathcal{B}$ is uniformly bounded, i.e., there exists a compact subset B of $\mathbb{R}^d$ such that $B(t) \subset B$ for each $t \in \mathbb{R}$. Then, $\mathcal{A}$ is also uniformly bounded with $A(t) \subset B$ for each $t \in \mathbb{R}$.

The ϕ-invariant family $\mathcal{A}$ with component subsets given by expression (2.3) in Theorem 3.3 does not necessarily reflect the full limiting dynamics in the forward time direction of the process. In particular, it need not be forward attracting in the sense of (3.4).

This is easily seen in the following example of a process ϕ on $\mathbb{R}^1$ generated by the switching ODE

$$\dot{x} = f(x) := \begin{cases} -x & : \quad t \leq 0, \\ x\left(1 - x^2\right) & : \quad t > 0. \end{cases} \tag{3.5}$$

This nonautonomous system is asymptotically autonomous in both time directions with the limiting autonomous systems equal to the component systems holding on the whole time set $\mathbb{R}$. In the linear autonomous system the steady state solution 0 is asymptotically stable, while in the nonlinear autonomous system it is unstable and the steady state solutions ± 1 are (locally) asymptotically stable. These autonomous systems have global autonomous attractors given by $\{0\}$ and $[-1, 1]$, respectively. (See [10, 11] for similar discrete time examples.)

The family $\mathcal{B}$ of constant sets $B(t) = \{x \in \mathbb{R} : |x| \leq 2\}$ is ϕ-positively invariant. The corresponding family $\mathcal{A}$ constructed by the pullback method in (2.3) has identical component subsets $A(t) \equiv \{0\}$, $t \in \mathbb{R}$, corresponding to the zero entire solution and is the only bounded entire solution of the process. (It is, in fact, the pullback attractor of the nonautonomous system generated by (3.5), but that is not of interest here.) It is obviously not forward asymptotically attracting. Indeed, the set of forward omega limit points is $[-1, 1]$, which is not invariant for the process. In particular, the boundary points ± 1 are not entire solutions of the process, so cannot belong to a nonautonomous attractor, forward or pullback, since these consist of entire solutions.

3.3. **Omega limit points of a process.** The asymptotic dynamics of the process ϕ inside the uniformly bounded ϕ-positively invariant family $\mathcal{B} = \{B(t), t \in \mathbb{R}\}$ of nonempty compact subsets of $\mathbb{R}^d$ will be investigated more closely here.

For each $t_0 \in \mathbb{R}$, the forward omega limit set with respect to $\mathcal{B}$ is defined by

$$\omega_{\mathcal{B}}(t_0) := \bigcap_{t \geq t_0} \overline{\bigcup_{s \geq t} \phi\left(s, t_0, B(t_0)\right)},$$

which is nonempty and compact as the intersection of nonempty nested compact subsets. In particular,

$$\lim_{t \to \infty} \mathrm{dist}_{\mathbb{R}^d}\left(\phi(t, t_0, B(t_0)), \omega_{\mathcal{B}}(t_0)\right) = 0, \quad (\text{fixed } t_0). \tag{3.6}$$

Since $A(t_0) \subset B(t_0)$ and $A(t) = \phi(t, t_0, A(t_0)) \subset \phi(t, t_0, B(t_0))$, it follows that

$$\lim_{t\to\infty} \mathrm{dist}_{\mathbb{R}^d} \left(A(t), \omega_{\mathcal{B}}(t_0)\right) = 0, \quad (\text{fixed } t_0). \tag{3.7}$$

Moreover, $\phi(t, t_0, B(t_0)) \subset B(t)$ for each $t \geq t_0$ since $\mathcal{B}$ is ϕ-positively invariant, so

$$\omega_{\mathcal{B}}(t_0) \subset \omega_{\mathcal{B}}(t_0') \subset B, \qquad t_0 \leq t_0',$$

where the final inclusion is from the uniform boundedness of $\mathcal{B}$. Hence the sets

$$\omega_{\mathcal{B}}^{-\infty} := \bigcap_{t_0\in\mathbb{R}} \omega_{\mathcal{B}}(t_0), \qquad \omega_{\mathcal{B}}^{+\infty} := \overline{\bigcup_{t_0\in\mathbb{R}} \omega_{\mathcal{B}}(t_0)}$$

are nonempty compact sets with $\omega_{\mathcal{B}}^{-\infty} \subset \omega_{\mathcal{B}}^{+\infty} \subset B$. From (3.6) it is clear that

$$\lim_{t\to\infty} \mathrm{dist}_{\mathbb{R}^d} \left(A(t), \omega_{\mathcal{B}}^{-\infty}\right) = 0. \tag{3.8}$$

Consider now the set of omega limit points for dynamics starting inside the family of sets $\mathcal{A} = \{A(t), t \in \mathbb{R}\}$ (note the sets here need not be nested), which is defined by

$$\omega_{\mathcal{A}} := \bigcap_{t_0\in\mathbb{R}} \overline{\bigcup_{t\geq t_0} A(t)} = \bigcap_{t_0\in\mathbb{R}} \overline{\bigcup_{t\geq t_0} \phi(t, t_0, A(t_0))}$$

which is nonempty and compact as a family of nested compact sets.

Obviously, $\omega_{\mathcal{A}} \subset \omega_{\mathcal{B}}^{-\infty} \subset \omega_{\mathcal{B}}^{+\infty} \subset B$. The inclusions here may be strict. For example, for the process generated by the ODE (3.5), $\omega_{\mathcal{A}} = \{0\}$, while $\omega_{\mathcal{B}}^{-\infty} = \omega_{\mathcal{B}}^{+\infty} = [-1, 1]$. The ODE (3.5) can be changed to include a second switching at, say, time $t = 10$ to the nonlinear ODE $\dot{x} = x(1 - x^2)(x^2 - 4)$. Then $\omega_{\mathcal{B}}^{-\infty} = [-1, 1]$, while $\omega_{\mathcal{B}}^{+\infty} = [-2, 2]$.

3.4. **Conditions ensuring forward convergence.** The existence of omega limit points in $\omega_{\mathcal{B}}^{+\infty}$ that are not in $\omega_{\mathcal{A}}$ means that $\mathcal{A}$ cannot be forward attracting from within $\mathcal{B}$. The converse also holds.

Theorem 3.4. *$\mathcal{A}$ is forward attracting from within $\mathcal{B}$, i.e., the forward convergence (3.4) holds, if and only if $\omega_{\mathcal{A}} = \omega_{\mathcal{B}}^{+\infty}$.*

Proof. Suppose that the forward convergence (3.4) does not hold. Then there is an $\varepsilon_0 > 0$ and a sequence $t_n \to \infty$ as $n \to \infty$ such that $\mathrm{dist}_{\mathbb{R}^d} \left(\phi(t_n, t_0, B(t_0)), A(t_n)\right) \geq \varepsilon_0$ for all $n \in \mathbb{N}$. Since the sets here are compact there exist points $\phi_n \in \phi(t_n, t_0, B(t_0))$ such that

$$\mathrm{dist}_{\mathbb{R}^d} \left(\phi_n, A(t_n)\right) = \mathrm{dist}_{\mathbb{R}^d} \left(\phi(t_n, t_0, B(t_0)), A(t_n)\right) \geq \varepsilon_0.$$

Then for any points $a_n \in A(t_n)$

$$\|\phi_n - a_n\| \geq \mathrm{dist}_{\mathbb{R}^d} \left(\phi_n, A(t_n)\right) \geq \varepsilon_0.$$

In particular, $\|\phi_n - a_n\| \geq \varepsilon_0$ for all $n \in \mathbb{N}$.

Now the ϕ_n and a_n belong to the compact set B, so there are convergent subsequences $\phi_{n_j} \to \bar{\phi} \in B$ and $a_{n_j} \to \bar{a} \in B$. It follows that $\|\bar{\phi} - \bar{a}\| \geq \varepsilon_0$. From the definitions $\bar{\phi} \in \omega_{\mathbb{B}}(t_0) \subset \omega_{\mathcal{B}}^{+\infty}$ and $\bar{a} \in \omega_{\mathcal{A}}$. Since the a_n and hence $\bar{a}$ were otherwise arbitrary, it follows that $\mathrm{dist}_{\mathbb{R}^d} \left(\bar{\phi}, \omega_{\mathcal{A}}\right) \geq \varepsilon_0$, so $\mathrm{dist}_{\mathbb{R}^d} \left(\omega_{\mathcal{B}}^{+\infty}, \omega_{\mathcal{A}}\right) \geq \varepsilon_0$ and hence $\omega_{\mathcal{A}} \neq \omega_{\mathcal{B}}^{+\infty}$. $\square$

Forward attraction also holds when the rate of pullback convergence from within $\mathcal{B}$ to construct the component sets $A(t)$ of $\mathcal{A}$ in Theorem 3.3 is uniform. In fact, it suffices that the convergence rate is eventually uniform.

Theorem 3.5. *The family $\mathcal{A}$ with component sets constructed by* (3.2) *is forward attracting from within $\mathcal{B}$, i.e., the forward convergence* (3.4) *holds, if the rate of pullback convergence from within $\mathcal{B}$ to the component sets $A(t)$ of $\mathcal{A}$ is eventually uniform in the sense that for every $\varepsilon > 0$ there exist $\tau(\varepsilon) \in \mathbb{R}$ and $T(\varepsilon) > 0$ such that for each $t \geq \tau(\varepsilon)$*

$$\operatorname{dist}_{\mathbb{R}^d}\left(\phi(t, t_0, B(t_0)), A(t)\right) < \varepsilon \tag{3.9}$$

holds for all $t_0 \leq t - T(\varepsilon)$.

Proof. Given any $t_0 \in \mathbb{R}$ and $\varepsilon > 0$, it is clear that the inequality (3.9) holds for every $t \geq \max\{t_0, \tau(\varepsilon)\} + T(\varepsilon)$, i.e., the forward convergence (3.4) holds. □

Theorem 3.5 is concerned only with the rate of pullback convergence starting from inside the positively invariant family $\mathcal{B}$. It makes no assumptions about what is happening outside of $\mathcal{B}$ such as pullback or forward absorption.

The family $\mathcal{A}$ for the simple switching system generated by the ODE (3.5) is not forward attracting. By considering any time $t > 0$, it is not hard to see that the rate of pullback convergence is not eventually uniform.

4. Asymptotically autonomous systems

The ODE (3.5) generates a simple process ϕ on $\mathbb{R}^1$ formed by switching between two autonomous systems. It is a special case of an asymptotically autonomous system (Hale [7], Kato et al. [8], LaSalle [14]). These have additional structure that provides an easier way to check the condition of Theorem 3.4 for forward convergence. The following theorem is a modification of a theorem in Kloeden & Simsen [13].

Theorem 4.1. *Let ϕ be a uniformly bounded process on $\mathbb{R}^d$ with a ϕ-positively invariant family $\mathcal{B} = \{B(t), t \in \mathbb{R}\}$ of nonempty compact subsets of $\mathbb{R}^d$ with all component subsets contained in a compact set B. Let π be an autonomous semi-dynamical system on $\mathbb{R}^d$ with a global attractor A_∞ in B. In addition, suppose that*

$$\phi(t + \tau, \tau, x_\tau) \to \pi(t, x_0) \quad \text{as } \tau \to +\infty \tag{4.1}$$

uniformly in $t \in \mathbb{R}^+$ whenever $x_\tau \in \mathcal{B}(\tau)$ and $x_\tau \to x_0$ as $\tau \to +\infty$.

Then $\omega_{\mathcal{B}}^{+\infty}(t_0) \subset A_\infty$ for each $t_0 \in \mathbb{R}$ and

$$\lim_{t \to +\infty} \operatorname{dist}_{\mathbb{R}^d}\left(A(t), A_\infty\right) = 0, \tag{4.2}$$

where $\mathcal{A} = \{A(t), t \in \mathbb{R}\}$ is the family of nonempty compact subsets of $\mathbb{R}^d$ with all component subsets defined by pullback characterization (3.2) *from within $\mathcal{B}$.*

Proof. Fix $t \in \mathbb{R}^+$ and $t_0 \in \mathbb{R}$ and define $s_n := t_n - t$, where the sequence $t_n \to \infty$ as $n \to \infty$. By the 2-parameter semi-group property and the ϕ-positive invariance of $\mathcal{B}$

$$\phi\left(t_n, t_0, B(t_0)\right) = \phi\left(t_n, s_n, \phi\left(s_n, t_0, B(t_0)\right)\right) \subset \phi\left(t_n, s_n, B(s_n)\right).$$

Hence $\phi\left(t + s_n, t_0, B(t_0)\right) \subset \phi\left(t + s_n, s_n, B(s_n)\right)$. By compactness of the sets involved, for each $n \in \mathbb{N}$, there exists a $b_n \in B(s_n) \subset B$ such that

$$\operatorname{dist}_{\mathbb{R}^d}\left(\phi\left(t + s_n, s_n, b_n\right), A_\infty\right) = \operatorname{dist}_{\mathbb{R}^d}\left(\phi\left(t + s_n, s_n, B(s_n)\right), A_\infty\right).$$

By the compactness of the set B there is a convergent subsequence (labelled for convenience as the original one) $b_n \to \bar{b} \in B$ as $n \to \infty$. Hence,

$$\begin{aligned} \mathrm{dist}_{\mathbb{R}^d}\left(\phi\left(t+s_n, s_n, b_n\right), A_\infty\right) &\leq \mathrm{dist}_{\mathbb{R}^d}\left(\phi\left(t+s_n, s_n, b_n\right), \pi\left(t, \bar{b}\right)\right) \\ &\quad + \mathrm{dist}_{\mathbb{R}^d}\left(\pi\left(t, \bar{b}\right), A_\infty\right) \\ &\leq \varepsilon + \mathrm{dist}_{\mathbb{R}^d}\left(\pi\left(t, \bar{b}\right), A_\infty\right) \end{aligned}$$

for any $\varepsilon > 0$ and $n \geq N(\varepsilon)$ uniformly in $t \in \mathbb{R}$ by the asymptotic autonomous condition (4.1). Since A_∞ is the global attractor of π for any $\varepsilon > 0$ there exists a $T(\varepsilon, B) \geq 0$ such that

$$\mathrm{dist}_{\mathbb{R}^d}\left(\pi\left(t, \bar{b}\right), A_\infty\right) \leq \mathrm{dist}_{\mathbb{R}^d}\left(\pi\left(t, B\right), A_\infty\right) < \varepsilon \quad \text{for all} \quad t \geq T(\varepsilon, B).$$

Combining the results gives

$$\lim_{t \to +\infty} \mathrm{dist}_{\mathbb{R}^d}\left(\phi\left(t, t_0, B(t_0)\right), A_\infty\right) = 0, \tag{4.3}$$

for each $t_0 \in \mathbb{R}$. This means that $\omega_{\mathcal{B}}^{+\infty}(t_0) \subset A_\infty$ for each $t_0 \in \mathbb{R}$.

The limit (4.2) then follows since $A(t) = \phi(t, t_0, A(t_0)) \subset \phi(t, t_0, B(t_0))$, so

$$\mathrm{dist}_{\mathbb{R}^d}\left(A(t), A_\infty\right) = \mathrm{dist}_{\mathbb{R}^d}\left(\phi\left(t, t_0, A(t_0)\right), A_\infty\right) \leq \mathrm{dist}_{\mathbb{R}^d}\left(\phi\left(t, t_0, B(t_0)\right), A_\infty\right).$$

□

Combining Theorem 3.4 and the limit (4.3) in the proof of Theorem 4.1 gives:

Corollary 4.2. *Suppose that the assumptions of Theorems* 3.4 *and* 4.1 *hold and that $\omega_{\mathcal{A}} = A_\infty$. Then $\mathcal{A}$ is forward attracting from within $\mathcal{B}$, i.e., the forward convergence* (3.4) *holds.*

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