

ESTIMATES OF BANDS FOR LAPLACIANS ON PERIODIC EQUILATERAL METRIC GRAPHS

EVGENY KOROTYAEV AND NATALIA SABUROVA

(Communicated by Joachim Krieger)

ABSTRACT. We consider Laplacians on periodic equilateral metric graphs. The spectrum of the Laplacian consists of an absolutely continuous part (which is a union of an infinite number of non-degenerate spectral bands) plus an infinite number of flat bands, i.e., eigenvalues of infinite multiplicity. We estimate the Lebesgue measure of the bands on a finite interval in terms of geometric parameters of the graph. The proof is based on spectral properties of discrete Laplacians.

1. INTRODUCTION AND MAIN RESULTS

We consider metric Laplacians Δ_M on \mathbb{Z}^d -periodic equilateral metric graphs (each edge has unit length). Such operators arise naturally as simplified models in mathematics, physics, chemistry, and engineering when one considers propagation of waves of various nature through a quasi-one-dimensional system that looks like a thin neighborhood of a graph.

It is well known that the spectrum of the Laplacian Δ_M consists of an absolutely continuous part plus an infinite number of flat bands (i.e., eigenvalues with infinite multiplicity). These and other properties of Δ_M are discussed, e.g., in [BK13], [P12] and references therein. The absolutely continuous spectrum consists of an infinite number of spectral bands separated by gaps. There is a known problem: to estimate the spectrum and gaps of the Laplacian on periodic metric graphs. Note that in the case of the Schrödinger operators $-\Delta + Q$ with a periodic potential Q in \mathbb{R}^d there are two-sided estimates of potentials in terms of gap lengths only at $d = 1$ in [K98], [K03]. We do not know other estimates. For the case of periodic graphs we know only three papers about estimates of spectrum and gaps:

(1) Lledó and Post [LP08] estimated the positions of spectral bands of Laplacians both on metric and discrete graphs in terms of eigenvalues of the operator on finite graphs (the so-called eigenvalue bracketing).

(2) Korotyaev and Saburova [KS15] described a localization of spectral bands and estimated the Lebesgue measure of the spectrum of Schrödinger operators on discrete graphs in terms of eigenvalues of Dirichlet and Neumann operators on a fundamental domain of the periodic graph.

Received by the editors February 22, 2015 and, in revised form, April 23, 2015.

2010 *Mathematics Subject Classification.* Primary 47A10.

Key words and phrases. Spectral bands, flat bands, Laplace operator, periodic equilateral metric graph.

This research was supported by the RSF grant no. 15-11-30007.

(3) Korotyaev and Saburova [KS14] considered a Schrödinger operator on a discrete periodic graph and estimated the Lebesgue measure of its spectrum in terms of geometric parameters of the graph only.

Our main goal is to estimate the spectral bands and gaps for the Laplacian on the metric graph in terms of geometric parameters of the graph. Due to Cattaneo [C97], in order to study the spectrum of the Laplacian Δ_M on the equilateral metric graph it is enough to consider only the spectral interval $[0, \pi^2]$.

1.1. Metric Laplacians. Let $\Gamma = (V, \mathcal{E})$ be a connected infinite graph, possibly having loops and multiple edges, where V is the set of its vertices and \mathcal{E} is the set of its unoriented edges. An edge connecting vertices u and v from V will be denoted as the unordered pair $(u, v)_e \in \mathcal{E}$ and is said to be *incident* to the vertices. Vertices $u, v \in V$ will be called *adjacent* and denoted by $u \sim v$, if $(u, v)_e \in \mathcal{E}$. We define the degree \varkappa_v of the vertex $v \in V$ as the number of all its incident edges from \mathcal{E} (here a loop is counted twice).

Below we consider locally finite \mathbb{Z}^d -periodic metric equilateral graphs Γ , $d \geq 2$, i.e., graphs satisfying the following conditions:

- 1) Γ is equipped with an action of the free abelian group \mathbb{Z}^d ;
- 2) the degree of each vertex is finite;
- 3) the quotient graph $\Gamma_* = \Gamma/\mathbb{Z}^d$ is compact;
- 4) all edges of the graph are assigned unit length.

Remark. 1) We do not assume the graph to be embedded into a Euclidean space. But in main applications such a natural embedding exists (e.g., in modeling waves in thin branching “graph-like” structures: narrow waveguides, quantum wires, photonic crystal, blood vessels, lungs; see [BK13], [P12]). In this case a simple geometric model is a graph Γ embedded into \mathbb{R}^d in such a way that it is invariant with respect to the shifts by integer vectors $m \in \mathbb{Z}^d$, which produce an action of \mathbb{Z}^d .

2) We also call the quotient graph $\Gamma_* = \Gamma/\mathbb{Z}^d$ the *fundamental graph* of the periodic graph Γ . If Γ is embedded into the space \mathbb{R}^d the fundamental graph Γ_* is a graph on the surface $\mathbb{R}^d/\mathbb{Z}^d$. The fundamental graph $\Gamma_* = (V_*, \mathcal{E}_*)$ has the vertex set $V_* = V/\mathbb{Z}^d$ and the set $\mathcal{E}_* = \mathcal{E}/\mathbb{Z}^d$ of unoriented edges.

Each edge \mathbf{e} of Γ will be identified with the segment $[0, 1]$. This identification introduces a local coordinate $t \in [0, 1]$ along each edge. Thus, we give an orientation on the edge. Note that the spectrum of Laplacians on metric graphs does not depend on the orientation of graph edges. For each function y on Γ we define a function $y_{\mathbf{e}} = y|_{\mathbf{e}}$, $\mathbf{e} \in \mathcal{E}$. We identify each function $y_{\mathbf{e}}$ on \mathbf{e} with a function on $[0, 1]$ by using the local coordinate $t \in [0, 1]$. Let $L^2(\Gamma)$ be the Hilbert space of all functions $y = (y_{\mathbf{e}})_{\mathbf{e} \in \mathcal{E}}$, where each $y_{\mathbf{e}} \in L^2(0, 1)$, equipped with the norm

$$\|y\|_{L^2(\Gamma)}^2 = \sum_{\mathbf{e} \in \mathcal{E}} \|y_{\mathbf{e}}\|_{L^2(0,1)}^2 < \infty.$$

We define the metric Laplacian Δ_M on $y = (y_{\mathbf{e}})_{\mathbf{e} \in \mathcal{E}} \in L^2(\Gamma)$ by

$$(\Delta_M y)_{\mathbf{e}} = -y_{\mathbf{e}}'', \quad \text{where } (y_{\mathbf{e}}'')_{\mathbf{e} \in \mathcal{E}} \in L^2(\Gamma)$$

and y satisfies the so-called Kirchhoff conditions:

$$(1.1) \quad y \text{ is continuous on } \Gamma, \quad \sum_{\mathbf{e}=(v,u) \in \mathcal{E}} \delta_{\mathbf{e}}(v) y'_{\mathbf{e}}(v) = 0, \quad \forall v \in V,$$

$$\delta_{\mathbf{e}}(v) = \begin{cases} 1, & \text{if } v \text{ is the terminal vertex of the edge } \mathbf{e}, \text{ i.e., } t = 1 \text{ at } v, \\ -1, & \text{if } v \text{ is the initial vertex of the edge } \mathbf{e}, \text{ i.e., } t = 0 \text{ at } v. \end{cases}$$

1.2. Discrete Laplacians. Let $\ell^2(V)$ be the Hilbert space of all square summable functions $f : V \rightarrow \mathbb{C}$, equipped with the norm

$$\|f\|_{\ell^2(V)}^2 = \sum_{v \in V} |f(v)|^2 < \infty.$$

We define the self-adjoint normalized Laplacian (i.e., the Laplace operator) Δ on $f \in \ell^2(V)$ by

$$(1.2) \quad (\Delta f)(v) = -\frac{1}{\sqrt{\varkappa_v}} \sum_{(v,u) \in \mathcal{E}} \frac{1}{\sqrt{\varkappa_u}} f(u), \quad v \in V,$$

where \varkappa_v is the degree of the vertex $v \in V$ and all loops in the sum (1.2) are counted twice. In the literature the operator $-\Delta$ is usually called *the transition operator* and the normalized Laplacian is defined by $I + \Delta$, where I is the identity operator. But since Δ and $I + \Delta$ differ only by a shift we will call Δ the normalized Laplacian.

We recall the basic facts about the Laplacian Δ (see [Ch97], [HS04], [MW89]), which hold true for both finite and periodic graphs:

(i) *the point -1 belongs to the spectrum $\sigma(\Delta)$, and $\sigma(\Delta)$ is contained in $[-1, 1]$, i.e.,*

$$(1.3) \quad -1 \in \sigma(\Delta) \subset [-1, 1].$$

(ii) *The points ± 1 are never eigenvalues of Δ on a periodic graph.*

Let $\nu = \#V_*$, where $\#A$ is the number of elements of the set A . We fix any ν vertices of the periodic graph Γ which are not \mathbb{Z}^d -equivalent to each other, and denote this vertex set by V_0 . We will call V_0 a *fundamental vertex set* of Γ . Denote by \mathcal{B} the set of all edges of Γ connecting the vertices from V_0 with the vertices from $V \setminus V_0$. These edges will be called *bridges*.

The discrete Laplacian Δ on $\ell^2(V)$ has the standard decomposition into a constant fiber direct integral by

$$(1.4) \quad \ell^2(V) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d}^{\oplus} \ell^2(V_*) d\vartheta, \quad U\Delta U^{-1} = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d}^{\oplus} \Delta(\vartheta) d\vartheta,$$

$\mathbb{T}^d = \mathbb{R}^d / (2\pi\mathbb{Z})^d$, for some unitary operator U . Here $\ell^2(V_*) = \mathbb{C}^\nu$ is the fiber space and $\Delta(\vartheta)$ is the Floquet fiber $\nu \times \nu$ matrix. The precise form of the matrix $\Delta(\vartheta) = \{\Delta_{jk}(\vartheta)\}_{j,k=1}^\nu$ was determined in [BKS13].

Each Floquet matrix $\Delta(\vartheta)$, $\vartheta \in \mathbb{T}^d$, has ν real eigenvalues $\lambda_n(\vartheta)$, $n \in \mathbb{N}_\nu = \{1, \dots, \nu\}$, which are labeled in increasing order (counting multiplicities) by

$$(1.5) \quad \lambda_1(\vartheta) \leq \dots \leq \lambda_\nu(\vartheta).$$

Each $\lambda_n(\cdot)$, $n \in \mathbb{N}_\nu$, is a continuous function on the torus \mathbb{T}^d and creates the spectral band $\sigma_n(\Delta)$ given by

$$(1.6) \quad \sigma_n = \sigma_n(\Delta) = [\lambda_n^-, \lambda_n^+] = \lambda_n(\mathbb{T}^d).$$

Some of the spectral bands may overlap. Note that if $\lambda_n(\cdot) = C_n = \text{const}$ on some set $\mathcal{B} \subset \mathbb{T}^d$ of positive Lebesgue measure, then the operator Δ on Γ has the

eigenvalue C_n with infinite multiplicity. We call C_n a *flat band*. The spectrum of the Laplace operator Δ on the periodic graph Γ has the form

$$(1.7) \quad \sigma(\Delta) = \bigcup_{n=1}^{\nu} \sigma_n(\Delta) = \sigma_{ac}(\Delta) \cup \sigma_{fb}(\Delta).$$

Here $\sigma_{ac}(\Delta)$ is the absolutely continuous spectrum, which is a union of non-degenerate intervals from (1.6), and $\sigma_{fb}(\Delta) = \{\mu_1, \dots, \mu_r\}$, $r < \nu$, is the set of all flat bands (eigenvalues of infinite multiplicity). An open interval between two neighboring non-degenerate bands is called a *gap*. Let λ^+ be the upper point of the absolutely continuous spectrum of Δ . If $\lambda^+ < 1$, then it is convenient for us to also call an open interval $(\lambda^+, 1)$ a gap of the operator Δ .

1.3. Main results. Instead of the Laplacian $\Delta_M \geq 0$ it is convenient for us to define the momentum operator $\sqrt{\Delta_M} \geq 0$. Due to the relationship between the spectrum of the metric Laplacian Δ_M and the spectrum of the discrete Laplacian Δ (see subsection 2.1) the spectrum of the operator $\sqrt{\Delta_M} \geq 0$ on Γ has the form

$$(1.8) \quad \sigma(\sqrt{\Delta_M}) = \sigma_{ac}(\sqrt{\Delta_M}) \cup \sigma_{fb}(\sqrt{\Delta_M}).$$

Both the sets $\sigma_{ac}(\sqrt{\Delta_M})$ and $\sigma_{fb}(\sqrt{\Delta_M})$ are 2π -periodic on the half-line $(0, \infty)$ and are symmetric on the interval $(0, 2\pi)$ with respect to the point π . Thus, in order to study Δ_M it is sufficient to study its restriction Ω on the spectral interval $[0, \pi]$ given by

$$(1.9) \quad \Omega = \sqrt{\Delta_M} \chi_{[0, \pi]}(\sqrt{\Delta_M}),$$

where $\chi_A(\cdot)$ is the characteristic function of the set A . Due to Theorem 2.1 (see subsection 2.1) the spectrum of the operator Ω on a periodic metric graph Γ has the form

$$(1.10) \quad \begin{aligned} \sigma(\Omega) &= \bigcup_{n=1}^{\nu} \sigma_n(\Omega) = \sigma_{ac}(\Omega) \cup \sigma_{fb}(\Omega), \\ \sigma_n(\Omega) &= [z_n^-, z_n^+], \quad -\cos(z_n^\pm) = \lambda_n^\pm, \quad n \in \mathbb{N}_\nu. \end{aligned}$$

Here $\sigma_{ac}(\Omega)$ is a union of non-degenerate spectral bands $\sigma_n(\Omega)$ with $z_n^- < z_n^+$ and $\sigma_{fb}(\Omega)$ is the flat band spectrum (for more details see subsection 2.1).

Now we formulate our main results. Let

$$(1.11) \quad \beta = \sum_{v \in V_0} \frac{\beta_v}{\varkappa_v},$$

where β_v is the bridge degree (the number of bridges incident to $v \in V_0$) and \varkappa_v is the degree of $v \in V_0$.

Theorem 1.1. *i) All spectral bands $\sigma_n(\Omega)$ and $\sigma_n(\Delta)$, $n \in \mathbb{N}_\nu$, of the momentum operator Ω and the discrete Laplacian Δ , respectively, satisfy*

$$(1.12) \quad |\sigma_n(\Delta)| \leq |\sigma_n(\Omega)| \leq \frac{\pi}{\sqrt{2}} |\sigma_n(\Delta)|^{\frac{1}{2}},$$

where $|A|$ denotes the Lebesgue measure of the set A .

ii) The Lebesgue measure $|\sigma(\Omega)|$ and $|\sigma(\Delta)|$ of the spectrum of Ω and Δ , respectively, satisfies

$$(1.13) \quad |\sigma(\Delta)| \leq |\sigma(\Omega)| \leq \frac{\pi}{\sqrt{2}} |\sigma(\Delta)|^{\frac{1}{2}} \leq \pi\sqrt{\beta},$$

where β is defined by (1.11). Moreover, if there exist spectral gaps $\gamma_1(\Omega), \dots, \gamma_s(\Omega)$, in the spectrum $\sigma(\Omega)$, then the following estimate holds true:

$$(1.14) \quad \sum_{n=1}^s |\gamma_n(\Omega)| \geq \pi(1 - \sqrt{\beta}).$$

iii) Let $\beta < 1$. Then the spectrum $\sigma(\Delta_M)$ has infinitely many gaps.

Remark. 1) The estimate $|\sigma(\Omega)| \leq \pi\sqrt{\beta}$ is not trivial iff $\beta < 1$. The condition $\beta < 1$ holds true when the number of bridges at each vertex $v \in V_0$ is sufficiently small compared to the degree of the vertex. The number of bridges depends on the choice of the fundamental vertex set V_0 . For example, we can choose the vertex set V_0 by two different ways:

- All vertices of V_0 are not adjacent to each other. Then each edge incident to a vertex $v \in V_0$ is a bridge and the number $\beta = \nu \geq 1$. In this case the estimate (1.13) is trivial.
- The vertex-induced subgraph $\Gamma_0 = (V_0, \mathcal{E}_0)$ (that is the vertex set V_0 together with any edges whose endpoints are both in this set) is connected. Then each edge of the subgraph Γ_0 is not a bridge.

In order to get the best estimate in (1.13) we have to choose the vertex set V_0 such that the number of bridges is minimal. For the most popular periodic graphs (the d -dimensional lattice, the hexagonal lattice, the Kagome lattice, the face-centered cubic lattice, the body-centered cubic lattice, etc.) the best choice of the fundamental vertex set V_0 is standard.

2) The Bethe-Sommerfeld conjecture states that each Schrödinger operator $-\Delta + Q$ with a periodic potential Q in \mathbb{R}^d , $d \geq 2$, has only finitely many gaps in the spectrum. This conjecture was proved by Skriganov [S85]. On an equilateral metric graph the spectrum of the Laplacian Δ_M has no gaps iff $\sigma(\Delta) = [-1, 1]$. If $\sigma(\Delta) \neq [-1, 1]$, then in the spectrum of the Laplacian Δ_M on a metric graph there exist **infinitely many gaps** $\gamma_1, \gamma_2, \dots$ and $|\gamma_n| \rightarrow \infty$ as $n \rightarrow \infty$.

Let $\langle \cdot, \cdot \rangle$ denote the standard inner product in \mathbb{R}^d .

Definition of loop graphs 1. i) A periodic graph Γ is called a loop graph if each bridge \mathbf{e} has the form $\mathbf{e} = (v, v + \tau(\mathbf{e}))$ for some $v \in V_0$ and $\tau(\mathbf{e}) \in \mathbb{Z}^d$.

ii) A loop graph Γ is called a precise loop graph if $\cos\langle \tau(\mathbf{e}), \vartheta_0 \rangle = -1$ for all bridges $\mathbf{e} \in \mathcal{B}$ and some $\vartheta_0 \in \mathbb{T}^d$. This point ϑ_0 is called a precise point of the loop graph Γ .

Remark. 1) If the number $\langle \vartheta_0, \tau(\mathbf{e}) \rangle / \pi$ is odd for all bridges $\mathbf{e} \in \mathcal{B}$ and some $\vartheta_0 \in \mathbb{T}^d$, then ϑ_0 is a precise point of the loop graph Γ .

2) The class of all precise loop graphs is quite large. The simplest example of precise loop graphs is the d -dimensional lattice. More complicated examples are discussed in Proposition 2.3 in [KS14].

3) There exists a loop graph which is not a precise loop graph. The simplest example of such a graph is the triangular lattice (see Proposition 2.3 in [KS14]).

We now describe all bands for precise loop periodic graphs.

Theorem 1.2. *i) Let Γ be a loop graph. Then the spectral bands $\sigma_n(\Omega) = [z_n^-, z_n^+]$ of the operator Ω satisfy*

$$(1.15) \quad -\cos(z_n^-) = \lambda_n(0) = \lambda_n^-, \quad \forall n \in \mathbb{N}_\nu.$$

ii) Let, in addition, Γ be precise with a precise point $\vartheta_0 \in \mathbb{T}^d$. Then

$$(1.16) \quad -\cos(z_n^+) = \lambda_n(\vartheta_0) = \lambda_n^+, \quad \forall n \in \mathbb{N}_\nu,$$

$$(1.17) \quad 2\beta = \sum_{n=1}^{\nu} |\sigma_n(\Delta)| \leq \sum_{n=1}^{\nu} |\sigma_n(\Omega)|,$$

where β is defined by (1.11).

We present the plan of our paper. In Section 2 we estimate the Lebesgue measure of the spectrum of the operator $\sqrt{\Delta_M}$ on the finite interval $[0, \pi]$ in terms of geometric parameters of the graph and discuss some spectral properties of metric Laplacians on loop graphs and bipartite periodic graphs. In the appendix we collect spectral properties of the discrete Laplacian from [KS14], needed to prove our main results.

2. PROOF OF THE MAIN THEOREMS

2.1. Relation between the spectra of metric and discrete Laplacians. The correspondence between the spectrum of the Laplacian Δ_M on the equilateral metric graph and the spectrum of the Laplacian Δ on the corresponding discrete graph was obtained in [B85], [C97], [BGP08]. For the sake of completeness and the reader's convenience we recall this correspondence.

Consider the eigenvalues problem with Dirichlet boundary conditions

$$(2.1) \quad -y'' = Ey, \quad y(0) = y(1) = 0.$$

It is known that the spectrum of this problem is given by $\sigma_D = \{(\pi n)^2 : n \in \mathbb{N}\}$. Here $(\pi n)^2$ is the so-called Dirichlet eigenvalue of the problem (2.1).

We formulate the results [C97], [BGP08] in the form convenient for us. This theorem gives a basis for describing the spectrum of the operator Δ_M in terms of Δ , and conversely.

Theorem 2.1. *i) The spectrum of the operator $\sqrt{\Delta_M} \geq 0$ on the periodic metric graph Γ has the form*

$$(2.2) \quad \sigma(\sqrt{\Delta_M}) = \sigma_{ac}(\sqrt{\Delta_M}) \cup \sigma_{fb}(\sqrt{\Delta_M}),$$

$$(2.3) \quad \sigma_{ac}(\sqrt{\Delta_M}) = \{z \in \mathbb{R}_+ : -\cos z \in \sigma_{ac}(\Delta)\},$$

$$(2.4) \quad \sigma_{fb}(\sqrt{\Delta_M}) = \{z \in \mathbb{R}_+ : -\cos z \in \sigma_{fb}(\Delta)\} \cup \{\pi n : n \in \mathbb{N}\}.$$

ii) Each flat band $2\pi n$, $n \in \mathbb{N}$, is embedded in the absolutely continuous spectrum $\sigma_{ac}(\sqrt{\Delta_M})$.

iii) Both the sets $\sigma_{ac}(\sqrt{\Delta_M})$ and $\sigma_{fb}(\sqrt{\Delta_M})$ are 2π -periodic on the half-line $(0, \infty)$ and are symmetric on the interval $(0, 2\pi)$ with respect to the point π .

iv) The spectrum of the operator Ω on a periodic metric graph Γ has the form

$$(2.5) \quad \begin{aligned} \sigma(\Omega) &= \bigcup_{n=1}^{\nu} \sigma_n(\Omega) = \sigma_{ac}(\Omega) \cup \sigma_{fb}(\Omega), \\ \sigma_n(\Omega) &= [z_n^-, z_n^+], \quad -\cos(z_n^\pm) = \lambda_n^\pm, \quad \forall n \in \mathbb{N}_\nu. \end{aligned}$$

Here $\sigma_{ac}(\Delta)$ is a union of non-degenerate spectral bands $\sigma_n(\Omega)$ with $z_n^- < z_n^+$. Moreover, the first spectral band $\sigma_1(\Omega) = [0, z_1^+]$ is non-degenerate. The flat band spectrum has the form

$$(2.6) \quad \sigma_{fb}(\Omega) = \{z_1, \dots, z_r, \pi\}, \quad -\cos(z_k) = \mu_k \neq 1, \quad k \in \mathbb{N}_r.$$

v) $\sigma(\Omega) = [0, \pi]$ iff $\sigma(\Delta) = [-1, 1]$.

vi) The spectrum of the operator Ω has exactly k gaps iff $\sigma(\Delta)$ has exactly k gaps.

Remark. 1) Cattaneo considered the Laplacian Δ_M on connected locally finite graphs (including periodic). She proved that if the graph has at least one even cycle (i.e., a closed path passing an even number of edges), then all points $(2\pi n)^2$, $n \in \mathbb{N}$, are eigenvalues of the Laplacian Δ_M . Since there is always an even cycle on \mathbb{Z}^d -periodic graphs ($d \geq 2$), $\sigma_D \subset \sigma_{fb}(\Delta_M)$ (see (2.4)).

2) For non-periodic graphs some points of the Dirichlet spectrum σ_D may not be in the spectrum of the Laplacian Δ_M (for more details see, e.g., [C97], [LP08]).

3) The relation between the spectra of Δ and $\sqrt{\Delta_M}$ is shown in Figure 1.

4) The flat bands πn , $n \in \mathbb{N}$, of the operator $\sqrt{\Delta_M}$ will be called *Dirichlet flat bands*.

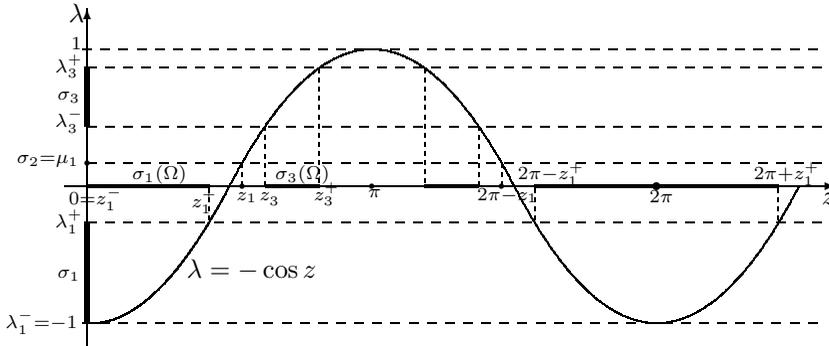


FIGURE 1. Relation between the spectra of Δ and $\sqrt{\Delta_M}$.

5) The number of flat bands of the operator Ω is $r + 1$. Flat bands z_1, \dots, z_r correspond to r flat bands of the discrete Laplacian and the flat band π is a Dirichlet flat band.

6) Let z^+ be the upper point of the absolutely continuous spectrum of the operator Ω . If $z^+ < \pi$, then there is a gap $(z^+, 2\pi - z^+)$ in the spectrum of $\sqrt{\Delta_M}$. For convenience the interval (z^+, π) is also called a gap of the operator Ω .

2.2. Proof of the main results.

Proof of Theorem 1.1. i) Consider the spectral band $\sigma_n(\Delta) = [\lambda_n^-, \lambda_n^+]$ of the discrete Laplacian Δ for some $n \in \mathbb{N}_\nu$. Due to Theorem 2.1.iv the corresponding spectral band $\sigma_n(\Omega)$ of the momentum operator Ω has the form

$$(2.7) \quad \sigma_n(\Omega) = [z_n^-, z_n^+], \quad \text{where} \quad \cos z_n^\pm = -\lambda_n^\pm.$$

Applying Proposition 3.1.i to the spectral band $\sigma_n(\Delta)$ and its preimage $\sigma_n(\Omega)$ under the function $\phi(z) = -\cos z$, $z \in [0, \pi]$, we obtain (1.12).

ii) The spectrum of the discrete Laplacian, defined by (1.7), is a union of the spectral bands $\sigma_1(\Delta), \dots, \sigma_\nu(\Delta)$, some of which may overlap. We rewrite the spectrum as a union of non-overlapping segments $\mathfrak{S}_1, \dots, \mathfrak{S}_{\nu_*}$, where $\nu_* \leq \nu$, i.e.,

$$(2.8) \quad \sigma(\Delta) = \bigcup_{n=1}^{\nu_*} \mathfrak{S}_n, \quad \mathfrak{S}_m \cap \mathfrak{S}_n = \emptyset, \quad m \neq n.$$

Then, due to Theorem 2.1.iv, the spectrum of the operator Ω has the form

$$(2.9) \quad \sigma(\Omega) = \bigcup_{n=1}^{\nu_*} \phi^{-1}(\mathfrak{S}_n), \quad \text{where } \phi^{-1}(\mathfrak{S}_m) \cap \phi^{-1}(\mathfrak{S}_n) = \emptyset, \quad m \neq n.$$

The identities (2.8), (2.9) give

$$(2.10) \quad |\sigma(\Delta)| = \sum_{n=1}^{\nu_*} |\mathfrak{S}_n|, \quad |\sigma(\Omega)| = \sum_{n=1}^{\nu_*} |\phi^{-1}(\mathfrak{S}_n)|.$$

Applying Proposition 3.1.i to each segment \mathfrak{S}_n , $n \in \mathbb{N}_{\nu_*}$, we obtain

$$(2.11) \quad |\mathfrak{S}_n| \leq |\phi^{-1}(\mathfrak{S}_n)|, \quad \forall n \in \mathbb{N}_{\nu_*}.$$

Summing these inequalities and using (2.10), we obtain

$$(2.12) \quad |\sigma(\Delta)| = \sum_{n=1}^{\nu_*} |\mathfrak{S}_n| \leq \sum_{n=1}^{\nu_*} |\phi^{-1}(\mathfrak{S}_n)| = |\sigma(\Omega)|,$$

which gives the first inequality in (1.13).

We show the second inequality in (1.13). Define the subset $\mathfrak{S}_\bullet \subset [-1, 1]$ by

$$(2.13) \quad \mathfrak{S}_\bullet = [-1, -\lambda_\bullet] \cup [\lambda_\bullet, 1], \quad \lambda_\bullet = 1 - \frac{|\sigma(\Delta)|}{2}, \quad 0 \leq \lambda_\bullet < 1.$$

Note that the Lebesgue measure $|\mathfrak{S}_\bullet| = |\sigma(\Delta)|$. The preimage of the subset \mathfrak{S}_\bullet under the function $\phi(z) = -\cos z$, $z \in [0, \pi]$, is given by

$$(2.14) \quad \phi^{-1}(\mathfrak{S}_\bullet) = [0, z_\bullet] \cup [\pi - z_\bullet, \pi], \quad \lambda_\bullet = \cos z_\bullet, \quad 0 < z_\bullet \leq \frac{\pi}{2}.$$

Combining Theorem 2.1.iv, Proposition 3.1 and the identity (2.13), we obtain

$$|\sigma(\Omega)| = |\phi^{-1}(\sigma(\Delta))| \leq |\phi^{-1}(\mathfrak{S}_\bullet)| \leq \frac{\pi}{\sqrt{2}} |\sigma(\Delta)|^{\frac{1}{2}}.$$

Thus, the second inequality in (1.13) has been proved. The last inequality in (1.13) follows from Theorem 3.2.i.

Now we will prove (1.14). Since $\sigma(\Omega) \subset [0, \pi]$, we have

$$(2.15) \quad \sum_{n=1}^s |\gamma_n(\Omega)| = \pi - |\sigma(\Omega)| \geq \pi(1 - \sqrt{\beta}).$$

Here we have used the estimate (1.13).

iii) Let $\beta < 1$. Then, due to (1.13), we have $|\sigma(\Omega)| < \pi$, which yields that in the spectrum of Ω there exists a gap. Hence, due to the periodicity of the spectrum of $\sqrt{\Delta_M}$, the spectrum of Δ_M has infinitely many gaps.

Now we give another proof of item iii). If $\beta < 1$, then Theorem 3.2.i gives that $|\sigma(\Delta)| < 2$, i.e., $\sigma(\Delta) \neq [-1, 1]$, which, by Theorem 2.1.iii,v, yields that the spectrum of Δ_M has infinitely many gaps. \square

Proof of Theorem 1.2. i) Let Γ be a loop graph. Then, due to Theorem 3.3.i, the spectral bands $\sigma_n(\Delta) = [\lambda_n^-, \lambda_n^+]$ of the Laplacian Δ satisfy $\lambda_n^- = \lambda_n(0), \forall n \in \mathbb{N}_\nu$. This and (2.5) give that the spectral bands $\sigma_n(\Omega) = [z_n^-, z_n^+]$ of the operator Ω satisfy (1.15).

ii) Let Γ be a precise loop graph with a precise point $\vartheta_0 \in \mathbb{T}^d$. Theorem 2.1.iv and the formula (3.17) give (1.16).

Combining (3.17), (3.18) and (1.16) with the simple estimate $\cos z_n^- - \cos z_n^+ \leq z_n^+ - z_n^-$, we obtain

$$2\beta = \sum_{n=1}^{\nu} |\sigma_n(\Delta)| = \sum_{n=1}^{\nu} (\lambda_n(\vartheta_0) - \lambda_n(0)) = \sum_{n=1}^{\nu} (\cos z_n^- - \cos z_n^+) \leq \sum_{n=1}^{\nu} (z_n^+ - z_n^-) = \sum_{n=1}^{\nu} |\sigma_n(\Omega)|.$$

Thus, (1.17) has been proved. □

A graph is called *bipartite* if its vertex set is divided into two disjoint sets (called *parts* of the graph) such that each edge connects vertices from distinct parts. It is known that for a connected locally finite graph Γ the following assertions are equivalent:

- 1) Γ is bipartite;
 - 2) the point $1 \in \sigma(\Delta)$;
 - 3) the spectrum $\sigma(\Delta)$ is symmetric with respect to the point 0
- (see, e.g., Proposition 2.3. in [LP08]).

Now we formulate some simple spectral properties of the Laplacian Δ_M and the momentum operator Ω on bipartite graphs.

Theorem 2.2. *The following statements hold true.*

i) If Γ is bipartite, then the flat bands $\pi^2(2n + 1)^2, n \in \mathbb{N}$, of the Laplacian Δ_M are embedded in $\sigma_{ac}(\Delta_M)$. If Γ is non-bipartite, then each flat band $\pi^2(n + 1)^2, n \in \mathbb{N}$, lies in a gap.

ii) On the interval $(0, \pi)$ the spectrum of Ω is symmetric with respect to the point $\pi/2$ iff Γ is bipartite iff the point $\pi \in \sigma_{ac}(\Omega)$.

iii) Let a fundamental graph Γ_* be bipartite. If ν is odd, then $z = \pi/2$ is a flat band of Ω .

iv) Let Γ be a loop bipartite graph (Γ_* is not bipartite, since there is a loop on Γ_*). Then each spectral band of the operator Ω on Γ has the form $\sigma_n(\Omega) = [z_n^-, z_n^+], n \in \mathbb{N}_\nu$, where

$$(2.16) \quad -\cos z_n^- = \lambda_n(0), \quad \cos z_n^+ = \lambda_{\nu-n+1}(0).$$

Proof. i) If Γ is bipartite, then $1 \in \sigma(\Delta)$. Since 1 is never a flat band, the point $1 \in \sigma_{ac}(\Delta)$. Then (2.3) gives that $\pi^2(n + 1)^2 \in \sigma_{ac}(\Delta_M)$ for all $n \in \mathbb{N}$. Similarly, if Γ is non-bipartite, then $1 \notin \sigma_{ac}(\Delta)$ and $\pi^2(n + 1)^2 \notin \sigma_{ac}(\Delta_M)$ for all $n \in \mathbb{N}$. Thus, each flat band $\pi^2(n + 1)^2, n \in \mathbb{N}$, lies in a gap.

ii) Firstly, we show that the graph Γ is bipartite iff the point $\pi \in \sigma_{ac}(\Omega)$. Indeed, the graph Γ is bipartite iff the point $1 \in \sigma_{ac}(\Delta)$. Due to (2.3), the condition $1 \in \sigma_{ac}(\Delta)$ is equivalent to $\pi \in \sigma_{ac}(\Omega)$.

Secondly, we prove that on the interval $(0, \pi)$ the spectrum of Ω is symmetric with respect to the point $\pi/2$ iff Γ is bipartite. Let Γ be bipartite. Then the spectrum of the discrete Laplacian Δ is symmetric with respect to 0. Assume that

$z \in (0, \pi)$ and $z \in \sigma(\Omega)$. Then Theorem 2.1 gives that $-\cos z \in \sigma(\Delta)$. Due to the symmetry of the spectrum $\sigma(\Delta)$, the point $-\cos(\pi - z) = \cos z \in \sigma(\Delta)$. This and (2.3) yield that $\pi - z \in \sigma(\Omega)$.

Conversely, let on the interval $(0, \pi)$ the spectrum of Ω be symmetric with respect to the point $\pi/2$. Since $-1 \in \sigma_{ac}(\Delta)$, due to Theorem 2.1, $0 \in \sigma_{ac}(\Omega)$. Then the symmetry of $\sigma(\Omega)$ gives that $\pi \in \sigma_{ac}(\Omega)$ and Γ is bipartite.

iii) Let a fundamental graph Γ_* be bipartite and let ν be odd. By Theorem 3.2.ii, $\mu = 0$ is a flat band of the discrete Laplacian Δ on Γ . This and Theorem 2.1.iv yield that $z = \pi/2$ is a flat band of Ω .

iv) By Theorem 3.3.iii each spectral band of the discrete Laplacian on Γ has the form $\sigma_n(\Delta) = [\lambda_n^-, \lambda_n^+]$, $n \in \mathbb{N}_\nu$, where λ_n^\pm are the eigenvalues of the matrix $\mp \Delta(0)$, i.e., $\lambda_n^- = \lambda_n(0)$, $\lambda_n^+ = -\lambda_{\nu-n+1}(0)$. Then Theorem 2.1.iv gives that each spectral band of the operator Ω has the form $\sigma_n(\Omega) = [z_n^-, z_n^+]$, $n \in \mathbb{N}_\nu$, where $-\cos z_n^- = \lambda_n^- = \lambda_n(0)$, $-\cos z_n^+ = \lambda_n^+ = -\lambda_{\nu-n+1}(0)$. \square

Remark. Item iii) gives a simple sufficient condition for existence of the flat band.

3. APPENDIX

We formulate some simple facts needed to prove our main results.

Proposition 3.1. *Let $\phi(z) = -\cos z$, $z \in [0, \pi]$ and let $|A|$ denote the Lebesgue measure of the set A . Then the following statements hold true.*

i) *If $\mathfrak{S} = [\lambda^-, \lambda^+] \subset [-1, 1]$, then*

$$(3.1) \quad |\mathfrak{S}| \leq |\phi^{-1}(\mathfrak{S})| \leq \frac{\pi}{\sqrt{2}} |\mathfrak{S}|^{\frac{1}{2}}.$$

ii) *If $\mathfrak{S} \subset [-1, 1]$ is any subset of the segment $[-1, 1]$ and if $\mathfrak{S}_\bullet \subset [-1, 1]$ is the subset given by*

$$(3.2) \quad \mathfrak{S}_\bullet = [-1, -\lambda_\bullet] \cup [\lambda_\bullet, 1], \quad \lambda_\bullet = 1 - \frac{1}{2} |\mathfrak{S}|,$$

having the same Lebesgue measure $|\mathfrak{S}_\bullet| = |\mathfrak{S}|$, then

$$(3.3) \quad |\phi^{-1}(\mathfrak{S})| \leq |\phi^{-1}(\mathfrak{S}_\bullet)| \leq \frac{\pi}{\sqrt{2}} |\mathfrak{S}|^{\frac{1}{2}}.$$

Proof. i) The function ϕ is an increasing bijection of the segment $[0, \pi]$ onto the segment $[-1, 1]$. The preimage of the segment $\mathfrak{S} = [\lambda^-, \lambda^+] \subset [-1, 1]$ under the function ϕ has the form

$$(3.4) \quad \phi^{-1}(\mathfrak{S}) = [z^-, z^+], \quad \text{where } z^\pm = \phi^{-1}(\lambda^\pm).$$

We have the simple inequality

$$(3.5) \quad |\mathfrak{S}| = \lambda^+ - \lambda^- = -\cos z^+ + \cos z^- = \int_{z^-}^{z^+} \sin t \, dt \leq z^+ - z^- = |\phi^{-1}(\mathfrak{S})|.$$

Thus, the first estimate in (3.1) has been proved.

We show that

$$(3.6) \quad |\phi^{-1}(\mathfrak{S})| \leq |\phi^{-1}(\mathfrak{S}_\bullet)|,$$

where $\mathfrak{S}_\bullet \subset [-1, 1]$ is given by

$$(3.7) \quad \mathfrak{S}_\bullet = [-1, -\lambda_\bullet] \cup [\lambda_\bullet, 1], \quad \lambda_\bullet = 1 - \frac{1}{2} (\lambda^+ - \lambda^-) \in [0, 1].$$

Note that from the definitions of \mathfrak{S} and \mathfrak{S}_\bullet it follows that

$$(3.8) \quad |\mathfrak{S}_\bullet| = |\mathfrak{S}|.$$

The preimage of the segment \mathfrak{S}_\bullet under the function ϕ has the form

$$(3.9) \quad \phi^{-1}(\mathfrak{S}_\bullet) = [0, z_\bullet] \cup [\pi - z_\bullet, \pi], \quad \text{where } z_\bullet = \phi^{-1}(-\lambda_\bullet) \in [0, \frac{\pi}{2}].$$

Using (3.4) and (3.9) we can rewrite the inequality (3.6) in the equivalent form

$$(3.10) \quad z^+ - z^- \leq 2z_\bullet.$$

In order to prove (3.10) we use the definition of z_\bullet in (3.9), (3.7) and consider the difference

$$\begin{aligned} \cos \frac{z^+ - z^-}{2} - \cos z_\bullet &= \cos \frac{z^+ - z^-}{2} - 1 + \frac{1}{2}(\lambda^+ - \lambda^-) \\ &= \cos \frac{z^+ - z^-}{2} - 1 + \frac{1}{2}(-\cos z^+ + \cos z^-) \\ &= \cos \frac{z^+}{2} \cos \frac{z^-}{2} + \sin \frac{z^+}{2} \sin \frac{z^-}{2} - \cos^2 \frac{z^+}{2} - \sin^2 \frac{z^-}{2}. \end{aligned}$$

From this identity, using the simple inequalities

$$(3.11) \quad \cos \frac{z^-}{2} \geq \cos \frac{z^+}{2} \geq 0, \quad \sin \frac{z^+}{2} \geq \sin \frac{z^-}{2} \geq 0, \quad 0 \leq z^- \leq z^+ \leq \pi,$$

we obtain

$$(3.12) \quad \cos \frac{z^+ - z^-}{2} \geq \cos z_\bullet \quad \Rightarrow \quad z_\bullet \geq \frac{z^+ - z^-}{2},$$

since $\frac{z^+ - z^-}{2}, z_\bullet \in [0, \frac{\pi}{2}]$ and the function $\cos z$ decreases on this interval. Thus, (3.10) (and, consequently, (3.6)) has been proved. Using the estimate $\sin x \geq \frac{2\sqrt{2}}{\pi} x$, $x \in [0, \frac{\pi}{4}]$, we obtain that the Lebesgue measure of the set \mathfrak{S}_\bullet , defined by (3.7), satisfies

$$(3.13) \quad |\mathfrak{S}_\bullet| = 2(1 - \lambda_\bullet) = 2(1 - \cos z_\bullet) = 4 \sin^2 \frac{z_\bullet}{2} \geq \frac{8}{\pi^2} z_\bullet^2 = \frac{2}{\pi^2} |\phi^{-1}(\mathfrak{S}_\bullet)|^2,$$

since $\frac{z_\bullet}{2} \in [0, \frac{\pi}{4}]$.

Combining (3.6), (3.13), and (3.8), we obtain

$$(3.14) \quad |\phi^{-1}(\mathfrak{S})|^2 \leq |\phi^{-1}(\mathfrak{S}_\bullet)|^2 = 4z_\bullet^2 \leq \frac{\pi^2}{2} |\mathfrak{S}_\bullet| = \frac{\pi^2}{2} |\mathfrak{S}|,$$

which yields the second estimate in (3.1).

ii) The preimage $\phi^{-1}(\mathfrak{S}_\bullet)$ of the subset \mathfrak{S}_\bullet is given by (3.9). Since the derivative ϕ' of the function $\phi(z) = -\cos z$ increases from the point 0 to the point $\frac{\pi}{2}$ and $|\phi^{-1}([-1, -\lambda_\bullet])| = |\phi^{-1}([\lambda_\bullet, 1])|$, we obtain the first inequality in (3.3).

The Identity (3.8) and the estimate (3.13) imply the second inequality in (3.3). □

We collect properties of the discrete Laplacian on periodic graphs, which we need to prove our main results. These properties and their proof for the combinatorial discrete Laplacian can be found in [KS14]. The proof of these properties for the normalized Laplacian repeats the proof for the combinatorial one.

Theorem 3.2. *i) The Lebesgue measure $|\sigma(\Delta)|$ of the spectrum of the Laplace operator Δ satisfies*

$$(3.15) \quad |\sigma(\Delta)| \leq \sum_{n=1}^{\nu} |\sigma_n(\Delta)| \leq 2\beta,$$

where β is defined by (1.11).

ii) *Let a fundamental graph Γ_* be bipartite. If ν is odd, then $\mu = 0$ is a flat band of the discrete Laplacian Δ on Γ .*

Remark. The proof of the identity (3.15) is based on the precise representation of fiber Laplacians constructed in [KS14] and some ideas from [Ku15].

Theorem 3.3. *i) Let Γ be a loop graph. Then the spectral bands $\sigma_n(\Delta) = [\lambda_n^-, \lambda_n^+]$ of the Laplace operator Δ satisfy*

$$(3.16) \quad \lambda_n^- = \lambda_n(0), \quad \forall n \in \mathbb{N}_\nu.$$

ii) Let Γ be a precise loop graph with a precise point $\vartheta_0 \in \mathbb{T}^d$. Then

$$(3.17) \quad \sigma_n(\Delta) = [\lambda_n^-, \lambda_n^+] = [\lambda_n(0), \lambda_n(\vartheta_0)], \quad \forall n \in \mathbb{N}_\nu,$$

$$(3.18) \quad \sum_{n=1}^{\nu} |\sigma_n(\Delta)| = 2\beta,$$

where β is defined by (1.11).

iii) Let in addition Γ be bipartite (Γ_ is not bipartite, since there is a loop on Γ_*). Then each spectral band of the Laplacian on Γ has the form $\sigma_n(\Delta) = [\lambda_n^-, \lambda_n^+]$, $n \in \mathbb{N}_\nu$, where λ_n^\pm are the eigenvalues of the matrix $\mp\Delta(0)$.*

REFERENCES

- [BKS13] A. Badanin; E. Korotyaev, N. Saburova, Laplacians on \mathbb{Z}^2 -periodic discrete graphs, preprint 2013.
- [B85] Joachim von Below, *A characteristic equation associated to an eigenvalue problem on c^2 -networks*, Linear Algebra Appl. **71** (1985), 309–325, DOI 10.1016/0024-3795(85)90258-7. MR813056 (87i:94030)
- [BK13] Gregory Berkolaiko and Peter Kuchment, *Introduction to quantum graphs*, Mathematical Surveys and Monographs, vol. 186, American Mathematical Society, Providence, RI, 2013. MR3013208
- [BGP08] Jochen Brüning, Vladimir Gejler, and Konstantin Pankrashkin, *Spectra of self-adjoint extensions and applications to solvable Schrödinger operators*, Rev. Math. Phys. **20** (2008), no. 1, 1–70, DOI 10.1142/S0129055X08003249. MR2379246 (2008m:47029)
- [C97] Carla Cattaneo, *The spectrum of the continuous Laplacian on a graph*, Monatsh. Math. **124** (1997), no. 3, 215–235, DOI 10.1007/BF01298245. MR1476363 (98j:35127)
- [Ch97] Fan R. K. Chung, *Spectral graph theory*, CBMS Regional Conference Series in Mathematics, vol. 92, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1997. MR1421568 (97k:58183)
- [HS04] Yusuke Higuchi and Tomoyuki Shirai, *Some spectral and geometric properties for infinite graphs*, Discrete geometric analysis, Contemp. Math., vol. 347, Amer. Math. Soc., Providence, RI, 2004, pp. 29–56, DOI 10.1090/conm/347/06265. MR2077029 (2006f:47040)
- [K98] Evgeny Korotyaev, *Estimates of periodic potentials in terms of gap lengths*, Comm. Math. Phys. **197** (1998), no. 3, 521–526, DOI 10.1007/s002200050462. MR1652779 (99h:34125)
- [K03] Evgeny Korotyaev, *Characterization of the spectrum of Schrödinger operators with periodic distributions*, Int. Math. Res. Not. **37** (2003), 2019–2031, DOI 10.1155/S1073792803209107. MR1995145 (2004e:34134)
- [KS14] Evgeny Korotyaev and Natalia Saburova, *Schrödinger operators on periodic discrete graphs*, J. Math. Anal. Appl. **420** (2014), no. 1, 576–611, DOI 10.1016/j.jmaa.2014.05.088. MR3229841
- [KS15] Evgeny Korotyaev and Natalia Saburova, *Spectral band localization for Schrödinger operators on discrete periodic graphs*, Proc. Amer. Math. Soc. **143** (2015), no. 9, 3951–3967, DOI 10.1090/S0002-9939-2015-12586-5. MR3359585
- [Ku15] Anton A. Kutsenko, *On the measure of the spectrum of direct integrals*, Banach J. Math. Anal. **9** (2015), no. 2, 1–8, DOI 10.15352/bjma/09-2-1. MR3296101
- [LP08] Fernando Lledó and Olaf Post, *Eigenvalue bracketing for discrete and metric graphs*, J. Math. Anal. Appl. **348** (2008), no. 2, 806–833, DOI 10.1016/j.jmaa.2008.07.029. MR2446037 (2010a:47076)

- [MW89] Bojan Mohar and Wolfgang Woess, *A survey on spectra of infinite graphs*, Bull. London Math. Soc. **21** (1989), no. 3, 209–234, DOI 10.1112/blms/21.3.209. MR986363 (90d:05162)
- [P12] Olaf Post, *Spectral analysis on graph-like spaces*, Lecture Notes in Mathematics, vol. 2039, Springer, Heidelberg, 2012. MR2934267
- [S85] M. M. Skriganov, *The spectrum band structure of the three-dimensional Schrödinger operator with periodic potential*, Invent. Math. **80** (1985), no. 1, 107–121, DOI 10.1007/BF01388550. MR784531 (86i:35107)

MATHEMATICAL PHYSICS DEPARTMENT, FACULTY OF PHYSICS, ULIANOVSKAYA 2, ST. PETERSBURG STATE UNIVERSITY, ST. PETERSBURG, 198904, RUSSIA

E-mail address: korotyaev@gmail.com

DEPARTMENT OF MATHEMATICAL ANALYSIS, ALGEBRA AND GEOMETRY, INSTITUTE OF MATHEMATICS, INFORMATION AND SPACE TECHNOLOGIES, URITSKOGO ST. 68, NORTHERN (ARCTIC) FEDERAL UNIVERSITY, ARKHANGELSK, 163002, RUSSIA

E-mail address: n.saburova@gmail.com