PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 144, Number 2, February 2016, Pages 575–586
http://dx.doi.org/10.1090/proc12724
Article electronically published on May 28, 2015

MULTIPLE SOLUTIONS FOR AN INDEFINITE ELLIPTIC PROBLEM WITH CRITICAL GROWTH IN THE GRADIENT

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(Communicated by Catherine Sulem)

Abstract. We consider the problem

$$(P) \qquad -\Delta u = c(x)u + \mu|\nabla u|^2 + f(x), \quad u \in H_0^1(\Omega) \cap L^\infty(\Omega),$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 3$, $\mu > 0$ and $c, f \in L^q(\Omega)$ for some $q > \frac{N}{2}$ with $f \gneqq 0$. Here c is allowed to change sign and we assume that $c^+ \not\equiv 0$. We show that when c^+ and μf are suitably small this problem has at least two positive solutions. This result contrasts with the case $c \leq 0$, where uniqueness holds. To show this multiplicity result we first transform (P) into a semilinear problem having a variational structure. Then we are led to the search of two critical points for a functional whose superquadratic part is indefinite in sign and has a so-called *slow growth* at infinity. The key point is to show that the Palais-Smale condition holds.

1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^N$ with $N \geq 3$. In this paper we are concerned with the boundary value problem

$$(P) \qquad -\Delta u = c(x)u + \mu|\nabla u|^2 + f(x), \quad u \in H_0^1(\Omega) \cap L^\infty(\Omega),$$

where

$$(\mathcal{H}) \qquad \mu > 0, \quad f \gneqq 0 \quad \text{and} \quad c, f \in L^q(\Omega) \text{ for some } q > \frac{N}{2}.$$

Quasilinear elliptic equations with a gradient dependence up to the critical growth $|\nabla u|^2$ were first studied by Boccardo, Murat and Puel in the 1980's [12–14] and have been an active field of research until now; see for example [2, 18, 19]. To situate our problem we underline that we are interested in bounded solutions. The main goal of this paper is to carry on the study of non-uniqueness of solutions for such problems, of which (P) is a prototype.

The sign of c plays in (P) a central role regarding uniqueness, as well as existence, of bounded solutions. We refer to [20] for a heuristic discussion on the influence of the sign of c on the nature of the problem. The case $c \leq -\alpha_0$ a.e. in Ω for some $\alpha_0 > 0$ is referred to as the coercive case. In this case, the existence of solutions holds under very general assumptions and it was shown in [9, 10] (see also [8, 11])

Received by the editors July 15, 2014 and, in revised form, December 25, 2014.
2010 *Mathematics Subject Classification.* Primary 35J20, 35J61, 35J91.
Key words and phrases. Indefinite variational problem, critical growth in the gradient, superlinear term with slow growth, Cerami condition.
The second author was supported by the FONDECYT project 11121567. This work has been carried out in the framework of the project NONLOCAL (ANR-14-CE25-0013) funded by the French National Research Agency (ANR).

that there is a unique bounded solution. When one just requires $c \leq 0$ (in particular when $c \equiv 0$) the situation is already more complex. The fact that restrictions on the data are necessary for (P) to have a solution was first observed in [16, 17]. Concerning uniqueness, some partial results are given in [9, 10], but it was only in [6] that uniqueness of bounded solutions was established under the mere condition $c \leq 0$. See also [7] for an extension to a larger class of problems.

The case $c \gneqq 0$ started to be studied only recently; surely in part because it was not accessible by the methods traditionally used in the coercive case. In [20] it was shown that when $c \gneqq 0$ and c, μ and f are sufficiently small in an appropriate sense, (P) has two solutions. See also [1, 3, 24] for related results. Note that the case where μ is allowed to be non-constant was treated in [6] leading also, when $c \gneqq 0$ and under appropriate conditions, to the existence of two bounded solutions.

In view of these results it remained to analyse the case where c is allowed to change sign, which is the aim of the present paper. Roughly speaking we shall show that the uniqueness is lost as soon as $c^+ \not\equiv 0$, where $c^+ = \max\{0, c\}$; see Theorem 1.1.

We first observe that (P) is equivalent to

$$(P') \qquad -\Delta w = c(x)w + |\nabla w|^2 + \mu f(x), \quad w \in H_0^1(\Omega) \cap L^\infty(\Omega).$$

Indeed, it is easy to check that u is a solution of (P) if and only if $w = \mu u$ is a solution of (P'). Now, we use the change of variable

$$v = e^w - 1, \tag{1.1}$$

which goes back to [21] and rids the gradient term of (P'), reducing it to a semilinear problem with a variational structure; namely,

$$(Q) \qquad -\Delta v - (c(x) + \mu f(x))v = c(x)g(v) + \mu f(x), \quad v \in H_0^1(\Omega) \cap L^\infty(\Omega),$$

where

$$g(s) = \begin{cases} (1+s)\ln(1+s) - s & \text{if } s \geq 0, \\ 0 & \text{if } s \leq 0. \end{cases} \tag{1.2}$$

We shall prove in Lemma 2.1 that if v is a non-negative solution of (Q), then w defined by (1.1) is a non-negative (and therefore positive, by Harnack inequality) solution of (P'). Solutions of (Q) will be obtained as critical points of the functional

$$I(v) = \frac{1}{2}\int_\Omega \left[|\nabla v|^2 - [c(x) + \mu f(x)](v^+)^2\right] - \int_\Omega c(x)G(v^+) - \mu \int_\Omega f(x)v$$

defined on $H_0^1(\Omega)$ and where $G(s) = \int_0^s g(t)\,dt$. Note that since $f \geq 0$, critical points of I are necessarily non-negative; see Lemma 2.1. Since g behaves essentially as $s \ln s$ for s large, the superquadratic part of I has at infinity a growth which is usually referred to as a *slowly superlinear growth*.

To obtain two critical points we start following the strategy used in [20]. Note that if the positive part of $c + \mu f$ is not 'too large' in a suitable sense (cf. Lemma 2.2), then

$$\int_\Omega \left[|\nabla v|^2 - [c(x) + \mu f(x)](v^+)^2\right]$$

is coercive. Moreover, as g is superlinear, we shall prove that I takes positive values on a sphere $\|v\| = \rho$ if either c or μf is sufficiently small. Moreover it is easily seen that since $f \not\equiv 0$, I takes negative values in the ball $B(0, \rho)$. Finally, since $c^+ \not\equiv 0$, it is possible to show that I takes a negative value at some point v_0 outside of

the ball $B(0,\rho)$. Thus I has a mountain-pass geometry and it is reasonable to search for a first critical point as a minimizer of I in $B(0,\rho)$ and a second one at the mountain-pass level. The existence of a minimizer will follow from a standard lower semicontinuity argument, whereas in the proof of the existence of a mountain-pass critical point we will face the difficulty of showing that Palais-Smale sequences are bounded.

We recall that the Palais-Smale condition holds for I if any sequence $(u_n) \subset H_0^1(\Omega)$ such that $(I(u_n)) \subset \mathbb{R}$ is bounded and $||I'(u_n)||_* \to 0$ admits a convergent subsequence. The boundedness of such sequences proves to be a delicate issue due to the fact that c is sign-changing and g has a slow growth at infinity. In particular g does not satisfy an Ambrosetti-Rabinowitz type condition. Let us recall that a non-linearity f is said to satisfy the Ambrosetti-Rabinowitz condition if

$$(\mathcal{AR}) \qquad \text{There exist } \theta > 2 \text{ and } s_1 > 0 \text{ such that } 0 < \theta F(s) \leq sf(s) \quad \forall s \geq s_1,$$

where $F(s) = \int_0^s f(t)\ dt$. This condition is known to be central when proving that Palais-Smale sequences are bounded. When the domain $\Omega \subset \mathbb{R}^N$ is bounded and the non-linearity is subcritical, the boundedness leads directly to the strong convergence of a subsequence.

In the case where the superquadratic term is positive, many efforts have been made to weaken the condition $(\mathcal{AR})$. However, to the best of our knowledge, this issue has not been considered for functionals of the type

$$J(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - c(x)F(u) \right), \quad u \in H_0^1(\Omega)$$

when c changes sign and f is a superlinear function not satisfying $(\mathcal{AR})$. A typical example of such a non-linearity is $f(s) = s\ln(s+1)$.

When $f(s) = s^{p-1}$ with $p \in [2, 2^*)$, using the homogeneity of f it is straightforward that J satisfies the Palais-Smale condition. When f is not powerlike, this issue becomes delicate, as shown in [5] (see also [4]), where the authors assume that f is superlinear and asymptotically powerlike at infinity, i.e.,

$$(\mathcal{G}) \qquad \text{There exists } p > 2 \text{ such that } \lim_{s\to\infty} \frac{f(s)}{s^{p-1}} = 1.$$

Note that this condition implies $(\mathcal{AR})$. Furthermore, in [5] one needs to assume the so-called *thick zero set* condition on $c \in \mathcal{C}(\overline{\Omega})$:

$$(\mathcal{AT}) \qquad \overline{(\Omega_+)} \cap \overline{(\Omega_-)} = \emptyset,$$

where

$$\Omega_+ := \{x \in \Omega;\ c(x) > 0\} \quad \text{and} \quad \Omega_- := \{x \in \Omega;\ c(x) < 0\}.$$

In [23], still under $(\mathcal{G})$, the authors were able to remove $(\mathcal{AT})$, but at the expense of some alternative strong conditions on c.

In our problem we prove that the Palais-Smale condition is satisfied without assuming $(\mathcal{AT})$ nor any special condition on c. Given $V \in L^q(\Omega)$, with $q > \frac{N}{2}$, we denote by $\lambda_1(V) = \lambda_1(V,\Omega)$ the first eigenvalue of the problem

$$\begin{cases} -\Delta u + V(x)u = \mu u & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Let us recall that $\lambda_1(V)$ is given by

$$\lambda_1(V) = \inf\left\{\int_\Omega \left(|\nabla v|^2 + V(x)v^2\right);\ u \in H_0^1(\Omega),\ \|u\|_2 = 1\right\}.$$

It is well known that $\lambda_1(V)$ is simple, so that it is achieved by a unique $\varphi_1 > 0$ such that $\|\varphi_1\|_2 = 1$; cf. [22].

Our main result is the following:

Theorem 1.1. *Assume* $(\mathcal{H})$ *and* $c^+ \not\equiv 0$. *Then* (P) *has two positive solutions if either one of the following conditions hold:*

(1) $\lambda_1(-\mu f) > 0$ *and* $\|c^+\|_q < K$, *where* K *is a constant depending on* f *and* μ.

(2) $\lambda_1(-c) > 0$ *and* $\|\mu f\|_q < K$, *where* K *is a constant depending on* c.

Remark 1.2. In [20, Theorem 2], assuming $c \gneqq 0$, it is proved that if

$$\|\mu f\|_{\frac{N}{2}} < C_N, \tag{1.3}$$

where C_N denotes the best Sobolev constant for the embedding $H_0^1(\Omega) \subset L^{2^*}(\Omega)$, then there exists $\overline{c} > 0$ such that (P) has at least two bounded solutions if $\|c\|_q < \overline{c}$. We observe that under (1.3) we have $\lambda_1(-\mu f) > 0$. Thus Theorem 1.1 (2) is consistent with [20, Theorem 2]. We point out however that f is allowed to be sign-changing in [20].

In [6] (see Corollary 3.2 and Remark 3.2) it is shown that when $c \equiv 0$, (P) has a solution if and only if $\lambda_1(-\mu f) > 0$. We now complement this result.

Lemma 1.3. *Assume* $(\mathcal{H})$.

(1) If $c \geq 0$, *then* $\lambda_1(-c - \mu f) > 0$ *is necessary for* (P) *to have a non-negative solution.*

(2) $\lambda_1(-c) > 0$ *is necessary for* (P) *to have a non-negative solution and under this condition every solution of* (P) *is non-negative.*

Remark 1.4. As far as non-negative solutions are concerned, Lemma 1.3 (1) shows that when $c \geq 0$ the condition $\lambda_1(-c - \mu f) > 0$ is necessary in Theorem 1.1. However, other kinds of solutions of (P), namely negative or sign-changing solutions, may exist if $\lambda_1(-c - \mu f) \leq 0$. See [15] in this direction.

This paper is organized as follows. In Section 2 we prove some preliminary results and show that the functional I has the geometry described above. Section 3 is devoted to the Palais-Smale condition for I. Finally in Section 4 we prove Theorem 1.1 and Lemma 1.3. Also in Remark 4.1 we discuss the necessity of some assumptions in Theorem 1.1.

1.1. **Notation.**

- The Lebesgue norm in $L^r(\Omega)$ will be denoted by $\|\cdot\|_r$ and the usual norm of $H_0^1(\Omega)$ by $\|\cdot\|$, i.e., $\|u\| = \|\nabla u\|_2$. The Holder conjugate of r is denoted by r'.
- The weak convergence is denoted by $\rightharpoonup$.
- The positive and negative parts of a function u are defined by $u^\pm := \max\{\pm u, 0\}$.
- We denote by $B(0, R)$ the ball of radius R centered at 0 in $H_0^1(\Omega)$.

2. Preliminaries

Lemma 2.1. *Assume* $(\mathcal{H})$.

(1) If v is a non-negative solution of (Q), then $w = \ln(1+v)$ is a non-negative solution of (P'). Similarly if w is a non-negative solution of (P'), then v given by (1.1) *is a non-negative solution of (Q).*

(2) If v is a critical point of I, then v is a non-negative solution of (Q).

(3) If u is a non-negative solution of (P), then u is positive.

Proof. Let $v \geq 0$ be a solution of (Q). From the expression of g it is seen that v solves

$$-\Delta v = c(x)(1+v)\ln(1+v) + \mu f(x)(1+v). \tag{2.1}$$

Let $w = \ln(1+v)$, i.e., $e^w = 1+v$. Since $v \geq 0$ and $\nabla w = \frac{\nabla v}{1+v}$, one may easily see that $w \in H_0^1(\Omega)$. If $\phi \in H_0^1(\Omega)$, then $\psi = \frac{\phi}{1+v} \in H_0^1(\Omega)$, so that (2.1) provides

$$\begin{aligned} \int_\Omega \nabla v \nabla \psi &= \int_\Omega c(x)\psi(1+v)\ln(1+v) + \mu \int_\Omega f(x)\psi(1+v) \\ &= \int_\Omega c(x)\phi \ln(1+v) + \mu \int_\Omega f(x)\phi. \end{aligned} \tag{2.2}$$

Now, from $\nabla v = e^w \nabla w$ and $\nabla \psi = \frac{\nabla \phi}{1+v} - \frac{\phi \nabla v}{(1+v)^2}$, we get

$$\begin{aligned} \int_\Omega \nabla v \nabla \psi &= \int_\Omega e^w \nabla w \left(\frac{\nabla \phi}{1+v} - \frac{\phi \nabla v}{(1+v)^2} \right) = \int_\Omega \nabla w \left(\nabla \phi - \frac{\phi \nabla v}{1+v} \right) \\ &= \int_\Omega \nabla w \left(\nabla \phi - \frac{\phi e^w \nabla w}{1+v} \right) = \int_\Omega \left(\nabla w \nabla \phi - |\nabla w|^2 \phi \right). \end{aligned}$$

Furthermore, we have

$$\int_\Omega c(x)\phi \ln(1+v) = \int_\Omega c(x) w \phi,$$

so we deduce from (2.2) that u is a solution of (P'). By similar arguments we prove the reverse statement. This proves (1).

To prove (2), let v be a critical point of I. Then

$$\int_\Omega \left[\nabla v \nabla \varphi - (c(x) + \mu f(x)) v^+ \varphi \right] - \int_\Omega c(x) g(v^+)\varphi - \mu \int_\Omega f(x)\varphi = 0 \tag{2.3}$$

for all $\varphi \in H_0^1(\Omega)$. Taking $\varphi = -v^-$ we get

$$\int_\Omega |\nabla v^-|^2 + \mu \int_\Omega f(x) v^- = 0.$$

Since $f \geq 0$, we get

$$\int_\Omega |\nabla v^-|^2 \leq 0$$

and it follows that $v^- \equiv 0$, i.e., $v \geq 0$. The proof that $v \in L^\infty(\Omega)$ can be found in [20, Lemma 13], so we omit it.

Finally, if $u \geq 0$ is a solution of (P), then, since $\mu > 0$ and $f \geq 0$, u is a bounded weak supersolution of

$$-\Delta u = c(x)u, \quad u \in H_0^1(\Omega).$$

By a standard argument relying on the Harnack inequality (see [25, Theorem 1.2]) we have either $u \equiv 0$ or $u > 0$. Since $f \gneqq 0$, we get $u > 0$. $\square$

We shall now prove that when $\lambda_1(-c-\mu f)>0$ the functional I takes positive values on a sphere centered at the origin if either $\|c^+\|_q$ or $\|\mu f\|_q$ is small enough.

Lemma 2.2. *Let $V\in L^q(\Omega)$, with $q>\frac{N}{2}$. If $\lambda_1(V)>0$, then there exists $K_1>0$ such that*

$$\int_\Omega \left(|\nabla v|^2+V(x)(v^+)^2\right)\geq K_1\|v\|^2 \quad \forall v\in H_0^1(\Omega). \tag{2.4}$$

Proof. Let us first prove that there exists a constant $K_1>0$ such that

$$\int_\Omega \left(|\nabla v|^2+V(x)v^2\right)\geq K_1\|v\|^2 \quad \forall v\in H_0^1(\Omega). \tag{2.5}$$

Indeed, assume by contradiction that there is a sequence $(v_n)\subset H_0^1(\Omega)$ such that

$$\int_\Omega \left(|\nabla v_n|^2+V(x)(v_n)^2\right)\leq \frac{\|v_n\|^2}{n}.$$

Setting $w_n=\frac{v_n}{\|v_n\|}$ we may assume that, up to a subsequence,

$$w_n\rightharpoonup w_0 \text{ in } H_0^1(\Omega) \quad \text{and} \quad w_n\to w_0 \text{ in } L^r(\Omega) \text{ for } r\in[1,2^*).$$

In particular since $q>\frac{N}{2}$ we have that $w_n\to w_0$ in $L^{2q'}(\Omega)$. Thus from

$$\int_\Omega \left(|\nabla w_n|^2+V(x)(w_n)^2\right)\leq \frac{1}{n} \tag{2.6}$$

it follows that

$$\int_\Omega \left(|\nabla w_0|^2+V(x)(w_0)^2\right)\leq 0. \tag{2.7}$$

We claim that $w_0\not\equiv 0$. Indeed, if $w_0\equiv 0$, then $w_n\to 0$ in $L^{2q'}(\Omega)$ and (2.6) yields $w_n\to 0$ in $H_0^1(\Omega)$, which is impossible since $\|w_n\|=1$. Hence $w_0\not\equiv 0$ and consequently (2.7) provides $\lambda_1(V)\leq 0$, which contradicts our assumption. Thus (2.5) is proved. Finally, we may assume that $K_1\leq 1$, so that

$$\begin{aligned}\int_\Omega \left(|\nabla v|^2+V(x)(v^+)^2\right) &= \int_\Omega |\nabla v^-|^2+\int_\Omega \left(|\nabla v^+|^2+V(x)(v^+)^2\right)\\ &\geq \|v^-\|^2+K_1\|v^+\|^2\geq K_1\|v\|^2.\end{aligned}$$

□

We are now ready to prove that I has the appropriate geometry. Note that g given by (1.2) satisfies

$$\lim_{s\to 0}\frac{g(s)}{s^p}=\lim_{s\to\infty}\frac{g(s)}{s^p}=0$$

if $p\in(1,2)$. As a consequence, there exists a constant $C>0$ such that

$$0\leq G(s)\leq Cs^{p+1}, \quad \forall s\in\mathbb{R}. \tag{2.8}$$

Proposition 2.3. *Assume that $\lambda_1(-c-\mu f)>0$. Given $R>0$ sufficiently large, there exist $K,M>0$ depending on R and such that:*

(1) If $\|c^+\|_q<K$, then $I(v)\geq M$ for every $v\in H_0^1(\Omega)$ with $\|v\|=R$.
(2) If $\|\mu f\|_q<K$, then $I(v)\geq M$ for every $v\in H_0^1(\Omega)$ with $\|v\|=R^{-1}$.

Proof. Since $\lambda_1(-c-\mu f)>0$, by Lemma 2.2 there exists $K_1>0$ such that

$$\int_\Omega \left(|\nabla v|^2-[c(x)+\mu f(x)](v^+)^2\right)\geq K_1\|v\|^2 \quad \forall v\in H_0^1(\Omega).$$

Let $p\in(1,2)$. By (2.8) we have

$$I(v)\geq K_1\|v\|^2-C_1\|c^+\|_q\|v\|^{p+1}-C_2\|\mu f\|_q\|v\|$$

for some $C_1,C_2>0$. If $\|v\|=R$ and $\|c^+\|_q\leq R^{-\beta}$, with $\beta>p-1$, then

$$I(v)\geq K_1R^2-C_1R^{p+1-\beta}-C_2\mu\|f\|_qR\geq R$$

for R sufficiently large. Thus (1) holds with $K=R^{-\beta}$ and $M=R$.

In a similar way, if now $\|v\|=R^{-1}$ and $\|\mu f\|_q\leq R^{-\beta}$, with $\beta>1$, then

$$I(v)\geq K_1R^{-2}-C_1\|c^+\|_qR^{-(p+1)}-C_2R^{-\beta-1}\geq R^{-3}$$

for R sufficiently large. Hence we may take $K=R^{-\beta}$ and $M=R^{-3}$ to get (2). □

3. The Palais-Smale condition

We set

$$\alpha_c=\inf\left\{\int_\Omega\left(|\nabla u|^2-\mu f(x)(u^+)^2\right);u\in H_0^1(\Omega),\ \|u\|_2=1,\ cu^+\equiv 0\right\}.$$

In the next proposition, we shall use an explicit expression of G; namely,

$$G(s)=\frac{s^2}{2}\ln(s+1)-\frac{3}{4}s^2+s\ln(s+1)-\frac{s}{2}+\frac{1}{2}\ln(s+1) \tag{3.1}$$

for $s>0$.

Proposition 3.1. *If $\alpha_c>0$, then I satisfies the Palais-Smale condition.*

Proof. Let (u_n) be a Palais-Smale sequence for I at the level $d\in\mathbb{R}$, i.e.,

$$I(u_n)\to d \quad \text{and} \quad \|I'(u_n)\|_*\to 0. \tag{3.2}$$

From (3.2) we have

$$\frac{1}{2}\int_\Omega\left[|\nabla u_n|^2-(c(x)+\mu f(x))(u_n^+)^2\right]-\int_\Omega c(x)G(u_n^+)-\mu\int_\Omega f(x)u_n=d+o(1) \tag{3.3}$$

and

$$\left|\int_\Omega\left[\nabla u_n\nabla\varphi-(c(x)+\mu f(x))u_n^+\varphi\right]-\int_\Omega c(x)g(u_n^+)\varphi-\mu\int_\Omega f(x)\varphi\right|\leq\varepsilon_n\|\varphi\| \tag{3.4}$$

for some sequence $\varepsilon_n\to 0$ and for every $\varphi\in H_0^1(\Omega)$. In particular, we have

$$|\langle I'(u_n),u_n\rangle|\leq\varepsilon_n\|u_n\|. \tag{3.5}$$

Let us assume that $\|u_n\|\to\infty$ and set $v_n=\frac{u_n}{\|u_n\|}$. Up to a subsequence, we have

$$v_n\rightharpoonup v_0 \text{ in } H_0^1(\Omega),\quad v_n\to v_0 \text{ in } L^r(\Omega),\ \forall r\in[1,2^*),\quad \text{and}\quad v_n\to v_0 \text{ a.e. in } \Omega.$$

We claim that $cv_0^+\equiv 0$. Indeed, from (3.4) we have, using the convergences above,

$$\int_\Omega c(x)\frac{g(u_n^+)}{\|u_n\|}\varphi=\int_\Omega\left[\nabla v_0\nabla\varphi-(c(x)+\mu f(x))v_0^+\varphi\right]+o(1)<\infty, \tag{3.6}$$

for every $\varphi \in H_0^1(\Omega)$. If $cv_0^+ \not\equiv 0$, then we may choose $\varphi \in H_0^1(\Omega)$ and a measurable subset $\Omega_\varphi \subset \Omega$ such that

$$|\Omega_\varphi| > 0, \quad cv_0^+\varphi > 0 \text{ on } \Omega_\varphi \subset \Omega, \quad \text{and} \quad cv_0^+\varphi = 0 \text{ on } \Omega \setminus \Omega_\varphi.$$

Now, using that $\lim_{s\to\infty} \frac{g(s)}{s} = \infty$, we have

$$\liminf c(x)\frac{g(u_n^+)}{\|u_n\|}\varphi = \liminf c(x)v_n^+\frac{g(\|u_n\|v_n^+)}{\|u_n\|v_n^+}\varphi = +\infty \quad \text{on} \quad \Omega_\varphi.$$

Fatou's lemma then yields a contradiction with (3.6). Therefore $cv_0^+ \equiv 0$. On the other hand, taking $\varphi = v_0$ in (3.4) and dividing it by $\|u_n\|$ we get

$$\int_\Omega \left[\nabla v_n \nabla v_0 - (c(x) + \mu f(x))v_n^+ v_0\right] \to 0,$$

so that, using $v_n \rightharpoonup v_0$ in $H_0^1(\Omega)$ and $cv_0^+ \equiv 0$, we get

$$\int_\Omega \left[|\nabla v_0|^2 - \mu f(x)(v_0^+)^2\right] = 0.$$

Thus $v_0 \equiv 0$ (otherwise $\alpha_c \le 0$). Now from (3.4) we have, taking $\varphi = u_n$ and using the definition (1.2) of g,

(3.7)
$$\left|\int_\Omega (|\nabla u_n|^2 - \mu f(x))(u_n^+)^2 - \int_\Omega c(x)(1+u_n^+)\ln(1+u_n^+)u_n^+ - \mu\int_\Omega f(x)u_n^+\right| \le \varepsilon_n\|u_n\|.$$

Dividing by $||u_n||^2$ and using that $v_n \to 0$ in $L^r(\Omega), \forall r \in [1, 2^*)$ we get

$$1 - \int_\Omega c(x)(v_n^+)^2 \ln(1 + ||u_n||v_n^+) \to 0.$$

Now, using the property $\ln(st) = \ln s + \ln t$, it follows that

$$1 - \ln(||u_n||)\int_\Omega c(x)(v_n^+)^2 - \int_\Omega c(x)(v_n^+)^2 \ln\left(v_n^+ + \frac{1}{||u_n||}\right) \to 0.$$

We claim that

$$\ln(||u_n||)\int_\Omega c(x)(v_n^+)^2 \to 0. \tag{3.8}$$

In that case we would get

$$\int_\Omega c(x)(v_n^+)^2 \ln\left(v_n^+ + \frac{1}{||u_n||}\right) \to 1,$$

which clearly contradicts the fact that $v_0 = 0$. To prove (3.8) we define for every $s > 0$

$$H(s) = \frac{1}{2}g(s)s - G(s).$$

From (1.2) and (3.1) it follows that

$$H(s) = \frac{s^2}{4} - s\ln(s+1) + \frac{s}{2} - \frac{1}{2}\ln(1+s). \tag{3.9}$$

From (3.5) we get

$$I(u_n) - \frac{1}{2}\langle I'(u_n), u_n\rangle = c + \varepsilon_n\|u_n\| + o(1),$$

which leads, using the definition of H, to

$$\int_\Omega c(x)H(u_n^+) - \frac{\mu}{2}\int_\Omega f(x)u_n = c + \varepsilon_n\|u_n\| + o(1). \tag{3.10}$$

Now, combining (3.9) and (3.10), we obtain

$$\begin{aligned}\frac{1}{4}\int_\Omega c(x)(u_n^+)^2 &= c + \varepsilon_n\|u_n\| + \frac{1}{2}\int_\Omega c(x)u_n^+ - \int_\Omega c(x)u_n^+\ln(1+u_n^+)\\ &\quad + \frac{1}{2}\int_\Omega c(x)\ln(1+u_n^+) + \frac{\mu}{2}\int_\Omega f(x)u_n + o(1).\end{aligned}$$

Hence

$$\begin{aligned}\ln(||u_n||)\int_\Omega c(x)(v_n^+)^2 = \frac{4\ln||u_n||}{||u_n||^2}\Big(&c + \varepsilon_n\|u_n\| + \frac{1}{2}\int_\Omega c(x)u_n^+ - \int_\Omega c(x)u_n^+\ln(1+u_n^+)\\ &+ \frac{1}{2}\int_\Omega c(x)\ln(1+u_n^+) + \frac{\mu}{2}\int_\Omega f(x)u_n + o(1)\Big) \to 0.\end{aligned}$$

Thus (3.8) is proved and we reach a contradiction. Therefore (u_n) must be bounded and, up to subsequence, we have $u_n \rightharpoonup u_0$ in $H_0^1(\Omega)$ and $u_n \to u_0$ in $L^p(\Omega)$ for $p \in [1, 2^*)$. At this point the strong convergence follows in a standard way. We refer to [20, Lemma 11] for a proof. □

Corollary 3.2. *If $\lambda_1(-c-\mu f) > 0$, then I satisfies the Palais-Smale condition.*

Proof. Let $\|u\|_2 = 1$ with $cu^+ \equiv 0$. Since $\lambda_1(-c-\mu f) > 0$, by Lemma 2.2 there is a constant $K_1 > 0$ such that

$$\begin{aligned}\int_\Omega \left(|\nabla u|^2 - \mu f(x)(u^+)^2\right) &= \int_\Omega \left(|\nabla u|^2 - (c(x)+\mu f(x))(u^+)^2\right)\\ &\geq K_1\|u\|^2 \geq SK_1\|u\|_2^2 = SK_1 > 0,\end{aligned}$$

where S is the best Sobolev constant for the embedding $H_0^1(\Omega) \subset L^2(\Omega)$. Thus $\alpha_c > 0$ and by Proposition 3.1 we get the conclusion. □

4. Proof of Theorem 1.1 and Lemma 1.3

We are now ready to prove our main results.

Proof of Theorem 1.1. First of all, we fix $K > 0$ such that $\lambda_1(-c-\mu f) > 0$ if either $\lambda_1(-\mu f) > 0$ and $\|c^+\|_q < K$ or $\lambda_1(-c) > 0$ and $\|\mu f\|_q < K$. This is possible in view of the continuity of $\lambda_1(V)$ with respect to $V \in L^q(\Omega)$. Decreasing K if necessary, we fix R sufficiently large so that, by Proposition 2.3, if $\|c^+\|_q < K$ (respect. $\|\mu f\|_q < K$), then $I(v) \geq M > 0$ for $\|v\| = R$ (respect. $\|v\| = R^{-1}$). We set $\rho = R$ if $\|c^+\|_q < K$ and $\rho = R^{-1}$ if $\|\mu f\|_q < K$. It is easily seen that if $f \not\equiv 0$, then I takes negative values in the ball $B(0,\rho)$. Therefore, by weak lower semi-continuity, we infer that if either $\|c^+\|_q < K$ or $\|\mu f\|_q < K$, then the infimum of I in $B(0,\rho)$ is achieved by some $w_0 \not\equiv 0$, which is a critical point of I. Furthermore, since $G(s)/s^2 \to \infty$ as $s \to \infty$, if $v \in H_0^1(\Omega)$ is such that $\int_\Omega c(x)G(v^+) > 0$, then $I(tv) \to -\infty$ as $t \to \infty$. We fix $t > 0$ and v such that $v_0 = tv$ satisfies $\|v_0\| > \rho$ and $I(v_0) < 0$. Now let

$$\Gamma := \{\gamma \in \mathcal{C}([0,1], H_0^1(\Omega));\ \gamma(0) = 0, \gamma(1) = v_0\}$$

and

$$d := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t)).$$

Since I satisfies the Palais-Smale condition, by the mountain-pass theorem it is straightforward that I has a critical point w_1, which, by Proposition 2.3, satisfies $I(w_1) = d > 0$. In particular, we have $w_0 \neq w_1$. Finally, from Lemma 2.1, we know that these two critical points provide two positive solutions of (P'), and consequently, two positive solutions of (P). □

Proof of Lemma 1.3. By Lemma 2.1, we know that if $u \geq 0$ is a solution of (P), then u is positive so that $w = \mu u$ is a positive solution of (P'). Thus v given by (1.1) is a positive solution of (Q). Taking $\phi > 0$, the first positive eigenfunction associated to $\lambda_1(-c - \mu f)$, as test function and using that $g \geq 0$ on $\mathbb{R}$ we obtain

$$\int_\Omega (\nabla v \nabla \phi - c(x) v \phi - \mu f(x) v \phi) = \int_\Omega (c(x) g(v) \phi + \mu f(x) \phi) > 0,$$

so that

$$\lambda_1(-c - \mu f) \int_\Omega v \phi > 0.$$

Thus $\lambda_1(-c - \mu f) > 0$.

Similarly, let $\varphi > 0$ be an eigenfunction associated to $\lambda_1(-c)$ and assume that $u \geq 0$ is a solution of (P). Taking $\varphi > 0$ as test function we get

$$\int_\Omega (\nabla u \nabla \varphi - c(x) u \varphi) = \int_\Omega \big(\mu |\nabla u|^2 \varphi + f(x) \varphi\big) > 0.$$

Thus

$$\lambda_1(-c) \int_\Omega u \varphi > 0,$$

so that $\lambda_1(-c) > 0$. Finally, let u be a solution of (P). Using u^- as test function in (P), we obtain

$$-\int_\Omega (|\nabla u^-|^2 - c(x)|u^-|^2) = \int_\Omega (\mu |\nabla u|^2 u^- + f(x) u^-) \geq 0.$$

Hence

$$\int_\Omega (|\nabla u^-|^2 - c(x)|u^-|^2) \leq 0,$$

so that under the condition $\lambda_1(-c) > 0$ we get $u^- \equiv 0$, i.e., $u \geq 0$. □

Our last result shows that when $\lambda_1(-c) > 0$ a restriction on the size of μf is necessary in Theorem 1.1.

Remark 4.1. Assume $(\mathcal{H})$, $\lambda_1(-c) > 0$, and $c \geq 0$ in some open set $\Omega_0 \subset \Omega$. Then there exist an $R > 0$ and an $f \in L^q(\Omega)$ with $\|\mu f\|_q = R$ such that (P) has no non-negative solutions.

Proof. Equivalently we shall prove that (P') has no non-negative solutions. We choose $\phi \in C_0^\infty(\Omega_0)$ and $f \in L^q(\Omega)$ such that $f > 0$ on $supp\, \phi$. In particular we have

$$\int_\Omega f(x) \phi^2 > 0. \tag{4.1}$$

By Cauchy-Schwartz inequality we have

$$\int_\Omega \nabla u \nabla (\phi^2) = \int_\Omega 2\phi \nabla u \nabla \phi \leq \int_\Omega |\nabla \phi|^2 + |\nabla u|^2 \phi^2. \tag{4.2}$$

Now assume that (P') has a non-negative solution. Using ϕ^2 as test function in (P') and (4.2) we get

$$\int_\Omega |\nabla\phi|^2 \geq \int_\Omega c(x)u\phi^2 + \mu\int_\Omega f(x)\phi^2 \geq \mu\int_\Omega f(x)\phi^2.$$

Because of (4.1) we get a contradiction for $\mu > 0$ large enough. □

Acknowledgment

The two authors warmly thank the referee for pointing out an important mistake in the initial version of this paper.

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