PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 144, Number 1, January 2016, Pages 103–108
http://dx.doi.org/10.1090/proc/12846
Article electronically published on August 5, 2015

ON WEAK COMPACTNESS IN LEBESGUE-BOCHNER SPACES

JOSÉ RODRÍGUEZ

(Communicated by Thomas Schlumprecht)

ABSTRACT. Let X be a Banach space, (Ω,Σ,μ) a probability space and K a weakly compact subset of $L^p(\mu,X)$, $1\leq p<\infty$. The following question was posed by J. Diestel: is there a weakly compactly generated subspace $Y\subset X$ such that $K\subset L^p(\mu,Y)$? We show that, in general, the answer is negative. We also prove that the answer is affirmative if either μ is separable or X is weakly sequentially complete.

1. Introduction

The study of weakly compact sets and weakly Cauchy sequences in Lebesgue-Bochner spaces has attracted the attention of many authors over the years. The papers by Diestel, Ruess and Schachermayer [7] (preceeded by Ülger [14]) and Talagrand [13] are fundamental contributions to the subject. However, several problems related to (conditional) weak compactness in Lebesgue-Bochner spaces are still open. In this note we address a question raised by Diestel in [6]. Throughout this paper X is a Banach space, (Ω,Σ,μ) a probability space and, for $1\leq p<\infty$, we consider the Lebesgue-Bochner space $L^p(\mu,X)$, that is, the Banach space of all (equivalence classes) of strongly measurable functions $f:\Omega\to X$ for which $\|f\|_{L^p(\mu,X)}=(\int_\Omega\|f(\cdot)\|^p\,d\mu)^{1/p}<\infty$. The following was asked in [6, (Q6)]: *given any weakly compact set $K\subset L^p(\mu,X)$, is there a weakly compactly generated subspace $Y\subset X$ such that $K\subset L^p(\mu,Y)$?* Clearly, the question has an affirmative answer whenever K is separable (in this case Y can be taken separable). It is also clear that, in order to look for a positive solution, it suffices to deal with $L^1(\mu,X)$ (since the formal inclusion from $L^p(\mu,X)$ to $L^1(\mu,X)$ is linear and continuous).

Our main results read as follows:

Theorem 1.1. *If μ is not separable and $1\leq p<\infty$, then there is a weakly compact set $K\subset L^p(\mu,\ell^\infty)$ such that $K\not\subset L^p(\mu,Y)$ for any separable subspace $Y\subset\ell^\infty$.*

Theorem 1.1 provides a negative answer to Diestel's question, because every weakly compact subset of ℓ^∞ is separable (see e.g. [8, p. 252, Theorem 13]).

Theorem 1.2. *Suppose either μ is separable or X is weakly sequentially complete. Let $1\leq p<\infty$. If $K\subset L^p(\mu,X)$ is weakly compact, then there is a weakly compactly generated subspace $Y\subset X$ such that $K\subset L^p(\mu,Y)$.*

Received by the editors October 28, 2014.

2010 *Mathematics Subject Classification.* Primary 46B50, 46G10.

Key words and phrases. Weakly compact set, weakly compactly generated Banach space, Lebesgue-Bochner space, separable measure.

This research was supported by MINECO and FEDER under project MTM2011-25377.

The proofs of Theorems 1.1 and 1.2 are given in Section 2.

We use standard terminology as can be found in [8] and [9]. The measure μ is called separable if the space $L^1(\mu)$ is separable. We write $\Sigma^+ = \{A \in \Sigma : \mu(A) > 0\}$. The characteristic function of a set S is denoted by 1_S. Our Banach spaces are real. By a subspace of a Banach space we mean a closed linear subspace. A Banach space Y is weakly compactly generated if there is a weakly compact set $L \subset Y$ such that $\overline{span}(L) = Y$ or, equivalently, there is a sequence (L_n) of weakly compact subsets of Y such that $\overline{span}(\bigcup_{n\in\mathbb{N}} L_n) = Y$.

2. Proofs

Lemma 2.1. *Let I be an infinite set and μ_I the usual probability measure on $\{0,1\}^I$. Then there is a set $\{f_i : i \in I\}$ of continuous functions on $\{0,1\}^I$ with values in $\{-1,1\}$ such that, for every Borel measurable set $A \subset \{0,1\}^I$, we have*

$$\Big(\int_A f_i \, d\mu_I \Big)_{i\in I} \in c_0(I).$$

Proof. For each $i \in I$, let $\pi_i : \{0,1\}^I \to \{0,1\}$ be the ith-coordinate projection and define the clopen $E_{i,k} := \{x \in \{0,1\}^I : \pi_i(x) = k\}$ for $k \in \{0,1\}$. A measurable cylinder is a set $C \subset \{0,1\}^I$ of the form $C = \bigcap_{i\in J} E_{i,k_i}$ for some finite set $J \subset I$ and some $k_i \in \{0,1\}$. Let $\mathcal{C}$ be the set of all measurable cylinders of $\{0,1\}^I$.

Define the continuous function $f_i := 1_{E_{i,0}} - 1_{E_{i,1}}$ for all $i \in I$. We claim that the set $\{f_i : i \in I\}$ satisfies the required property. Indeed, note first that given any $C = \bigcap_{i\in J} E_{i,k_i} \in \mathcal{C}$ as above, then for every $i \in I \setminus J$ we have

$$\mu_I(C \cap E_{i,0}) = \mu_I(C \cap E_{i,1}) = 2^{-(card(J)+1)}$$

and so $\int_C f_i \, d\mu_I = 0$. Now let $A \subset \{0,1\}^I$ be an arbitrary Borel set and fix $\varepsilon > 0$. Since $L^1(\mu_I) = \overline{span}\{1_C : C \in \mathcal{C}\}$ (see e.g. [10, 254Q]), we can find finitely many $C_1, \dots, C_p \in \mathcal{C}$ and $a_1, \dots, a_p \in \mathbb{R}$ such that

$$\Big\| 1_A - \sum_{k=1}^{p} a_k 1_{C_k} \Big\|_{L^1(\mu_I)} \le \varepsilon.$$

Take a finite set $J \subset I$ such that

$$\int_{C_k} f_i \, d\mu_I = 0 \quad \text{for every } i \in I \setminus J \text{ and every } k \in \{1, \dots, p\}.$$

Write $g := \sum_{k=1}^{p} a_k 1_{C_k}$. For every $i \in I \setminus J$ we have $\int_{\{0,1\}^I} g f_i \, d\mu_I = 0$ and so

$$\Big| \int_A f_i \, d\mu_I \Big| = \Big| \int_{\{0,1\}^I} 1_A f_i \, d\mu_I \Big| = \Big| \int_{\{0,1\}^I} (1_A - g) f_i \, d\mu_I \Big| \le \|1_A - g\|_{L^1(\mu_I)} \le \varepsilon.$$

This proves that $(\int_A f_i \, d\mu_I)_{i\in I} \in c_0(I)$. □

A set Q in a Banach space is called conditionally weakly compact (or weakly precompact) if every sequence in Q has a weakly Cauchy subsequence; thanks to Rosenthal's ℓ^1-theorem (see e.g. [9, Theorem 5.37]), this is equivalent to saying that Q is bounded and contains no sequence equivalent to the usual basis of ℓ^1.

We include a proof of the following known fact for the reader's convenience.

Lemma 2.2. *There exist conditionally weakly compact subsets of ℓ^∞ which are not separable.*

Proof. By Rosenthal's ℓ^1-theorem, it suffices to find a non-separable subspace X of ℓ^∞ not containing subspaces isomorphic to ℓ^1. Such a space can be obtained by taking $X = C(K)$ where K is any separable, non-metrizable, scattered compact space (see e.g. [9, Theorem 14.25]). A concrete example of this kind is given by the Johnson-Lindenstrauss space JL_0, which is defined as

$$JL_0 := \overline{span}\big(c_0 \cup \{1_{B_\alpha} : \alpha < \mathfrak{c}\}\big) \subset \ell^\infty,$$

where $\{B_\alpha : \alpha < \mathfrak{c}\}$ is an almost disjoint family of infinite subsets of $\mathbb{N}$ (see e.g. [9, Theorem 14.54]). $\square$

Proof of Theorem 1.1. Given any $D \in \Sigma^+$, write $\Sigma_D := \{A \cap D : A \in \Sigma\}$ and consider the restriction $\mu|_{\Sigma_D}$. We have a natural isometric embedding of $L^p(\mu|_{\Sigma_D}, X)$ as a complemented subspace of $L^p(\mu, X)$ (for an arbitrary Banach space X). Therefore, it suffices to find $D \in \Sigma^+$ and a weakly compact set $K \subset L^p(\mu|_{\Sigma_D}, \ell^\infty)$ such that $K \not\subset L^p(\mu|_{\Sigma_D}, Y)$ for any separable subspace $Y \subset \ell^\infty$.

Since μ is not separable, Maharam's theorem (see e.g. [12, Theorems 7 and 8]) ensures the existence of $D \in \Sigma^+$ and an uncountable set I such that the measure algebras of $\frac{1}{\mu(D)}\mu|_{\Sigma_D}$ and μ_I are isomorphic. Bearing in mind the previous paragraph, we can assume without loss of generality that $D = \Omega$. By Lemma 2.1, there exists a set $\{g_i : i \in I\} \subset L^\infty(\mu)$ with $|g_i| = 1$ for all $i \in I$ such that, for every $A \in \Sigma$, we have $(\int_A g_i \, d\mu)_{i \in I} \in c_0(I)$. Fix an injective map $\phi : \omega_1 \to I$.

Let $\{x_\alpha : \alpha < \omega_1\}$ be any conditionally weakly compact subset of ℓ^∞ which is norm discrete and has cardinality ω_1 (its existence is ensured by Lemma 2.2). We shall check that the set

$$K := \overline{\{g_{\phi(\alpha)}(\cdot)\, x_\alpha : \alpha < \omega_1\}}^{weak} \subset L^p(\mu, \ell^\infty)$$

satisfies the required properties. Here we write $g_{\phi(\alpha)}(\cdot)\, x_\alpha$ for the function given by $t \mapsto g_{\phi(\alpha)}(t)\, x_\alpha$ (which is sometimes denoted by $g_{\phi(\alpha)} \otimes x_\alpha$).

On one hand, K is weakly compact. Indeed, if (α_n) is any sequence of distinct elements of ω_1, then $(g_{\phi(\alpha_n)})$ is normalized and weakly null in $L^p(\mu)$ (note that $\int_A g_{\phi(\alpha_n)} \, d\mu \to 0$ for all $A \in \Sigma$). Since in addition $\{x_{\alpha_n} : n \in \mathbb{N}\}$ is conditionally weakly compact, the sequence $(g_{\phi(\alpha_n)}(\cdot)\, x_{\alpha_n})$ is weakly null in $L^p(\mu, \ell^\infty)$; see [4, Proposition 4.1]. This shows that $\{g_{\phi(\alpha)}(\cdot)\, x_\alpha : \alpha < \omega_1\}$ is weakly relatively sequentially compact and so K is weakly compact in $L^p(\mu, \ell^\infty)$.

On the other hand, if $Y \subset \ell^\infty$ is any subspace such that $K \subset L^p(\mu, Y)$, then we have $x_\alpha \in Y$ for all $\alpha < \omega_1$ (because $g_{\phi(\alpha)}$ takes values in $\{-1, 1\}$), hence Y is not separable. The proof is finished. $\square$

Remark 2.3. The idea of Theorem 1.1 can be adapted to obtain counterexamples to Diestel's question even if X is assumed to be a *subspace* of a weakly compactly generated space. Indeed, there exist Banach spaces X such that:

- X contains no subspace isomorphic to ℓ^1,
- X is not weakly compactly generated,
- X is a subspace of a weakly compactly generated Banach space;

see [1] and [2].

Lemma 2.4. *If $f : \Omega \to X$ is Bochner integrable, then*

$$f(t) \in \overline{span}\Big\{ \int_A f \, d\mu : \, A \in \Sigma \Big\}$$

for μ-a.e. $t \in \Omega$.

Proof. Write $Y := \overline{span}\{\int_A f \, d\mu : A \in \Sigma\} \subset X$. Fix $n \in \mathbb{N}$. Since f is strongly measurable, there is a countable collection $\mathcal{A}$ of pairwise disjoint elements of Σ^+ such that $\mu(\bigcup \mathcal{A}) = 1$ and

$$\sup_{A \in \mathcal{A}} \sup_{t,t' \in A} \|f(t) - f(t')\| \leq \frac{1}{n}$$

(see e.g. [8, p. 42, Corollary 3]). This implies that for every $A \in \mathcal{A}$ and every $t \in A$ we have

$$\Big\| f(t) - \frac{1}{\mu(A)} \int_A f \, d\mu \Big\| = \Big\| \frac{1}{\mu(A)} \int_A (f(t) - f) \, d\mu \Big\| \leq \frac{1}{\mu(A)} \int_A \|f(t) - f\| \, d\mu \leq \frac{1}{n}.$$

Thus, the function $g_n : \Omega \to Y$ defined by

$$g_n := \sum_{A \in \mathcal{A}} \Big(\frac{1}{\mu(A)} \int_A f \, d\mu \Big) 1_A$$

satisfies $\|f - g_n\| \leq \frac{1}{n}$ μ-a.e. As $n \in \mathbb{N}$ is arbitrary, $f(t) \in Y$ for μ-a.e. $t \in \Omega$. $\square$

Proof of Theorem 1.2. As explained in the introduction, it is enough to consider the case $p = 1$. Write

$$X_K := \overline{span}\Big\{ \int_A f \, d\mu : \, f \in K, \, A \in \Sigma \Big\} \subset X.$$

Since $K \subset L^1(\mu, X_K)$ (by Lemma 2.4), it suffices to check that X_K is weakly compactly generated. The proof is divided into two cases.

Case 1: *μ is separable.* Fix a countable set $\mathcal{D} \subset \Sigma$ such that

$$\inf_{D \in \mathcal{D}} \mu(A \triangle D) = 0 \quad \text{for every } A \in \Sigma.$$

Then

$$X_K = \overline{span}\Big\{ \int_D f \, d\mu : \, f \in K, \, D \in \mathcal{D} \Big\} = \overline{span}\Big(\bigcup_{D \in \mathcal{D}} I_D(K) \Big), \tag{1}$$

where $I_D : L^1(\mu, X) \to X$ is the integration operator given by $I_D(f) := \int_D f \, d\mu$. Since K is weakly compact, the sets $I_D(K)$ are weakly compact in X. From (1) and the fact that $\mathcal{D}$ is countable it follows that X_K is weakly compactly generated.

Case 2: *X is weakly sequentially complete.* Since K is conditionally weakly compact in $L^1(\mu, X)$, the same holds for $\hat{K} := \{f 1_A : \, f \in K, \, A \in \Sigma\}$; see [5, Proposition 10] (cf. [4, Corollary 4.2] for a simpler proof). By applying the integration operator $I_\Omega : L^1(\mu, X) \to X$, $I_\Omega(f) := \int_\Omega f \, d\mu$, we conclude that

$$I_\Omega(\hat{K}) = \Big\{ \int_A f \, d\mu : \, f \in K, \, A \in \Sigma \Big\}$$

is conditionally weakly compact in X. The weak sequential completeness of X ensures that $I_\Omega(\hat{K})$ is relatively weakly compact. It follows that $X_K = \overline{span}(I_\Omega(\hat{K}))$ is weakly compactly generated and the proof is over. $\square$

Remark 2.5. The proof of Theorem 1.2 shows that, in general:

(i) If κ is the Maharam type of μ (i.e. the density character of $L^1(\mu)$) and $K \subset L^p(\mu, X)$ is weakly compact, then there is a family $\mathcal{C}$ of weakly compact subsets of X such that $card(\mathcal{C}) \leq \kappa$ and $K \subset L^p(\mu, \overline{span}(\bigcup \mathcal{C}))$. We refer the reader to [3] for an account on Banach spaces generated by κ-many weakly compact sets, where κ is an uncountable cardinal.

(ii) If $K \subset L^p(\mu, X)$ is conditionally weakly compact, then there is a conditionally weakly compact set $Q \subset X$ such that $K \subset L^p(\mu, \overline{span}(Q))$. The class of Banach spaces generated by conditionally weakly compact sets was introduced in [11, Section 6].

Acknowledgement

We thank A. Avilés for valuable conversations on this topic.

References

[1] S. A. Argyros, *Weakly Lindelöf determined Banach spaces not containing $\ell^1(N)$*, preprint (1993), arXiv:math/9210210v1.

[2] Spiros A. Argyros and Sophocles Mercourakis, *Examples concerning heredity problems of WCG Banach spaces*, Proc. Amer. Math. Soc. **133** (2005), no. 3, 773–785 (electronic), DOI 10.1090/S0002-9939-04-07532-X. MR2113927 (2005m:46018)

[3] Antonio Avilés, *The number of weakly compact sets which generate a Banach space*, Israel J. Math. **159** (2007), 189–204, DOI 10.1007/s11856-007-0042-6. MR2342477 (2008m:46023)

[4] Jürgen Batt and Wolfgang Hiermeyer, *On compactness in $L_p(\mu, X)$ in the weak topology and in the topology $\sigma(L_p(\mu, X), L_q(\mu, X'))$*, Math. Z. **182** (1983), no. 3, 409–423, DOI 10.1007/BF01179760. MR696537 (84m:46039)

[5] J. Bourgain, *An averaging result for l^1-sequences and applications to weakly conditionally compact sets in L^1_X*, Israel J. Math. **32** (1979), no. 4, 289–298, DOI 10.1007/BF02760458. MR571083 (81i:46021)

[6] Joe Diestel, *Some problems arising in connection with the theory of vector measures*, Séminaire Choquet, 17e année (1977/78), Initiation à l'analyse, Fasc. 2, Secrétariat Math., Paris, 1978, pp. Exp. No. 23, 11. MR522987 (80d:46077)

[7] J. Diestel, W. M. Ruess, and W. Schachermayer, *On weak compactness in $L^1(\mu, X)$*, Proc. Amer. Math. Soc. **118** (1993), no. 2, 447–453, DOI 10.2307/2160321. MR1132408 (93g:46033)

[8] J. Diestel and J. J. Uhl Jr., *Vector measures*, American Mathematical Society, Providence, R.I., 1977. With a foreword by B. J. Pettis; Mathematical Surveys, No. 15. MR0453964 (56 #12216)

[9] Marián Fabian, Petr Habala, Petr Hájek, Vicente Montesinos, and Václav Zizler, *Banach space theory*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, Springer, New York, 2011. The basis for linear and nonlinear analysis. MR2766381 (2012h:46001)

[10] D. H. Fremlin, *Measure theory. Vol. 2*, Torres Fremlin, Colchester, 2003. Broad foundations; Corrected second printing of the 2001 original. MR2462280 (2011a:28001)

[11] Richard Haydon, *Nonseparable Banach spaces*, Functional analysis: surveys and recent results, II (Proc. Second Conf. Functional Anal., Univ. Paderborn, Paderborn, 1979), Notas Mat., vol. 68, North-Holland, Amsterdam-New York, 1980, pp. 19–30. MR565396 (81e:46016)

[12] H. Elton Lacey, *The isometric theory of classical Banach spaces*, Springer-Verlag, New York-Heidelberg, 1974. Die Grundlehren der mathematischen Wissenschaften, Band 208. MR0493279 (58 #12308)

[13] Michel Talagrand, *Weak Cauchy sequences in $L^1(E)$*, Amer. J. Math. **106** (1984), no. 3, 703–724, DOI 10.2307/2374292. MR745148 (85j:46062)

[14] A. Ülger, *Weak compactness in $L^1(\mu, X)$*, Proc. Amer. Math. Soc. **113** (1991), no. 1, 143–149, DOI 10.2307/2048450. MR1070533 (92g:46035)

Departamento de Matemática Aplicada, Facultad de Informática, Universidad de Murcia, 30100 Espinardo (Murcia), Spain

E-mail address: joserr@um.es