

TWIST SPINNING KNOTTED TRIVALENT GRAPHS

J. SCOTT CARTER AND SEUNG YEOP YANG

(Communicated by Martin Scharlemann)

ABSTRACT. In 1965, E. C. Zeeman proved that the (± 1) -twist spin of any knotted sphere in $(n-1)$ -space is unknotted in the n -sphere. In 1991, Y. Marumoto and Y. Nakanishi gave an alternate proof of Zeeman's theorem by using the moving picture method. In this paper, we define a knotted 2-dimensional foam which is a generalization of a knotted sphere and prove that a (± 1) -twist spin of a knotted trivalent graph may be knotted. We then construct some families of knotted graphs for which the (± 1) -twist spins are always unknotted.

1. INTRODUCTION

We initiate a study of knotted foams herein. A knotted foam is to a knotted surface in 4-dimensional space as a knotted trivalent graph is to a classical knot in 3-dimensions. We present the basic definitions and facts necessary for our study in Section 2.1. We quickly review twist spinning as a construction of knotted spheres in Definition 2.2. These ideas are generalized to construct the twist spin of a knotted trivalent graph in Section 2.2. There we study (± 1) -twist spins. We work in the smooth (or piecewise linear) category throughout this paper.

The birth of higher-dimensional knot theory occurred in 1925, when Artin [1] introduced the notion of spinning of a classical knot which is a method of making a knotted surface from a given classical knot. In 1965, Zeeman [11] introduced the notion of twist spun knots and showed that (± 1) -twist spun knots are equivalent to trivial 2-knots. More generally twist spun knots are fibered with the fiber being the branched covers of the punctured 3-sphere with branch set the given knot. For more details on the history of higher-dimensional knot theory see [4].

A *knotted (spatial) trivalent graph* or simply *KTG* is an embedding into 3-space of a graph, for which every vertex has degree 3. In this paper we will study the notion of n -twist spin of a knotted trivalent graph as a generalization of Artin and Zeeman's studies [1, 11]. The twist spins of a KTG are examples of knotted foams (this concept will be defined in a moment). This paper is a study of a limited class of examples of knotted foams; even so the results are surprising enough to indicate that the study of knotted foams is potentially interesting.

2. KNOTTED 2-DIMENSIONAL FOAMS (2-FOAMS)

2.1. Preliminaries. A *knotted sphere* or simply *2-knot* is the image of a smooth (or PL locally flat) embedding $f : S^2 \rightarrow \mathbb{R}^4$. A *knotted 2-dimensional foam* is analogous to a knotted sphere in the same way that a knotted trivalent graph is analogous to a classical knot. A formal definition follows.

Received by the editors November 12, 2014 and, in revised form, March 17, 2015.
2010 *Mathematics Subject Classification*. Primary 57Q45.

Definition 2.1. A *2-dimensional foam* or simply *2-foam* is a compact topological space such that a neighborhood of any point is homeomorphic to a neighborhood of a point in Y^2 that is depicted in Figure 1.

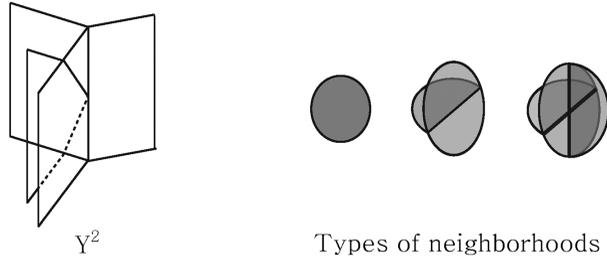


FIGURE 1. Neighborhoods of a point in Y^2

For more specificity, we define $Y_1^0 = (0, 1/2, 1/2)$, $Y_2^0 = (1/2, 0, 1/2)$, and $Y_3^0 = (1/2, 1/2, 0)$. The space Y^1 is the cone on $Y_1^0 \cup Y_2^0 \cup Y_3^0$ where the cone point is the barycenter $(1/3, 1/3, 1/3)$ of the standard equilateral triangle $\Delta^2 = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x, y, z \text{ \& } x + y + z = 1\}$. Let

$$\Delta^3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \sum_{i=1}^4 x_i = 1 \text{ \& } 0 \leq x_i \text{ for } i = 1, 2, 3, 4\}.$$

Let Δ_i^2 denote the i th face in which $x_i = 0$. Consider a copy $Y_i^1 \subset \Delta_i^2$ for each $i = 1, 2, 3, 4$. The space Y^2 is the cone on $\bigcup_{i=1}^4 Y_i^1$ where the cone point is the barycenter $b = (1/4, 1/4, 1/4, 1/4)$. Clearly, the space Y^2 generalizes to all dimensions. In such generalizations, one can construct notions of n -dimensional foams.

According to the definition of a foam, there are three types of neighborhoods of a point in a 2-foam (Figure 1), and the boundary of a 2-foam is a trivalent graph. A 2-foam with no boundary is said to be *closed*.

A 2-foam is an analogue of a trivalent graph. So aspects of embeddings of trivalent graphs will be reviewed. An embedded knotted graph Γ in \mathbb{R}^3 (or S^3) is *unknotted* if it is ambiently isotopic to an embedding in \mathbb{R}^2 . A subgraph of Γ which is a knot in \mathbb{R}^3 is called a *constituent knot* of Γ , and a mutually disjoint union of constituent knots of Γ is said to be a *constituent link* of Γ .

Any θ -curve embedded in \mathbb{R}^3 has three constituent knots. Although all three constituent knots may be unknotted on a θ -curve, the θ -curve may be knotted. For example, Kinoshita's θ -curve (in Figure 2) is a knotted θ -curve for which all constituent knots are unknotted. A graph Γ in \mathbb{R}^3 is said to be *almost trivial* if Γ is a non-trivial planar graph and every proper subgraph of Γ is unknotted [8].

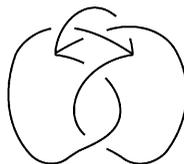


FIGURE 2. Kinoshita's θ -curve

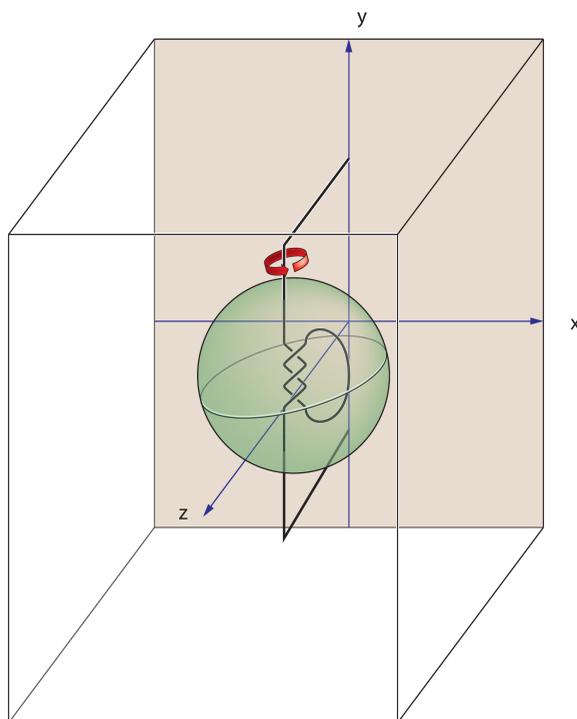


FIGURE 3. Twist spinning a classical knot

Definition 2.2. The k -twist spin of a classical knot K is obtained as follows. Let $\mathbb{R}_+^3 = \{(x, y, z) : 0 \leq z\}$ and, for $\theta \in [0, 2\pi)$, let $\mathbb{R}_+^3(\theta) = \{(x, y, z \cos \theta, z \sin \theta) : 0 \leq z\}$. Consider once and for all a standard identification between \mathbb{R}_+^3 and $\mathbb{R}_+^3(\theta)$. Embed a $(1, 1)$ tangle whose closure is K into the ball $B = \{(x, y, z) : x^2 + y^2 + (z - 3)^2 \leq 1\}$ such that the end points of the tangle are at $(0, \pm 1, 3)$. Now join these by L-shaped arcs to the points $(0, \pm 3, 0)$ in the plane $z = 0$; see Figure 3 for details. As θ varies, rotate the space in the ball B a total of k full twists. Since \mathbb{R}^4 can be thought of as the union of $\mathbb{R}_+^3(\theta)$ as θ varies and since the tangle twists within the ball B , the result is a sphere embedded in \mathbb{R}^4 . The half spaces $\mathbb{R}_+^3(\theta)$ are called the *pages of an open book decomposition* of \mathbb{R}^4 .

Note that an n -twist spin of a KTG is defined similarly, but the graph is broken along one edge and embedded in the ball B in a way similar to the tangle representation of K .

We now discuss the analogues of triviality for 2-foams. We say that an embedded 2-foam in 4-space is *unknotted* if it is ambiently isotopic to an embedding in 3-space. *In this case, the fundamental group of the complement of the foam in 4-space is a free group.*

2.2. Is a (± 1) -twist spin of a knotted trivalent graph unknotted? The following theorem due to Zeeman is remarkable, and is the motivation for studying twist spins of KTGs. First we need to develop some notation.

A knot is called *fibred* if its complement is a fiber bundle over S^1 . If a 2-knot is fibred, its fiber is a 3-manifold whose boundary is homeomorphic to S^2 . The closed 3-manifold obtained from such a fiber by attaching a 3-ball along its boundary is called the *closed fiber*.

Theorem 2.3 ([11]). *The $(\pm k)$ -twist spin of any knot $K \subset S^{n-1}$ is fibred where the fiber is the k -fold branched covering of S^{n-1} in which the branch set is the knot K .*

In the case of the (± 1) -twist spin of a knot, the 1-fold branched cover of S^3 branched along the knot is tautologically S^3 . Thus the closed fiber is a 3-sphere. So in this case the resulting knotted surface is trivial.

When we spin a knotted trivalent graph it is surprising that the result can be non-trivial. The following theorem is one of our main results.

Theorem 2.4. *There is a non-trivial foam obtained by a (± 1) -twist spinning of a knotted θ -curve. That is, a (± 1) -twist spun KTG is not always unknotted.*

We remind the reader (Chapter 9, [7]) that a knotted surface can be represented in a normal form as a classical knot diagram with rectangular bands that are labeled u or d (for up or down). The bands represent saddle points of a critical level embedding and are resolved as indicated in Figure 4. After all up or down resolutions, the resulting knot diagrams are unlinks.

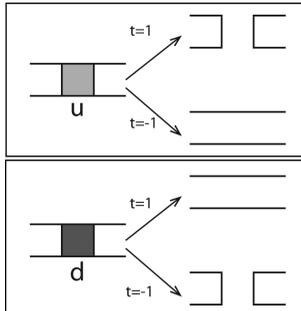


FIGURE 4. Resolving up and down bands

In 1991, Y. Marumoto and Y. Nakanishi [9] gave an alternate proof of the triviality of (± 1) -twist spins by using the motion picture method. Since we will use their idea to prove our main theorem, we will look at their proof in some detail.

By Zeeman’s twist spinning method, we can construct the (± 1) -twist spin of K , denoted by $t_{\pm 1}(K)$, from the given classical knot K . Since all knots can be deformed into a trivial knot by performing a finite number of crossing changes, we only need to show that $t_{\pm 1}(K)$ and $t_{\pm 1}(K')$ are ambiently isotopic when K' is the result of changing a crossing of K .

Let $*$ be the crossing at which the crossing change will occur. Pull the crossing close to the plane \mathbb{R}^2 that is fixed during spinning as indicated in case 1 of Figure 5. The crossing then becomes a minimal point. If the crossing is opposite, then we can deform this crossing to the first case crossing by using the second Reidemeister move as in case 2 of Figure 5. Then the first case’s and second case’s diagrams are locally the same nearby the fixed plane.

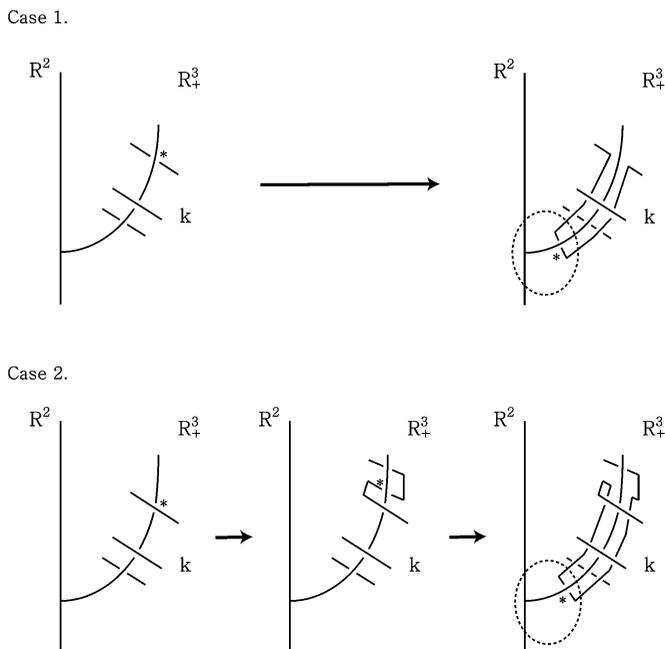


FIGURE 5. Marumoto and Nakanishi's idea I

Then the normal form of the (± 1) -twist spin of the diagram nearby the fixed plane is the first picture in Figure 6. The first picture and the second picture are ambiently isotopic. In the second picture, if we push the upper saddle band to the level of lower saddle band and the lower saddle band to the level of upper saddle band, then the saddle bands may switch their positions, and the saddle bands now can stretch up and down. We can deform the third picture to the last one by an ambient isotopy of \mathbb{R}^4 . In the last normal form in Figure 6, we see that the crossings $*$ are changed. Therefore the crossing changes can be realized by an ambient isotopy of \mathbb{R}^4 .

We next consider an important difference between spun knots and spun KTGs. Consider a knot K in S^3 , and take a small 3-dimensional ball B^3 on the knot K . Then we can consider the closure of $S^3 \setminus B^3$ as one page in an open book decomposition. Since a knot K is a 1-dimensional manifold, the 3-ball on the knot K can travel anywhere on the knot K . So spinning the classical knot does not depend on which arc that is split; see Figure 7. But since a KTG has vertices, the 3-ball may not move anywhere on the KTG. *Therefore spinning the KTG depends upon the splitting arc.*

When a KTG is spun or twist spun, it has constituents that are analogous to the constituents of a KTG. A constituent is either a knotted sphere or a knotted torus.

Theorem 2.5. *Every foam obtained by a (± 1) -twist spin of a knotted θ -curve has exactly two trivial S^2 -constituent components.*

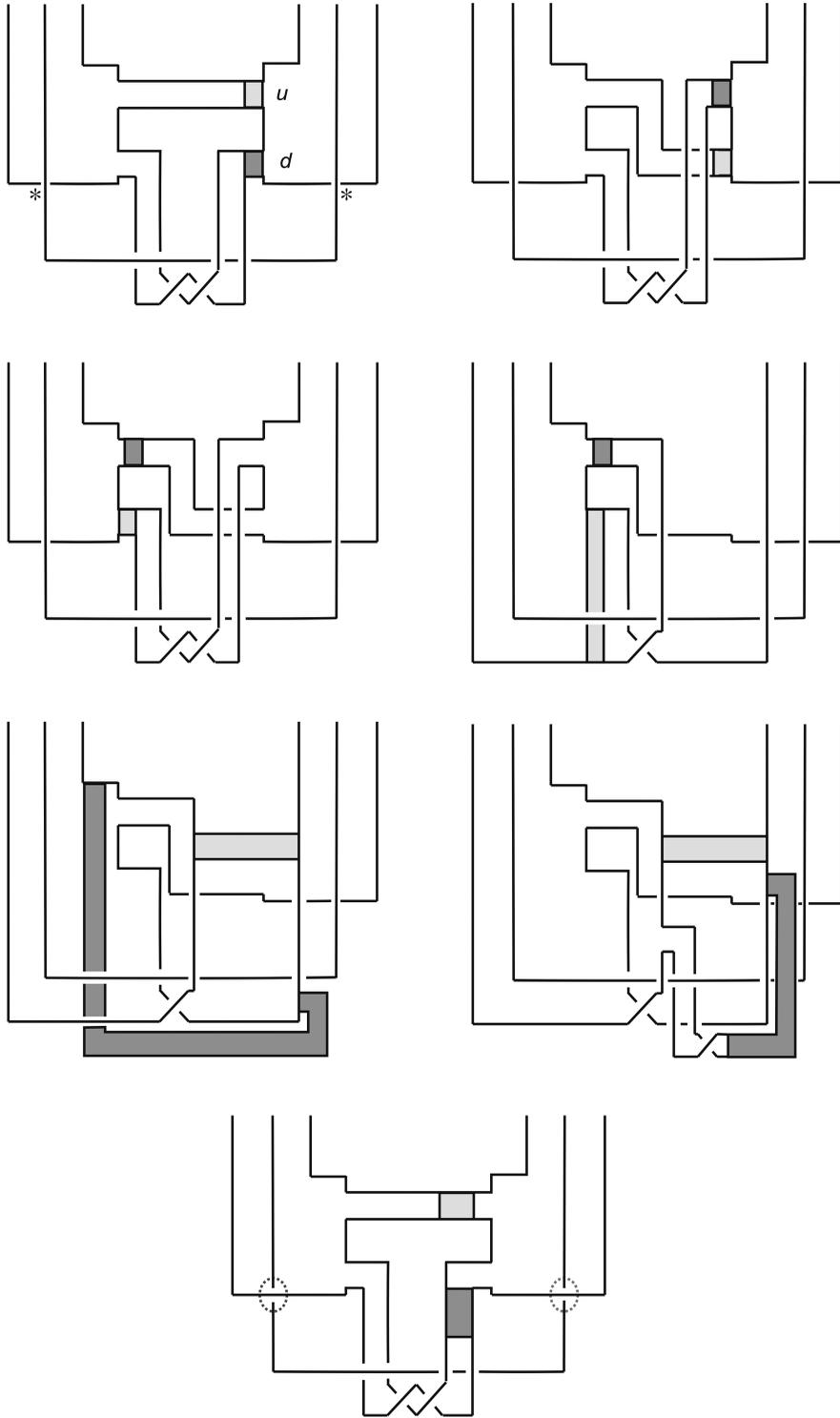


FIGURE 6. Marumoto and Nakanishi's idea II

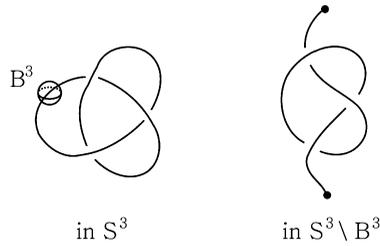


FIGURE 7. Spinning the knot does not depend on the splitting arc

Proof. Let Γ be a knotted θ -curve. Label its edges $e_1, e_2,$ and e_3 . There are three constituent knots of Γ ; these are $e_1 \cup e_2, e_2 \cup e_3,$ and $e_3 \cup e_1$. Choose an edge e_i of the knotted θ -curve Γ , cut at a point on e_i , and attach the end points to the plane \mathbb{R}^2 . Then by using Marumoto and Nakanishi’s technique, we can deform this broken KTG to satisfy the following conditions (see Figure 8):

- (1) There are no crossings on the broken edges e_i .
- (2) For each $j \neq i, e_j$ is an unknotted edge.

Denote this deformed KTG by \tilde{G} and the (± 1) -twist of \tilde{G} with respect to e_i by $t_{e_i}(\tilde{G})$. Then the (± 1) -twist spins of $e_1 \cup e_2, e_2 \cup e_3,$ and $e_3 \cup e_1$ are constituent components of $t_{e_i}(\tilde{G})$. Since $e_i \cup e_j$ is an unknotted arc for each $j \neq i, (\pm 1)$ -twist spinning $e_i \cup e_j$ makes a trivial S^2 -constituent component of Γ . \square

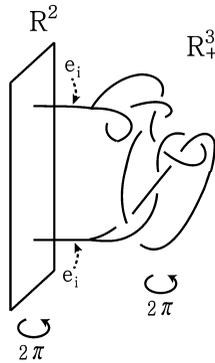


FIGURE 8. A deformed broken KTG

Example 2.6. Let Γ be the knotted θ -curve depicted in Figure 9. We take the edge e_1 of Γ and construct a deformed KTG with respect to the edge e_1 as in the following steps. We first pull the edge e_3 close to the plane and change the crossing by using Marumoto and Nakanishi’s technique. This process gives the second diagram. The second, third, and fourth pictures are ambiently isotopic. In the fourth picture, we see that (± 1) -twist spinning $e_1 \cup e_2$ and $e_1 \cup e_3$ construct two trivial S^2 -constituent components.

We are going to prove that there is a KTG such that a (± 1) -twist spin of the KTG is a knotted foam in \mathbb{R}^4 . First we need to recall from [6] the definition of a G -family of quandles.

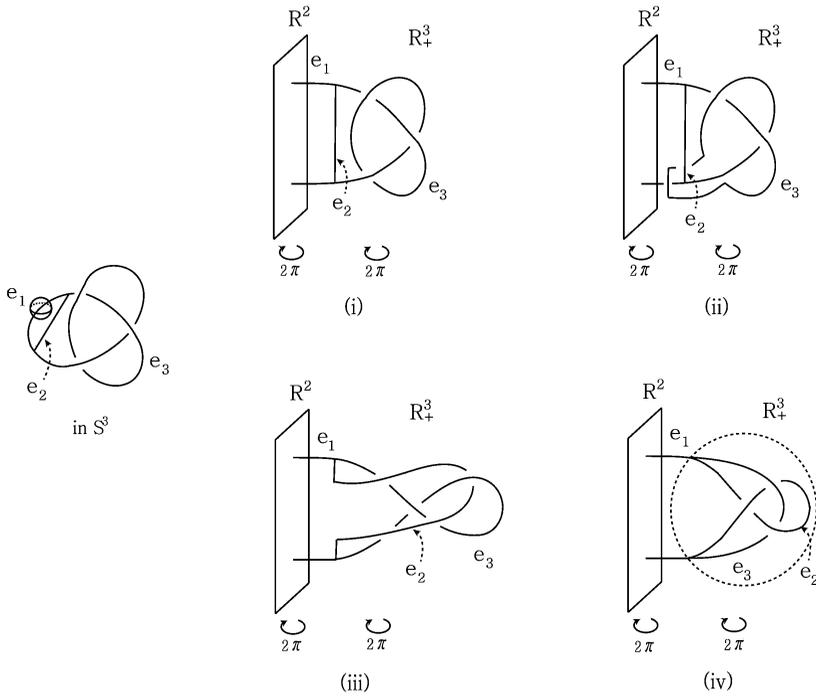


FIGURE 9. A (± 1) -twist spin of a knotted θ -curve has exactly two trivial S^2 -constituent components

Definition 2.7. Let G denote a group and X a set. Suppose that for each $g \in G$, there is a binary operation $\triangleleft_g : X \times X \rightarrow X$ that satisfies the properties:

- $x \triangleleft_g x = x$ for all $x \in X$ and $g \in G$,
- $(x \triangleleft_g y) \triangleleft_h y = x \triangleleft_{gh} y$ for all $x, y \in X$ and $g, h \in G$,
- $x \triangleleft_1 y = x$ for all $x, y \in X$ where $1 \in G$ denotes the identity, and
- $(x \triangleleft_g y) \triangleleft_h z = (x \triangleleft_h z) \triangleleft_{h^{-1}gh} (y \triangleleft_h z)$ for all $x, y, z \in X$ and $g, h \in G$.

Then $(X, \{\triangleleft_g : g \in G\})$ is called a G -family of quandles. In this case there is a quandle structure on $X \times G$ that is defined by $(x, g) \triangleleft (y, h) = (x \triangleleft_h y, h^{-1}gh)$. This structure is called *the associated quandle of a G -family of quandles*.

The concept of G -family of quandles was invented to color handlebody knots and KTGs, but the notion of coloring extends easily to colorings of knotted foams. When a foam is given, for example, as a movie of a KTG, then the coloring condition is that each still in the movie is colored as a KTG, and that the colorings obey consistency through Reidemeister moves.

Here we are only interested in the colorings by the quandle $Q = \mathbb{Z}/3 \times \mathbb{Z}_2$, with quandle operation

$$(x, \epsilon) \triangleleft (y, 1) = (2y - x, \epsilon), \quad (x, \epsilon) \triangleleft (y, 0) = (x, \epsilon).$$

We think of the set $\{0, 1, 2\} = \mathbb{Z}/3$ as the set of colors $\{R, G, B\}$ which are stand-ins for *Red*, *Green*, and *Blue*. The elements of Q are of the form $(C, 0)$ or $(C, 1)$ where $C \in \{R, G, B\}$. For each color C , abuse notation and let $C := (C, 1)$ and

$c := (C, 0)$. In our diagrams, an arc that is colored with $(C, 0) = c$ is indicated as a dotted arc, and those colored by $(C, 1) = C$ are indicated by solid arcs.

The coloring conditions for trivalent vertices and crossings are as follows.

- The first component of the colors (C, ϵ) at a vertex are all the same.
- Either all three arcs are dotted, or exactly one is dotted.
- If a solid arc with color $(C, 1)$ crosses over another arc of a different color (whether or not the under-arc is solid or dotted), the colors change on the under-arc.
- The colors on an under-arc remain unchanged if crossed over by a dotted arc or if crossed over by a solid arc of the same color.

Figure 10 indicates these conventions.

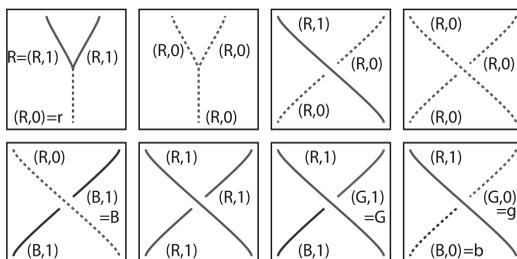


FIGURE 10. Coloring conditions at crossings

Theorem 2.8. *If there exists a non-trivial coloring of the given foam by the associated quandle of a G -family of quandles, then the foam is knotted.*

Proof. Such a coloring gives rise to a non-trivial representation of the fundamental group of the complement of the foam into the associated group of the quandle. \square

Theorem 2.9. *There is a non-trivial foam obtained by a (± 1) -twist spinning of a knotted θ -curve. That is, a (± 1) -twist spun KTG is not always unknotted.*

Proof. We consider the knotted θ -curve Γ in Figure 9. We choose the edge e_1 of Γ to construct a 1-twist spun Γ with respect to the edge e_1 . Following techniques of Satoh and Shima [10] (see Figure 11, p. 1019, therein), we construct the 1-twist spin by means of a movie as illustrated schematically in Figure 11. Notice in this figure of ours that we have specified a non-trivial coloring of each still of the movie of 1-twist spun Γ by the associated quandle $Q = \mathbb{Z}/3 \times \mathbb{Z}_2$. The consistency of the coloring through stills indicates that the coloring extends to the entire foam. This extension exists since the unbroken sheets of the diagram of the foam are constructed from the time-elapsed over-arcs in the movie. The situation is completely analogous to the knotted surface case. See [2, 4, 5] or [10] for details.

Thus, a (-1) -twist spun Γ with respect to the edge e_1 has a non-trivial coloring. By the italicized remark immediately above Section 2.2 and by Theorem 2.8, the (± 1) -twist spinning of Γ with respect to the edge e_1 is a knotted foam. \square

Theorem 2.10. *Let Γ be a spatial graph and e an edge of Γ . Assume that $\Gamma \setminus \{e\}$ is unknotted. Then the (± 1) -twist spin of Γ with respect to the edge e is unknotted.*

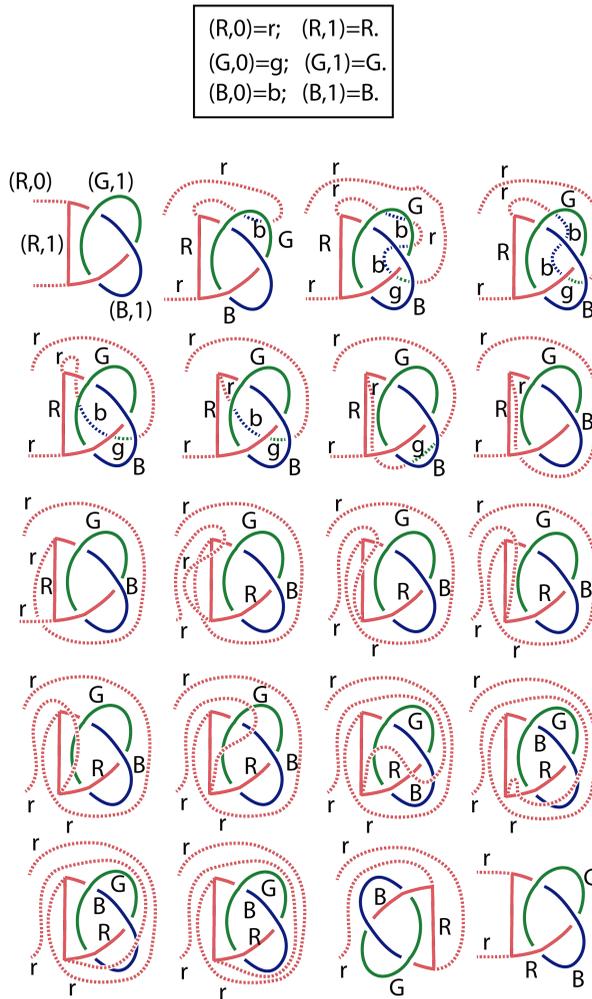


FIGURE 11. A (± 1) -twist spun KTG is not always unknotted

Proof. To construct (± 1) -twist spun Γ with respect to the edge e , we cut the edge e , and pull the end points close to the plane. When pulling the end points we can make the end points cross over the edges transversely but not cross over the vertices since Γ is finite. Now, choose one of the end points and trace on the edge until we come to a crossing. If we trace on the under-strand at this crossing, then we change the crossing. Otherwise we do not. Keep doing this process until we arrive at a vertex. Next, we apply the same strategy to another end point. Note that these crossing changes do not deform the (± 1) -twist spun Γ by Y. Marumoto and Y. Nakanishi's technique. Denote the deformed KTG after these crossing changes by $\tilde{\Gamma}$. Since only the crossings with respect to the edge e are changed in this process, $\tilde{\Gamma} \setminus \{e\}$ is still unknotted. So, we can deform $\tilde{\Gamma}$ to be unknotted by using Reidemeister moves and Y. Marumoto and Y. Nakanishi's technique if we need. The (± 1) -twist spin of $\tilde{\Gamma}$

with respect to e , therefore, is unknotted, so the (± 1) -twist spin of Γ with respect to e is also unknotted. The proof is complete. \square

Corollary 2.11. *Any (± 1) -twist spin of an unknotted or almost trivial graph is unknotted.*

Proof. If a graph is unknotted, then clearly (± 1) -twist spin of the given graph is unknotted. Let Γ be an almost trivial graph. Since Γ is almost trivial, the subgraph $\Gamma \setminus \{e\}$ of Γ is unknotted for each edge e . By Theorem 2.10, the (± 1) -twist spin of Γ with respect to the edge e is unknotted for each edge e . Hence any (± 1) -twist spin of Γ is unknotted. \square

By the above corollary, we can easily check the following example.

Example 2.12. Any (± 1) -twist spin of Kinoshita’s θ -curve is unknotted.

We do not know if the converse, “If any (± 1) -twist spin of a θ -curve is unknotted, then the θ -curve is either unknotted or almost trivial,” of the above corollary holds.

By a *Fox coloring* or an n -*coloring* of a knotted surface, we mean an assignment of the regions of the 2-dimensional regions of the knotted surface diagram by elements of the dihedral quandle $R_n = \{0, 1, 2, \dots, n - 1\}$ such that if a and c are the colors on two contiguous under-sheets that are separated by an over-sheet colored b , then $c = 2b - a$; this last equation is interpreted modulo n . See [10] or [4] for further details.

Theorem 2.13. *If any (± 1) -twist spin of a θ -curve Γ is unknotted and $\Gamma \setminus \{e\}$ has a non-trivial Fox coloring for each e , then the θ -curve is either unknotted or almost trivial.*

Proof. Suppose that a θ -curve Γ is knotted and is not almost trivial. Then there exists an edge e such that $\Gamma \setminus \{e\}$ is knotted. Since $\Gamma \setminus \{e\}$ is n -colorable for some positive integer n , the (± 1) -twist spin of Γ with respect to e is colored by the associated quandle $\mathbb{Z}/n \times \mathbb{Z}_2$ in a manner similar to Figure 11. This is a contradiction. \square

We can also construct a knotted θ -curve such that every (± 1) -twist spin of the knotted θ -curve is knotted. Since every constituent knot of the knotted θ -curve in Figure 12 is the connected sum of a positive trefoil knot and itself (a granny knot) which is 3-colorable, the (± 1) -twist spin is knotted.

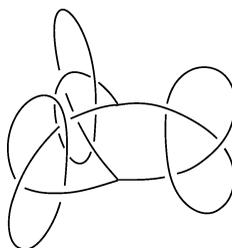


FIGURE 12. A knotted θ -curve for which every (± 1) -twist spin is knotted

ACKNOWLEDGEMENTS

The authors would like to thank Seiichi Kamada for some helpful advice. Yongju Bae introduced the authors of this paper. The paper grew out of the master thesis of the second author at the University of South Alabama.

REFERENCES

- [1] Emil Artin, *Zur Isotopie zweidimensionaler Flächen im R_4* (German), Abh. Math. Sem. Univ. Hamburg **4** (1925), no. 1, 174–177, DOI 10.1007/BF02950724. MR3069446
- [2] L. H. Kauffman and V. O. Manturov, *New Ideas in Low Dimensional Topology*, Series on Knots and Everything, vol. 56, World Scientific, Singapore, 2015.
- [3] J. S. Carter and A. Ishii, *A knotted 2-dimensional foam with non-trivial cocycle invariant*, Preprint (2012), <http://arxiv.org/abs/1206.4750>
- [4] Scott Carter, Seiichi Kamada, and Masahico Saito, *Surfaces in 4-space*, Encyclopaedia of Mathematical Sciences, vol. 142, Springer-Verlag, Berlin, 2004. Low-Dimensional Topology, III. MR2060067 (2005e:57065)
- [5] J. Scott Carter and Masahico Saito, *Knotted surfaces and their diagrams*, Mathematical Surveys and Monographs, vol. 55, American Mathematical Society, Providence, RI, 1998. MR1487374 (98m:57027)
- [6] Atsushi Ishii, Masahide Iwakiri, Yeonhee Jang, and Kanako Oshiro, *A G -family of quandles and handlebody-knots*, Illinois J. Math. **57** (2013), no. 3, 817–838. MR3275740
- [7] Seiichi Kamada, *Braid and knot theory in dimension four*, Mathematical Surveys and Monographs, vol. 95, American Mathematical Society, Providence, RI, 2002. MR1900979 (2003d:57050)
- [8] Akio Kawauchi, *A survey of knot theory*, Birkhäuser Verlag, Basel, 1996. Translated and revised from the 1990 Japanese original by the author. MR1417494 (97k:57011)
- [9] Yoshihiko Marumoto and Yasutaka Nakanishi, *A note on the Zeeman theorem*, Kobe J. Math. **8** (1991), no. 1, 67–71. MR1134705 (92h:57039)
- [10] Shin Satoh and Akiko Shima, *The 2-twist-spun trefoil has the triple point number four*, Trans. Amer. Math. Soc. **356** (2004), no. 3, 1007–1024 (electronic), DOI 10.1090/S0002-9947-03-03181-7. MR1984465 (2004k:57032)
- [11] E. C. Zeeman, *Twisting spun knots*, Trans. Amer. Math. Soc. **115** (1965), 471–495. MR0195085 (33 #3290)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTH ALABAMA, MOBILE, ALABAMA 36688
E-mail address: carter@southalabama.edu

DEPARTMENT OF MATHEMATICS, THE GEORGE WASHINGTON UNIVERSITY, WASHINGTON, DC 20052
E-mail address: syyang@gwu.edu