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LIVŠIC MEASURABLE RIGIDITY FOR $\mathcal{C}^1$ GENERIC VOLUME-PRESERVING ANOSOV SYSTEMS

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Abstract. In this paper, we prove that for $\mathcal{C}^1$ generic volume-preserving Anosov diffeomorphisms of a compact connected Riemannian manifold, the Livšic measurable rigidity theorem holds. We also give a parallel result for $\mathcal{C}^1$ generic volume-preserving Anosov flows.

1. Introduction

Let $T : M \to M$ be a diffeomorphism on a compact Riemannian manifold M. We consider a **cocycle** $\mathcal{A} : \mathbb{Z} \times M \to \mathbb{R}$; that is, a map satisfying the cocycle relation

$$\mathcal{A}(n_1 + n_2, x) = \mathcal{A}(n_1, T^{n_2}(x)) + \mathcal{A}(n_2, x),$$

for every $n_1, n_2 \in \mathbb{Z}$ and every $x \in M$. Following the definition in cohomological algebra, we call a cocycle $\mathcal{A}$ **a coboundary** if it satisfies the cohomological equation:

$$\mathcal{A}(n, x) = \Phi(T^n(x)) - \Phi(x), \tag{1}$$

where $\Phi : M \to \mathbb{R}$ is a function. Furthermore, two cocycles are called cohomologous if their difference is a coboundary. It is easy to see that a coboundary $\mathcal{A}$ must have trivial periodic data, i.e.

$$\mathcal{A}(n, x) = 0, \quad \forall x \in M, \;\; T^n(x) = x. \tag{2}$$

Livšic took the lead in considering the following three questions for the case when T is a transitive Anosov diffeomorphism on a compact Riemannian manifold M in [11,12]:

(1) Is the necessary condition, trivial periodic data, also a sufficient condition?
(2) **Measurable rigidity:** If the cocycle $\mathcal{A} : \mathbb{Z} \times M \to \mathbb{R}$ is Hölder continuous, can we get a Hölder continuous solution Φ to equation (1) from a measurable solution?
(3) Higher regularity: If the cocycle $\mathcal{A} : \mathbb{Z} \times M \to \mathbb{R}$ is $\mathcal{C}^r$ for some $1 \leq r \leq \infty$ or $r = \omega$, is a continuous solution to equation (1) also $\mathcal{C}^r$?

Thus, we call results answering the above questions Livšic theorems. Current research is usually concerned with two variations on this subject, namely altering the base system T and altering the group $\mathbb{R}$. Some of the highlights are [6,13,18]. We refer the reader to a survey [14] and a book [9] for some of the most recent results

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and overview of historical development in this area. The following theorem is the classical result.

Theorem 1.1 ([6, 11, 12, 17])**.** *Let $T : M \to M$ be a $\mathcal{C}^1$ transitive Anosov diffeomorphism on a compact Riemannian manifold M and let $\phi : M \to \mathbb{R}$ be a Hölder continuous function.*

(1) **Existence of solutions.** *$\phi = \Phi(T) - \Phi$ has a Hölder continuous solution Φ if and only if $\sum_{x \in \mathcal{O}} \phi(x) = 0$, for every T-periodic orbit $\mathcal{O}$.*
(2) **Measurable rigidity.** *For any Gibbs measure μ with Hölder continuous potential, if there exists a μ-measurable solution Φ to $\phi = \Phi(T) - \Phi$, then there is a continuous solution Ψ, with $\Phi = \Psi, a.e.\mu$.*

In this paper, we are only concerned with measurable rigidity. The proof of measurable rigidity in [17] is based on the Markov partitions and Livšic type theorems for cocycles over shifts of finite type [15], which depend heavily on the equipment of Gibbs measures with Hölder continuous potential. For the definition of Gibbs measures, we refer the reader to a classical and short book [4] by Bowen (see Chapter 1). For other measures, measurable rigidity may not hold. In this paper, we consider the measurable rigidity for the special measure, volume measure m. It is known that for $\mathcal{C}^{1+\alpha}$ volume-preserving Anosov diffeomorphisms, the volume measure is a Gibbs measure with the Hölder continuous potential

$$\varphi = -\log \det(DT|E^u).$$

However, the volume measure for $\mathcal{C}^1$ volume-preserving Anosov diffeomorphisms may not be a Gibbs measure with Hölder continuous potential. There are results answering the existence of Gibbs measure (or equilibrium measure) for $\mathcal{C}^1$ generic expanding maps on a circle. In a recent paper of Campbell and Quas [5], it was shown that for a $\mathcal{C}^1$ generic expanding map T on a circle, there is a unique equilibrium state for the potential $-\log T'$. However, this potential may also not be Hölder continuous.

Instead of arguing from the point of Markov partition, there is another direct proof of measurable rigidity for $\mathcal{C}^{1+\alpha}$ volume-preserving Anosov diffeomorphisms in [6] (see p. 80). The idea is, starting from the measurable solution ϕ, to define a new function ψ along the stable foliations and the unstable foliations. Based on the $\mathcal{C}^{1+\alpha}$ condition, we obtain Hölder regularity and absolute continuity of the foliations. Then, it follows that ψ can be extended uniformly to the whole manifold. However, for $\mathcal{C}^1$ diffeomorphisms, we cannot use this regularity argument anymore. This kind of *a priori* regularity argument is also widely used in the proof of ergodicity.

In this paper, under a $\mathcal{C}^1$ generic hypothesis, we have the following result.

Theorem 1.2. *There exists a residual subset $\mathcal{G}$ of $\mathcal{C}^1$ Anosov volume-preserving diffeomorphisms on a compact connected Riemannian manifold M such that for any $T \in \mathcal{G}$ and any Hölder continuous function $\phi : M \to \mathbb{R}$, if $\phi(x) = \Phi(T(x)) - \Phi(x), a.e.$ for some measurable function Φ, then there exists a continuous function Ψ such that $\phi(x) = \Psi(T(x)) - \Psi(x)$ and moreover $\Phi = \Psi, a.e.$*

We also get a parallel result for Anosov flows.

Theorem 1.3. *There exists a residual subset $\mathcal{G}$ of $\mathcal{C}^1$ Anosov volume-preserving flows on a compact connected Riemannian manifold M such that for any flow $\{T^t\} \in \mathcal{G}$ and any Hölder continuous function $\phi : M \to \mathbb{R}$, if $\phi = \Phi'_\xi$ almost* *everywhere for a measurable function Φ differentiable along the flow, then there exists a Hölder continuous function Ψ differentiable along the flow such that $\phi = \Psi'_\xi$ and moreover $\Phi = \Psi$, a.e.*

The argument for Anosov flows proceeds in an almost identical fashion as in the proof of Theorem 1.2, *mutatis mutandis*. So we only give the statement for this flow version. Instead of Theorem 1.1 and Theorem 2.1, the Livšic theorem for Anosov flows [11, 12] and the Central Limit Theorem for Anosov flows [16] are needed in the proof of Theorem 1.3.

2. Preliminaries

Assume M to be a compact Riemannian manifold. Recall that a diffeomorphism $T : M \to M$ is called Anosov if there exist a T-invariant splitting

$$TM = E^s \oplus E^u$$

and constants $C, \rho < 1$ such that

$$\forall v \in E^s, \|DT^n v\| \le C\rho^n \|v\|,$$

$$\forall v \in E^u, \|DT^{-n} v\| \le C\rho^n \|v\|.$$

Now we formulate the Central Limit Theorem for $\mathcal{C}^2$ volume-preserving Anosov diffeomorphisms. Its proof involves the construction of the Markov partition of Anosov diffeomorphisms and the corresponding statistical property of subshifts of finite type.

Theorem 2.1 (Central Limit Theorem (Section 1.27 in [4]))**.** *Let T be a $\mathcal{C}^2$ Anosov volume-preserving diffeomorphism on the compact Riemannian manifold M. Let m be the volume measure on M. Let ϕ be a Hölder continuous function on M with no measurable solution Φ to the equation:*

$$\phi(x) - \int \phi(x)\, dx = \Phi(T(x)) - \Phi(x).$$

Then ϕ satisfies the Central Limit Theorem with respect to T, i.e. there exists a constant $\sigma > 0$ such that for any $-\infty < \alpha < +\infty$,

$$\lim_{n\to+\infty} m\left\{x \in M : \frac{\sum_{i=0}^{n-1} \phi(T^i(x)) - n\int_M \phi(x)dm}{\sigma\sqrt{n}} < \alpha\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-\frac{1}{2}u^2}\, du.$$

What's more,

$$\sigma^2 = \lim_{n\to+\infty} \frac{\int (\sum_{i=0}^{n-1} (\phi(T^i(x)) - \int_M \phi(x)dm))^2\, dx}{n},$$

and σ is called the variance with respect to ϕ.

The Central Limit Theorem plays a central role in the proof of Theorem 1.2. It allows us to perform a Baire argument. More precisely, since we already know that what we want does hold for the $\mathcal{C}^2$ region, our goal is to extract some Baire property from $\mathcal{C}^2$ Anosov volume-preserving systems. The Baire property in this paper is called $(C, \tilde{C}, \varepsilon, p)$-type. The Central Limit Theorem helps us to prove that $\mathcal{C}^2$ Anosov volume-preserving systems are $(C, \tilde{C}, \varepsilon, p)$-type (Proposition 3.2).

3. Proof of Theorem 1.2

We now begin the proof of Theorem 1.2. First we state an essential definition.

Definition 3.1. Let T be a $\mathcal{C}^1$ Anosov volume-preserving diffeomorphism on a compact Riemannian manifold M. For any given constants $C > 0, \tilde{C} > 0, \varepsilon > 0$ and any given periodic point $p \in M$ with period $P(p)$, set

$$\begin{aligned}\mathcal{F}_T(\tilde{C}, \varepsilon, p) \quad = \quad & \Bigg\{ \phi \,|\, \phi \text{ is an } \alpha\text{-Hölder continuous function on } M, \int_M \phi \, dx = 0, \\ & \|\phi\|_\alpha \le \tilde{C}, |\sum_{i=0}^{P(p)-1} \phi(T^i(p))| \ge \varepsilon \Bigg\}.\end{aligned}$$

We say T is of $(C, \tilde{C}, \varepsilon, p)$**-type** if there exists a common time N, such that for any $\phi \in \mathcal{F}_T(\tilde{C}, \varepsilon, p)$, there exists at least one moment $1 \le k \le N$ such that

$$\mu\{x \in M : \sum_{i=0}^{k-1} \phi(T^i(x)) > C\} > \frac{1}{2} - \varepsilon. \tag{3}$$

It is easy to see that if $C_1 \le C_2$, then $(C_2, \tilde{C}, \varepsilon, p)$-type implies $(C_1, \tilde{C}, \varepsilon, p)$-type. In the following proposition, we use Central Limit Theorem 2.1 to prove that for any $(C, \tilde{C}, \varepsilon, p)$, $\mathcal{C}^2$ Anosov volume-preserving diffeomorphisms are of $(C, \tilde{C}, \varepsilon, p)$-type.

Proposition 3.2. *Let T be a $\mathcal{C}^2$ Anosov volume-preserving diffeomorphism on a compact connected manifold M. For any $(C, \tilde{C}, \varepsilon, p)$, T is of $(C, \tilde{C}, \varepsilon, p)$-type.*

Proof. Fix constants $(C, \tilde{C}, \varepsilon)$ and a periodic point p arbitrarily. According to the $\mathcal{C}^2$ case of Theorem 1.1 and Theorem 2.1, for any $\phi \in \mathcal{F}_T(\tilde{C}, \varepsilon, p)$, there exists $\sigma > 0$, such that for any $\alpha_0 > 0$, there exists $N_0 \in \mathbb{N}$ satisfying, for any $n \ge N_0$,

$$m\left\{ x \in M : \frac{\sum_{i=0}^{n-1} \phi(T^i(x))}{\sigma\sqrt{n}} > \alpha_0 \right\} \ge \frac{1}{\sqrt{2\pi}} \int_{\alpha_0}^{+\infty} e^{-\frac{1}{2}u^2} du - \frac{\varepsilon}{2}.$$

Choose α_0 small enough such that $\frac{1}{\sqrt{2\pi}} \int_{\alpha_0}^{+\infty} e^{-\frac{1}{2}u^2} du - \frac{\varepsilon}{2} \ge \frac{1}{2} - \varepsilon$. Assume N_1 to be an integer satisfying $\frac{C}{\sigma\sqrt{N_1}} \le \alpha_0$. Let $N(\phi) := \max\{N_0, N_1\}$. Then, for any $n \ge N(\phi)$,

$$\begin{aligned} m\left\{ x \in M : \sum_{i=0}^{n-1} \phi(T^i(x)) > C \right\} \quad & \ge \quad m\left\{ x \in M : \frac{\sum_{i=0}^{n-1} \phi(T^i(x))}{\sigma\sqrt{n}} > \alpha_0 \right\} \\ & > \quad \frac{1}{2} - \varepsilon. \end{aligned}$$

For this fixed time $N(\phi)$, there exists a small neighborhood $\mathcal{U}(\phi)$ of ϕ such that for any function $\psi \in \mathcal{U}(\phi)$, we have

$$m\left\{ x \in M : \sum_{i=0}^{N(\phi)-1} \psi(T^i(x)) > C \right\} > \frac{1}{2} - \varepsilon. \tag{4}$$

Due to the compactness (in the $\mathcal{C}^0$ topology) of the set $\mathcal{F}(\tilde{C}, \varepsilon, p)$, there exists a finite cover $\mathcal{P} = \{\mathcal{U}(\phi_i)\}_{i=0}^K$ of $\mathcal{F}_T(\tilde{C}, \varepsilon, p)$ and thus a common time

$$N = \max_{1 \leq i \leq K} N(\phi_i).$$

This common time N satisfies the condition we want. $\square$

Now we prove that $(C, \tilde{C}, \varepsilon, p)$-type implies some measurable rigidity.

Proposition 3.3. *Let T be a $\mathcal{C}^1$ Anosov volume-preserving diffeomorphism on a compact connected Riemannian manifold M. Assume that for any $C > 0$, $\tilde{C} > 0$, $\varepsilon > 0$ and any periodic point p, T is $(C, \tilde{C}, \varepsilon, p)$-type. Consider any α-Hölder continuous function $\phi : M \to \mathbb{R}$ satisfying $\phi(x) = \Phi(T(x)) - \Phi(x), a.e.$ for some measurable function Φ. Then there exists a continuous function Ψ such that $\phi(x) = \Psi(T(x)) - \Psi(x)$.*

Proof. Assume Φ is a measurable solution to $\phi(x) = \Phi(T(x)) - \Phi(x), a.e.$ Therefore, for any small number $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$m\{x \in M : |\Phi(x)| \leq C_\varepsilon\} > 1 - \frac{\varepsilon}{2}.$$

Thus, by the identity $\sum_{i=0}^{n-1} \phi(f^i(x)) = \Phi(T^n(x)) - \Phi(x),\ a.e.$, it follows that

$$\begin{aligned}
& m\{x \in M : \sum_{i=0}^{n-1} \phi(f^i(x)) \leq 2C_\varepsilon\} \\
= \;& m\{x \in M : \Phi(T^n(x)) - \Phi(x) \leq 2C_\varepsilon\} \\
> \;& m\{x \in M : |\Phi(T^n(x))| + |\Phi(x)| \leq 2C_\varepsilon\} \\
> \;& m(\{x \in M : |\Phi(T^n(x))| \leq C_\varepsilon\} \cap \{x \in M : |\Phi(x)| \leq C_\varepsilon\}) \\
> \;& 1 - \varepsilon,\ \forall\, n \geq 1.
\end{aligned}$$

Since the nonwandering set of $\mathcal{C}^1$ conservative Anosov diffeomorphisms should be the whole space, $\mathcal{C}^1$ conservative Anosov diffeomorphisms are transitive. According to the first result in Theorem 1.1, if there is no continuous solution for $\phi(x) = \Phi(T(x)) - \Phi(x)$, there must exist a periodic point p and $\varepsilon > 0$ such that $|\sum_{i=0}^{P(p)-1} \phi(T^i(p))| \geq \varepsilon$. On the other hand, since T is of $(C, \tilde{C}, \varepsilon, p)$-type, for $C > 2C_\varepsilon$ and the function ϕ, there exists a time $1 \leq k \leq N$, such that $m\{x \in M : \sum_{i=0}^{k-1} \phi(f^i(x)) \geq C\} \geq \frac{1}{2} - \varepsilon$. Therefore,

$$\begin{aligned}
1 \;&\geq\; m\{x \in M : \sum_{i=0}^{k-1} \phi(f^i(x)) \leq 2C_\varepsilon\} + m\{x \in M : \sum_{i=0}^{k-1} \phi(f^i(x)) \geq C\} \\
&\geq\; (1 - \varepsilon) + (\frac{1}{2} - \varepsilon) \\
&\geq\; \frac{3}{2} - \varepsilon,
\end{aligned}$$

which is a contradiction. So there exists a continuous solution $\Psi : M \to \mathbb{R}$ such that $\phi(x) = \Psi(T(x)) - \Psi(x)$, for any $x \in M$. $\square$

From Proposition 3.2, we have that $\mathcal{C}^2$ Anosov volume-preserving diffeomorphisms are good type. Recall A. Avila's result [1] about the regularization of volume-preserving maps.

Theorem 3.4 ([1]). *Smooth maps are $\mathcal{C}^1$ dense in $\mathcal{C}^1$ volume-preserving maps.*

By Theorem 3.4 and the $\mathcal{C}^1$ stability of Anosov systems, we can get that $\mathcal{C}^2$ Anosov volume-preserving diffeomorphisms are dense in $\mathcal{C}^1$ Anosov volume-preserving diffeomorphisms, which is a key point in our proof. (See Theorem 2.2 in [3] for the flow case.)

Theorem 3.5. *There exists a residual subset $\mathcal{G}$ of $\mathcal{C}^1$ Anosov volume-preserving diffeomorphisms on a compact connected Riemannian manifold M such that for any $T \in \mathcal{G}$ and any $\phi : M \to \mathbb{R}$ Hölder continuous, if $\phi(x) = \Phi(T(x)) - \Phi(x)$, a.e. for some measurable function Φ, then there exists a continuous function Ψ such that $\phi(x) = \Psi(T(x)) - \Psi(x)$.*

Proof. Take a countable basis $\mathcal{V} = \{\mathcal{V}_1, \mathcal{V}_2, \cdots\}$ of M. Denote

$$\begin{aligned} H_{k_1,k_2,k_3} \quad = \quad & \{G \text{ is a } \mathcal{C}^1 \text{ Anosov volume-preserving diffeomorphism on } M| \\ & \text{there exists a } \mathcal{C}^1 \text{ neighborhood } U(G) \text{ of } G \text{ in the } \mathcal{C}^1 \text{ Anosov} \\ & \text{volume-preserving diffeomorphisms on } M, \text{ such that for any} \\ & G_1 \in U(G), \text{ for any } 0 < C \leq k_1,\ \tilde{C} > 0, \varepsilon > 0 \text{ and for any} \\ & \text{periodic point of } G_1,\ p \in \mathcal{V}_{k_2} \in \mathcal{V} \text{ with period } P(p) \leq k_3, \\ & G_1 \text{ is of } (C, \tilde{C}, \varepsilon, p)\text{-type}\}. \end{aligned}$$

It is easy to see that H_{k_1,k_2,k_3} is open. Set

$$\mathcal{G} := \bigcap_{k_1,k_2,k_3 \in \mathbb{N}} H_{k_1,k_2,k_3}.$$

Now we prove $\mathcal{G}$ is the generic set we want.

In order to proof the density of H_{k_1,k_2,k_3}, we prove the following claim first.

Claim. $\mathcal{C}^2$ Anosov volume-preserving diffeomorphisms are contained in H_{k_1,k_2,k_3}, for any $k_1, k_2, k_3 \geq 1$.

Let T be any $\mathcal{C}^2$ Anosov volume-preserving diffeomorphism on M. Fix k_1, k_2, k_3. We finish our proof by choosing some smaller and smaller neighborhoods of T. By Proposition 3.2, for any $(C, \tilde{C}, \varepsilon, p)$, T is of $(C, \tilde{C}, \varepsilon, p)$-type. Thus, there exists N such that: for any $\phi \in \mathcal{F}_T(\tilde{C}, \varepsilon, p)$ where p is periodic for T, there exists $1 \leq i \leq N$ such that,

$$m\{x \in M : \sum_{j=0}^{i-1} \phi(T^j(x)) > 2k_1\} > \frac{1}{2} - \varepsilon.$$

Consider the set

$$\begin{aligned} \mathcal{F}_G(\tilde{C}, \varepsilon, p) \quad = \quad & \Bigg\{ \phi \,|\, \phi \text{ is an } \alpha\text{-Hölder continuous function on } M, \int_M \phi \, dx = 0, \\ & \|\phi\|_\alpha \leq \tilde{C}, |\sum_{i=0}^{P(p)-1} \phi(G^i(p))| \geq \varepsilon \Bigg\}, \end{aligned}$$

where G is a $\mathcal{C}^1$ Anosov volume-preserving diffeomorphism $\mathcal{C}^1$-close to T and $p \in \mathcal{V}_{k_2}$ is a periodic point with period $P(p) \leq k_3$ for G.

There exists a small neighborhood $W(T)$ of T such that for any $G \in W(T)$, $\mathcal{F}_G(\tilde{C}, \varepsilon, p) \subset \mathcal{F}_T(\tilde{C}, \frac{\varepsilon}{2}, q)$, where q is the continuation of p given by structure stability with the same period $P(p) \leq k_3$. Thus, there exists N such that for any $\phi \in \mathcal{F}_G(\tilde{C}, \varepsilon, p) \subset \mathcal{F}_T(\tilde{C}, \frac{\varepsilon}{2}, q)$, we have a time $1 \leq i \leq N$ such that

$$m\{x \in M : \sum_{j=0}^{i-1} \phi(T^j(x)) > 2k_1\} > \frac{1}{2} - \frac{\varepsilon}{2}.$$

Next, there exists a smaller neighborhood $V(T) \subset W(T)$ of T such that for any $G \in V(T)$, there exists N such that for any $\phi \in \mathcal{F}_G(\tilde{C}, \varepsilon, p) \subset \mathcal{F}_T(\tilde{C}, \frac{\varepsilon}{2}, q)$, there exists $1 \leq i \leq N$ such that

$$m\left\{x \in M : \sum_{j=0}^{i-1} \phi(G^j(x)) > k_1\right\} > \frac{1}{2} - \frac{\varepsilon}{2}.$$

Taking the uniform hyperbolicity of T into account, there are only finitely many $p \in \mathcal{V}_{k_2}$ with period $P(p) \leq k_3$ for every $G \in V(T)$. Thus we get another smaller neighborhood $U(T) \subset V(T)$ such that for any $G \in U(T)$, any constants $0 < C \leq k_1, \tilde{C} > 0, \varepsilon > 0$ and any periodic point $p \in \mathcal{V}_k$ with period $P(p) \leq k_3$, G is $(C, \tilde{C}, \varepsilon, p)$-type.

Thus, $T \in H_{k_1,k_2,k_3}$. This completes the proof of the claim.

So the above claim implies that H_{k_1,k_2,k_3} is $\mathcal{C}^1$ dense in $\mathcal{C}^1$ Anosov volume-preserving diffeomorphisms and then we get that $\mathcal{G}$ is a generic set.

It is easy to see from the definition that for every diffeomorphism $G \in \mathcal{G}$ and every tuple $(C, \tilde{C}, \varepsilon, p)$, G is of $(C, \tilde{C}, \varepsilon, p)$-type. By Proposition 3.3, we finish the proof. □

Proof of Theorem 1.2. To complete the proof of Theorem 1.2, it remains to prove that for $\mathcal{C}^1$ generic Anosov volume-preserving diffeomorphisms, the continuous solution Ψ we get in Theorem 3.5 satisfies $\Psi = \Phi, a.e.$ From $\Phi(T(x)) - \Phi = \Psi(T(x)) - \Psi, a.e.$ we obtain $(\Phi - \Psi)(T(x)) - (\Phi - \Psi)(x) = 0, a.e.$ Indeed we only need to prove $\Phi - \Psi = 0, a.e.$, which directly comes from the fact that $\mathcal{C}^1$ generic Anosov volume-preserving diffeomorphisms on a compact connected Riemannian manifold M are ergodic [2]. □

At the end of this paper, the author would like to address three interesting questions related to Theorem 3.5 and Theorem 1.2:

Question 3.6. Are there any counter-examples, i.e. for some C^1 Anosov volume-preserving diffeomorphism T and some Hölder continuous function ϕ, there is only a measurable solution Φ to $\phi(x) = \Phi(T(x)) - \Phi(x), a.e.$, but with no continuous solution?

Question 3.7. Is it true that for $\mathcal{C}^1$ generic Anosov volume-preserving systems, the Central Limit Theorem holds?

The third one is a well-known open question about ergodicity.

Question 3.8. Is it true that $\mathcal{C}^1$ Anosov volume-preserving diffeomorphisms are ergodic?

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