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OPERATOR LIPSCHITZ ESTIMATES IN THE UNITARY SETTING

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Abstract. We develop a Lipschitz estimate for unitary operators. More specifically, we show that for each $p \in (1, \infty)$ there exists a constant d_p such that $\|f(U) - f(V)\|_p \leq d_p \|U - V\|_p$ for all Lipschitz functions $f : \mathbb{T} \to \mathbb{C}$ and unitary operators U and V.

1. Preliminaries

We consider a problem in perturbation theory arising from a 1964 conjecture of M. G. Krein [1]. Although the conjecture did not hold [2], an entire class of new problems was brought to light. In fact, it was only recently that D. Potapov and F. Sukochev found a solution in the self-adjoint setting, as follows (notation will be explained below).

Theorem 1 (Potapov–Sukochev [3, Theorem 1])**.** *For each $p \in (1, \infty)$, there is a constant c_p such that*

$$\|f(A) - f(B)\|_p \leq c_p \|f\|_{\mathrm{Lip}} \|A - B\|_p \tag{1}$$

for all Lipschitz functions $f : \mathbb{R} \to \mathbb{C}$ and all self-adjoint operators A and B.

The intention of this paper is to translate the result above from the self-adjoint to the unitary setting. We now state our main result.

Theorem 2. *For each $p \in (1, \infty)$, there is a constant d_p such that*

$$\|f(U) - f(V)\|_p \leq d_p \|f\|_{\mathrm{Lip}} \|U - V\|_p \tag{2}$$

for all Lipschitz functions $f : \mathbb{T} \to \mathbb{C}$ and all unitary operators U and V. Further, $d_p \leq 32(c_p + 9)$.

Our method is straightforward and improves the constant d_p significantly over the existing estimates [4, Corollary 6.1].

We now explain the notation. Recall that a function $f : \mathbb{R} \to \mathbb{C}$ is said to be Lipschitz continuous if there is a constant C such that

$$|f(x) - f(y)| \leq C |x - y| \quad \forall x, y \in \mathbb{R};$$

it is well known that this is equivalent to the (distributional) derivative f' being in $L^\infty(\mathbb{R})$ and $\|f'\|_\infty \leq C$. The Lipschitz "norm" $\|f\|_{\mathrm{Lip}}$ of f is the smallest possible value of C in either of these inequalities. Evidently the Lipschitz "norm" of a

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constant function is 0; however, we may add a constant to the function f in the theorems above without changing either side of (1) or (2).

Throughout, we deal with a fixed Hilbert space $\mathcal{H}$ and linear operators thereon. For $p \in [1, \infty)$, the Schatten p-norm of the operator T on $\mathcal{H}$ is given by

$$\|T\|_p = \Big(\sum_{n\in\mathbb{N}} s_n(T)^p\Big)^{1/p},$$

where the numbers $s_n(T)$ are the eigenvalues of $(T^*T)^{1/2}$.

Of course, we also deal with functions $f : \mathbb{T} \to \mathbb{C}$, where $\mathbb{T}$ denotes the unit circle in the complex plane. For such a function f, we write f_r for the corresponding 2π-periodic function on $\mathbb{R}$, namely,

$$f_r(\theta) = f(\exp(i\theta)) \quad \forall \theta \in \mathbb{R},$$

and we define $\|f\|_{\mathrm{Lip}}$ to be $\|f_r\|_{\mathrm{Lip}}$.

By Fourier analysis, we may write any (sufficiently smooth) function $f : \mathbb{T} \to \mathbb{C}$ in the form

$$f(z) = \sum_{n\in\mathbb{Z}} \hat{f}(n)\, z^n \quad \forall z \in \mathbb{T},$$

where

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{-i\theta})\, e^{-in\theta}\, d\theta.$$

We write $\exp(i[a, b])$ for the arc $\{e^{i\theta} : a \le \theta \le b\}$.

2. Proof of the main result

We begin with a simpler version of the result.

Lemma 3. *Suppose that $f : \mathbb{T} \to \mathbb{C}$ is such that $\sum_{n\in\mathbb{Z}} \left|n\hat{f}(n)\right| < \infty$. Then*

$$\|f(U) - f(V)\|_q \le \sum_{n\in\mathbb{Z}} \left|n\hat{f}(n)\right| \|U - V\|_q$$

for all unitary operators U and V and all $q \in [1, \infty]$.

Proof. Clearly, if $n > 0$, then $U^n - V^n = \sum_{k=0}^{n-1} U^k(U - V)V^{n-1-k}$, and so

$$\|U^n - V^n\|_q \le \sum_{k=0}^{n-1} \left\|U^k\right\| \|U - V\|_q \left\|V^{n-1-k}\right\| = n\,\|U - V\|_q\,;$$

the case where $n < 0$ is similar, whence

$$\|f(U) - f(V)\|_q \le \sum_{n\in\mathbb{Z}} \left|\hat{f}(n)\right| \|U^n - V^n\|_q \le \sum_{n\in\mathbb{Z}} \left|n\hat{f}(n)\right| \|U - V\|_q\,,$$

as required. □

Lemma 4. *Suppose that $f : \mathbb{T} \to \mathbb{C}$ is a Lipschitz function and that*

$$f(z) = \sum_{n\in\mathbb{Z}} \hat{f}(n)\, z^n \quad \forall z \in \mathbb{T},$$

where $\hat{f}(0) = 0$. Then $\|f\|_\infty \le 2\,\|f\|_{\mathrm{Lip}}$.

Proof. Clearly,

$$f'(z) = \sum_{n\in\mathbb{Z}\setminus\{0\}} n\hat{f}(n)\, z^n \quad \forall z \in \mathbb{T},$$

and so

$$\begin{aligned}\|f\|_\infty &\le \sum_{n\in\mathbb{Z}\setminus\{0\}} \left|\hat{f}(n)\right| \\ &\le \left(\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n^2}\right)^{1/2} \left(\sum_{n\in\mathbb{Z}\setminus\{0\}} \left|n\hat{f}(n)\right|^2\right)^{1/2} \\ &\le 2\,\|f'\|_2\,,\end{aligned}$$

as claimed. □

Proof of Theorem 2. First, we use a partition of unity argument to reduce to the case where $\operatorname{supp}(f) \subseteq \exp(i[\frac{\pi}{6}, \frac{5\pi}{6}])$, at the cost of increasing the constant d_p by the factor 16.

We take the function $\phi : \mathbb{T} \to [0,1]$ such that ϕ_r is linear on both $[\frac{\pi}{6}, \frac{\pi}{3}]$ and $[\frac{2\pi}{3}, \frac{5\pi}{6}]$, and

$$\phi(e^{i\theta}) = \begin{cases} 1 & \text{if } \theta \in [\frac{\pi}{3}, \frac{2\pi}{3}], \\ 0 & \text{if } \theta \in [\frac{5\pi}{6}, \frac{13\pi}{6}]; \end{cases}$$

then $\|\phi'\|_\infty = 6/\pi$ and $\phi_1 + \phi_2 + \phi_3 + \phi_4 = 1$, where $\phi_k(e^{i\theta}) = \phi(i^k e^{i\theta})$ for all $\theta \in \mathbb{R}$. Without loss of generality, we suppose that f has mean 0, and so Lemma 4 implies that

$$\|f\phi_k\|_{\mathrm{Lip}} \le \|f'\phi_k\|_\infty + \|f(\phi_k)'\|_\infty \le 4\,\|f\|_{\mathrm{Lip}}\,.$$

Since $f = f\phi_1 + f\phi_2 + f\phi_3 + f\phi_4$, it suffices to show Theorem 2 for each $f\phi_k$. By a simple argument using rotations, it suffices to treat the case where $k = 0$.

Suppose that $\operatorname{supp}(f) \subseteq \exp(i[\frac{\pi}{6}, \frac{5\pi}{6}])$. We define the symmetrized function $\tilde{f} : \mathbb{T} \to \mathbb{C}$ by

$$\tilde{f}(e^{i\theta}) = f(e^{i\theta}) + f(e^{-i\theta}) \quad \forall \theta \in \mathbb{R}.$$

The corresponding function $\tilde{f}_r$ on $\mathbb{R}$ is given by $\tilde{f}_r(\theta) = f_r(\theta) + f_r(-\theta)$ for all $\theta \in \mathbb{R}$. The functions $f_r(\cdot)$ and $f_r(-\cdot)$ have disjoint supports, so $\|\tilde{f}\|_{\mathrm{Lip}} = \|f\|_{\mathrm{Lip}}$. We now show that

$$\left\|\tilde{f}(U) - \tilde{f}(V)\right\|_p \le 2\,c_p\,\|f\|_{\mathrm{Lip}}\,\|U - V\|_p \tag{3}$$

for all unitary operators U and V. Define $g : \mathbb{R} \to \mathbb{C}$ by

$$g(\lambda) = \begin{cases} \tilde{f}_r(\cos^{-1}(\lambda)) & \text{if } \lambda \in [-1,1], \\ 0 & \text{otherwise.} \end{cases}$$

If $\lambda \in [\frac{\sqrt{3}}{2}, 1]$, then $\cos^{-1}(\lambda) \in [0, \frac{\pi}{6}]$, and so $g(\lambda) = 0$. Similarly, $g(\lambda) = 0$ if $\lambda \in [-1, -\frac{\sqrt{3}}{2}]$. Now $g'(\lambda) = 0$ unless $\lambda \in [-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}]$, and for almost all these λ,

$$|g'(\lambda)| = \left|\tilde{f}_r'(\cos^{-1}(\lambda))\frac{1}{(1-\lambda^2)^{1/2}}\right| \le 2\,\|f\|_{\mathrm{Lip}}\,;$$

that is, $\|g\|_{\mathrm{Lip}} \le 2\,\|f\|_{\mathrm{Lip}}$. Further, by definition, for all $\theta \in \mathbb{R}$,

$$g(\tfrac{1}{2}(e^{i\theta} + e^{-i\theta})) = g(\cos(\theta)) = \tilde{f}_r(\theta) = \tilde{f}(e^{i\theta}).$$

For any unitary operator W, the operator $\frac{1}{2}(W+W^*)$ is self-adjoint, and by the spectral theorem,

$$g(\tfrac{1}{2}(W+W^*)) = \tilde{f}(W).$$

Theorem 1 therefore implies that

$$\begin{aligned}\left\|\tilde{f}(U)-\tilde{f}(V)\right\|_p &\le \left\|g(\tfrac{1}{2}(U+U^*))-g(\tfrac{1}{2}(V+V^*))\right\|_p \\ &\le c_p\,\|g\|_{\mathrm{Lip}}\left\|\tfrac{1}{2}(U+U^*)-\tfrac{1}{2}(V+V^*)\right\|_p \\ &\le 2\,c_p\left\|\tilde{f}\right\|_{\mathrm{Lip}}\|U-V\|_p\,.\end{aligned}$$

We continue to suppose that $\mathrm{supp}(f)\subseteq \exp(i[\frac{\pi}{6},\frac{5\pi}{6}])$. By Lemma 4, $\|f\|_\infty \le 2\,\|f\|_{\mathrm{Lip}}$. To conclude, we show that

$$\|f(U)-f(V)\|_p \le (2c_p+18)\,\|f\|_{\mathrm{Lip}}\,\|U-V\|_p\,.$$

Define the function $\psi:\mathbb{T}\to[0,1]$ by requiring that ψ_r is linear on both $[-\frac{\pi}{18},\frac{\pi}{18}]$ and $[\frac{17\pi}{18},\frac{19\pi}{18}]$, and

$$\psi(e^{i\theta}) = \begin{cases} 1 & \text{if } \theta\in[\frac{\pi}{18},\frac{17\pi}{18}]\,, \\ 0 & \text{if } \theta\in[\frac{19\pi}{18},\frac{35\pi}{18}]. \end{cases}$$

Further, let $\chi:\mathbb{T}\to\mathbb{R}$ be the step function such that $\chi(e^{i\theta})=9$ if $\theta\in[-\frac{\pi}{9},\frac{\pi}{9}]$ and $\chi(e^{i\theta})=0$ otherwise, and define ϕ to be the convolution $\chi*\psi$; that is,

$$\phi(e^{i\theta}) = \frac{9}{2\pi}\int_{-\pi/9}^{\pi/9}\psi(e^{i(\theta-\eta)})\,d\eta \quad \forall\theta\in\mathbb{R}.$$

We see easily that $\phi(e^{i\theta})=1$ if $\theta\in[\frac{\pi}{6},\frac{5\pi}{6}]$ and $\phi(e^{i\theta})=0$ if $\theta\in[\frac{7\pi}{6},\frac{11\pi}{6}]$; further, $\hat{\phi}=\hat{\psi}\,\hat{\chi}$, and a routine computation shows that

$$\begin{aligned}&\sum_{n\in\mathbb{Z}}\left|n\hat{\phi}(n)\right| \\ &\quad= \sum_{n\in\mathbb{Z}\setminus\{0\}}|n|\left|\frac{9}{n\pi}\sin\Big(\frac{n\pi}{9}\Big)\right|\left|\frac{9e^{-in\pi/2}}{n^2\pi^2}\left(\cos\Big(\frac{17n\pi}{18}\Big)-\cos\Big(\frac{19n\pi}{18}\Big)\right)\right| \\ &\quad\le \sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{162}{n^2\pi^3} = \frac{54}{\pi} < 18.\end{aligned}$$

Then $f=\phi\tilde{f}$. Further, $\phi(U)$ is well defined and $\|\phi(U)\|_\infty=1$ by the spectral theorem. Thus, by (3) and Lemma 3,

$$\begin{aligned}\|f(U)-f(V)\|_p &= \left\|\phi(U)\tilde{f}(U)-\phi(V)\tilde{f}(V)\right\|_p \\ &\le \left\|\phi(U)(\tilde{f}(U)-\tilde{f}(V))\right\|_p + \left\|(\phi(U)-\phi(V))\tilde{f}(V)\right\|_p \\ &\le \left\|\phi\right\|_\infty\left\|\tilde{f}(U)-\tilde{f}(V)\right\|_p + \left\|\phi(U)-\phi(V)\right\|_p\left\|\tilde{f}\right\|_\infty \\ &\le 2\,c_p\left\|f\right\|_{\mathrm{Lip}}\left\|U-V\right\|_p + 18\left\|U-V\right\|_p\left\|f\right\|_{\mathrm{Lip}},\end{aligned}$$

as desired. This step introduces a factor of $2(c_p+9)$ into d_p. □

Our final remark concerns the dependence of the constant d_p on p. It was shown in [5] that $c_p = O(p^2/(p-1))$. It is clear that the behaviour of the constant d_p is similar.

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