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ON n-MAXIMAL SUBALGEBRAS OF LIE ALGEBRAS

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Abstract. A 2-maximal subalgebra of a Lie algebra L is a maximal subalgebra of a maximal subalgebra of L. Similarly we can define 3-maximal subalgebras, and so on. There are many interesting results concerning the question of what certain intrinsic properties of the maximal subalgebras of a Lie algebra L imply about the structure of L itself. Here we consider whether similar results can be obtained by imposing conditions on the n-maximal subalgebras of L, where $n > 1$.

1. Introduction

Throughout L will denote a finite-dimensional Lie algebra over a field F. There will be no restrictions on F unless specified. A chain of subalgebras $S_0 < S_1 < \ldots < S_n = L$ is a *maximal chain* if each S_i is a maximal subalgebra of S_{i+1}. The subalgebra S_0 in such a series is called an *n-maximal* subalgebra. Relationships between certain properties of maximal subalgebras of a Lie algebra L and the structure of L itself have been studied by a number of authors. For example: all maximal subalgebras are ideals of L if and only if L is nilpotent (see [2]); all maximal subalgebras of L are c-ideals of L if and only if L is solvable (see [15]); if L is solvable, then all maximal subalgebras have codimension one in L if and only if L is supersolvable (see [3]); L can be characterised when its maximal subalgebras satisfy certain lattice-theoretic conditions, such as modularity (see [17]). Our purpose here is to consider whether similar results can be obtained by imposing conditions on the n-maximal subalgebras of L, where $n > 1$.

Similar studies have proved fruitful in group theory (see, for example, [5], [7] and [9]). In Lie algebras, a special type of n-maximal subalgebra, in which each element of the chain has codimension one in the next, has been studied in [1]. They call such subalgebras *flag Lie algebras* and they give a classification of them in [1, Theorem 4.7]. The following result was also established by Stitzinger.

Theorem 1.1 (Stitzinger, [11, Theorem]). *Every 2-maximal subalgebra of L is an ideal of L if and only if one of the following holds:*

(i) *L is nilpotent and $\phi(L) = \phi(M)$ for all maximal subalgebras M of L;*
(ii) $\dim L = 2$*; or*
(iii) *L is simple and every proper subalgebra is one-dimensional.*

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In the above result $\phi(L)$ denotes the Frattini ideal of L; that is, the largest ideal contained in the intersection of the maximal subalgebras of L. Our first objective in the next section is to find a similar characterisation of Lie algebras in which all 2-maximal subalgebras are subideals, and then those in which they are nilpotent. In section three we consider when all 3-maximals are ideals, and when they are subideals. In the final section we look at the situation where every n-maximal subalgebra is a subideal.

2. 2-maximal subalgebras

Recall that the factor algebra A/B is called a *chief factor* of L if B is an ideal of L and A/B is a minimal ideal of L/B. First, the following observations will be useful.

Lemma 2.1. *Let A/B be a chief factor of L with $A \subseteq \phi(L)$. Then A/B is an irreducible $L/\phi(L)$-module.*

Proof. The nilradical, N, of L is the intersection of the centralizers of the factors in a chief series of L, by [4, Lemma 4.3]. Since $\phi(L) \subseteq N$ this implies that $[A, \phi(L)] \subseteq B$ and so the multiplication of L on A induces a module action of $L/\phi(L)$ on A/B. Hence A/B can be viewed as an irreducible $L/\phi(L)$-module. □

We will refer to a chief factor such as is described in Lemma 2.1 as being *below* $\phi(L)$. We call I a *subideal* of a Lie algebra L if there is a chain of subalgebras

$$I = I_0 < I_1 < \ldots < I_n = L,$$

where I_j is an ideal of I_{j+1} for each $0 \leq j \leq n-1$.

Lemma 2.2. *If every n-maximal subalgebra of L is a subideal of L, then every $(n-1)$-maximal subalgebra is nilpotent.*

Proof. Let J be an $(n-1)$-maximal subalgebra of L. Then every maximal subalgebra I of J is an n-maximal subalgebra of L and so is a subideal of L, and thus of J. It follows that I is an ideal of J, and hence that J is nilpotent, by [2]. □

Theorem 2.3. *Every 2-maximal subalgebra of L is a subideal of L if and only if one of the following holds:*

(i) *L is nilpotent;*
(ii) *$L = N + Fx$ where N is the nilradical, $N^2 = 0$ and $\operatorname{ad} x$ acts irreducibly on N; or*
(iii) *L is simple with every proper subalgebra one-dimensional.*

Proof. Let every 2-maximal subalgebra of L be a subideal of L. If L is simple, then (iii) holds with $\phi(L) = 0$. So suppose that N is a maximal ideal of L. Since N will be contained in a maximal subalgebra of L it will be nilpotent, by Lemma 2.2.

Suppose first that $N \not\subseteq \phi(L)$. Then there is a maximal subalgebra M of L such that $L = N + M$. Since M is nilpotent, L is solvable. Moreover, L is nilpotent or minimal non-nilpotent. Suppose that L is minimal non-nilpotent and $\phi(L) \neq 0$, so $L = N + Fx$ where N is the nilradical of L, $N^2 = \phi(L)$ and $\operatorname{ad} x$ acts irreducibly on N/N^2, by [14, Theorem 2.1]. But now $\phi(L) + Fx$ is a maximal subalgebra of L and any 2-maximal subalgebra of L containing Fx would have to be contained in a proper ideal of L, which would be nilpotent, by Lemma 2.2, and so contained in N. It follows that $\phi(L) = 0$. Hence either (i) or (ii) holds.

So suppose now that $N \subseteq \phi(L)$. Then $N = \phi(L)$ and $L/\phi(L)$ is simple with every proper subalgebra one-dimensional. Now $N + Fs$ is a maximal subalgebra of L for every $s \in S$, and, as in the preceding paragraph, any 2-maximal subalgebra containing Fs would be contained in N. It follows that $N = 0$.

Conversely, let L satisfy (i), (ii) or (iii). If L is nilpotent, then every subalgebra of L is a subideal of L. If (ii) holds, then the maximal subalgebras of L are N and Fx, and so the 2-maximal subalgebras are inside N and so are subideals of L. If (iii) holds, then the only 2-maximal subalgebra is the trivial subalgebra. □

Note that the class of algebras given by this theorem is strictly larger than that given by Theorem 1.1 as the following examples show.

Example 2.1. The three-dimensional Heisenberg algebra H_1 over any field F with basis e_1, e_2, e_3 and product $[e_1, e_2] = e_3$ (other products being zero) has $\phi(H_1) = Fe_3$, but every maximal subalgebra M of H_1 is abelian, and so $\phi(M) = 0$. Thus H_1 is an example of type (i) in the above theorem but does not satisfy Stitzinger's result.

Example 2.2. Let N be any abelian Lie algebra with basis $Fe_1 + \ldots + Fe_n$, and put $L = N + Fx$ with multiplication $[e_1, x] = e_2, \ldots, [e_{n-1}, x] = e_n, [e_n, x] = a_0e_1 + \ldots + a_{n-1}e_n$ where $a_0 \neq 0$ and $p(x) = x^n - a_{n-1}x^{n-1} - \ldots - a_1x - a_0$ is irreducible in $F[x]$. Then all Lie algebras of type (ii) in the above theorem are of this form, and these also do not occur in Stitzinger's result unless F is algebraically closed, in which case $\dim L = 2$. Over the rational field there is no bound on the dimension of such an algebra, as there are irreducible polynomials of arbitrary degree over $\mathbb{Q}$.

Over a perfect field F of characteristic zero or $p > 3$, for L to satisfy condition (iii) in Theorem 1.1, it must be three-dimensional and $\sqrt{F} \not\subseteq F$, by [16, Theorem 3.4].

Next we consider when all of the 2-maximal subalgebras are nilpotent. We consider the non-solvable and solvable cases separately, as for the former case we require restrictions on the field F.

Theorem 2.4. *Let L be a non-solvable Lie algebra over an algebraically closed field F of characteristic different from $2, 3$. Then every 2-maximal subalgebra of L is nilpotent if and only if $L/\phi(L) \cong sl(2)$ and $sl(2)$ acts nilpotently on $\phi(L)$. If F has characteristic zero or if L is restricted, then $\phi(L) = 0$.*

Proof. Suppose that every 2-maximal subalgebra of L is nilpotent, and let M be a maximal subalgebra of L. If M is not nilpotent, then there is an element $x \in M$ such that $\operatorname{ad} x|_M$ has a non-zero eigenvalue, λ say. But now $M = Fx + Fy$ since this is not nilpotent. Hence, every maximal subalgebra of L is nilpotent or two-dimensional; in particular, they are all solvable. If F has characteristic $p > 3$, it follows from [19, Proposition 2.1] that $L/\phi(L) \cong sl(2)$. Moreover, all maximal subalgebras of $sl(2)$ are two-dimensional, so $\phi(L) + Fx$ is nilpotent for every $x \in sl(2)$. The claim for characteristic zero is well known; that for the case when L is restricted is [19, Corollary 2.13].

The converse is easy. □

Theorem 2.5. *Let L be a solvable Lie algebra over a field F. Denote the image of a subalgebra S of L under the canonical homomorphism onto $L/\phi(L)$ by $\bar{S}$. Then all*

2-maximal subalgebras of L are nilpotent if and only if one of the following occurs:

(i) *L is nilpotent;*
(ii) *L is minimal non-nilpotent, and so is as described in [14];*
(iii) *$\bar{L} = \bar{A} \dot{+} F\bar{b}$, where $\bar{A}$ is the unique minimal abelian ideal of $\bar{L}$ and $\phi(L) + Fb$ is minimal non-nilpotent;*
(iv) *$\bar{L} = \bar{A} \dot{+} (F\bar{b_1} + F\bar{b_2})$, where $\bar{A}$ is a minimal abelian ideal of $\bar{L}$, $\bar{B} = F\bar{b_1} + F\bar{b_2}$ is a subalgebra of $\bar{L}$ and $L/\phi(L)$ acts nilpotently on $\phi(L)$; or*
(v) *$\bar{L} = (\bar{A}_1 \oplus \bar{A}_2) \dot{+} F\bar{b}$, where $\bar{A}_1$ and $\bar{A}_2$ are minimal abelian ideals of $\bar{L}$ and $L/\phi(L)$ acts nilpotently on $\phi(L)$.*

Proof. Suppose that all 2-maximal subalgebras of L are nilpotent. Then $\bar{L} = (\bar{A}_1 \oplus \ldots \oplus \bar{A}_n) \dot{+} \bar{B}$, where $\bar{A}_i$ is a minimal abelian ideal of $\bar{L}$ for each $i = 1, \ldots, n$, $\bar{A}_1 \oplus \ldots \oplus \bar{A}_n$ is the nilradical, $\bar{N}$, of $\bar{L}$ and $\bar{B}$ is a subalgebra of $\bar{L}$, by [13, Theorem 7.3]. If $n > 2$ we have $\bar{A}_i + \bar{B}$ is nilpotent for each $i = 1, \ldots, n$. But then $\bar{L}$, and hence L, is nilpotent, by [13, Theorem 6.1]. Suppose that $\dim \bar{B} > 2$ and let $\bar{C}$ be a minimal ideal of $\bar{B}$. If $\dim \bar{C} = 1$, then $\bar{N} + \bar{C}$ is a nilpotent ideal of $\bar{L}$, contradicting the fact that $\bar{N}$ is the nilradical of $\bar{L}$; if $\dim \bar{C} > 1$ we have that $\bar{N} + F\bar{c}$ is nilpotent for each $\bar{c} \in \bar{C}$ which again implies that $\bar{N} + \bar{C}$ is a nilpotent ideal of $\bar{L}$. Finally, if $n = 2$ and $\dim \bar{B} = 2$ a similar argument produces a contradiction.

So suppose next that $n = 1$ and $\dim \bar{B} = 1$. Then the maximal subalgebras of L are A and $\phi(L) + Fx$, where $x \notin N$. If $\phi(L) + Fb$ is nilpotent we have case (ii); if it is minimal non-nilpotent we have case (iii).

Next let $n = 1$ and $\dim \bar{B} = 2$. If $\bar{B} = F\bar{b_1} + F\bar{b_2}$, then $A + Fb_1$ and $A + Fb_2$ are maximal subalgebras of L, and so $\phi(L) + Fb_1$ and $\phi(L) + Fb_2$ are 2-maximal subalgebras. It follows that B acts nilpotently on $\phi(L)$ and we have case (iv).

Finally, suppose that $n = 2$ and $\dim \bar{B} = 1$. Maximal subalgebras are $A_1 \oplus A_2$, $A_1 + Fb$ and $A_2 + Fb$, and $\phi(L) + Fb$ is a 2-maximal subalgebra. It follows that Fb acts nilpotently on $\phi(L)$ and we have case (v).

The converse is straightforward. □

If S is a subalgebra of L, the *centralizer* of S in L is $C_L(S) = \{x \in L \colon [x, S] = 0\}$.

Corollary 2.6. *With the notation of Theorem 2.5, if L is solvable and F is algebraically closed, then all 2-maximal subalgebras of L are nilpotent if and only if one of the following occurs:*

(a) *L is nilpotent;*
(b) *$\dim L \leq 3$;*
(c) *F has characteristic p, $\bar{L} = \bigoplus_{i=0}^{p-1} F\bar{a}_i + F\bar{b_1} + F\bar{b_2}$, where $[\bar{a}_i, \bar{b_1}] = \bar{a_{i+1}}$, $[\bar{a}_i, \bar{b_2}] = (\alpha + i)\bar{a}_i$ for $i = 0, \ldots, p-1$ ($\alpha \in F$, suffices modulo p), $[\bar{b_1}, \bar{b_2}] = \bar{b_1}$ and $L/\phi(L)$ acts nilpotently on $\phi(L)$; or*
(d) *$\bar{L} = F\bar{a_1} + F\bar{a_2} + F\bar{b}$, where $[\bar{b}, \bar{a_1}] = \bar{a_1}$, $[\bar{b}, \bar{a_2}] = \alpha\bar{a_2}$ ($\alpha \in F$), $[\bar{a_1}, \bar{a_2}] = 0$ and $L/\phi(L)$ acts nilpotently on $\phi(L)$.*

Proof. We consider in turn each of the cases given in Theorem 2.5. Clearly case (i) gives (a), and if case (ii) holds, then $\dim L = 2$ (see [14]), which is included in (b). If case (iii) holds, then $\bar{A}$ and $\phi(L)$ are both one-dimensional, and so we have (b) again.

Next consider case (iv). Suppose first that $\bar{B}$ is abelian. Then $\dim \bar{A} = 1$, by [12, Lemma 5.6]. But now $\dim \bar{L}/C_{\bar{L}}(F\bar{a}) \leq 1$ so $\dim C_{\bar{L}}(F\bar{a}) \geq 2$, contradicting the fact that $C_{\bar{L}}(F\bar{a}) = F\bar{a}$. Thus $\bar{B}$ cannot be abelian.

If $\bar{B}$ is non-abelian, then $\bar{B} = F\bar{b_1} + F\bar{b_2}$ where $[\bar{b_1}, \bar{b_2}] = \bar{b_1}$. If F has characteristic zero, then $\dim \bar{A} = 1$, by Lie's Theorem. But now, as in the previous paragraph, $\dim C_{\bar{L}}(F\bar{a}) \geq 2$, yielding the same contradiction. Hence F has characteristic $p > 0$. Then this algebra has a unique p-map making it into a restricted Lie algebra: namely $\bar{b_1}^{[p]} = 0, \bar{b_2}^{[p]} = \bar{b_2}$ (see [12]); its irreducible modules are of dimension one or p, by [12, Example 1, page 244]. Once again we can rule out the possibility that $\dim \bar{A} = 1$. So suppose now that $\dim \bar{A} = p$. Let α be an eigenvalue for $\operatorname{ad} \bar{b_2}|_{\bar{A}}$, so $[\bar{a}, \bar{b_2}] = \alpha\, \bar{a}$ for some $\bar{a} \in \bar{A}$. Then $[\bar{a}(\operatorname{ad} \bar{b_1})^i, \bar{b_2}] = (\alpha + i)\bar{a}(\operatorname{ad} \bar{b_1})^i$ for every i, so putting $\bar{a_i} = \bar{a}(\operatorname{ad} \bar{b_1})^i$ we see that $F\bar{a_0} + \cdots + F\bar{a_{p-1}}$ is $\bar{B}$-stable and hence equal to $\bar{A}$. We then have the multiplication given in (c).

Finally, consider case (v). Then $\bar{A_1}$ and $\bar{A_2}$ are one-dimensional. Moreover, if L is not nilpotent, then $\bar{b}$ must act non-trivially on at least one of them, $\bar{A_1} = F\bar{a_1}$, say. This gives the multiplication described in (d). □

3. 3-MAXIMAL SUBALGEBRAS

We first consider Lie algebras all of whose 3-maximal subalgebras are ideals. We shall need the following lemma, which is an easy generalisation of [11, Lemma 2].

Lemma 3.1. *Suppose that every n-maximal subalgebra of L is an ideal of L. Then every $(n-1)$-maximal subalgebra of L is nilpotent and is either an ideal or is one-dimensional.*

Proof. Let K be an $(n-1)$-maximal subalgebra of L. The fact that K is nilpotent follows from Lemma 2.2. Suppose that $\dim K > 1$. Then K has at least two distinct maximal subalgebras J_1 and J_2, by [11, Lemma 1]. These are n-maximal subalgebras of L and so are ideals of L. Moreover, $K = J_1 + J_2$ and so is an ideal of L. □

Theorem 3.2. *Let L be a solvable Lie algebra over a field F. Then every 3-maximal subalgebra of L is an ideal of L if and only if one of the following holds:*

(i) *L is nilpotent and $\phi(K) = \phi(M)$ for every 2-maximal subalgebra K of L and every maximal subalgebra M of L containing it; or*
(ii) $\dim L \leq 3$.

Proof. Suppose that every 3-maximal subalgebra of L is an ideal of L. Then Lemma 3.1 shows that L is given by Theorem 2.5. We consider each of the cases in turn, and use the notation of that result. Suppose first that L is nilpotent and let J be a 3-maximal subalgebra of L, K be any 2-maximal subalgebra of L containing it, and M be any maximal subalgebra of L containing K. Then J is an ideal of L and M/J is two-dimensional. It follows that $M^2 \subseteq J$ and so $M^2 \subseteq \phi(K) = K^2 \subseteq M^2$. Hence $\phi(K) = M^2 = \phi(M)$.

Now suppose that $\bar{L} = \bar{A} \dot{+} \bar{B}$, where $\bar{A}$ is the unique minimal ideal of $\bar{L}$ and $\bar{B}$ is a subalgebra of $\bar{L}$ with $\dim \bar{B} \leq 2$. This covers cases (ii), (iii) and (iv) of Theorem 2.5. If $\dim \bar{A} > 2$, then there is a proper subalgebra $\bar{C}$ of $\bar{A}$ which is a 3-maximal subalgebra of $\bar{L}$, and so an ideal of $\bar{L}$, contradicting the minimality of $\bar{A}$. If $\dim \bar{B} = 2$, then $A + Fb$ is a maximal subalgebra of L for each $\bar{0} \neq \bar{b} \in \bar{B}$. It follows that $\phi(L) + Fb$ is a 2-maximal subalgebra of L. If this is an ideal of L, then $F\bar{b}$ is a minimal ideal of $\bar{L}$, contradicting the uniqueness of $\bar{A}$. It follows from Lemma 3.1 that it has dimension one, and so $\phi(L) = 0$. Similarly, $\dim \bar{A} = 2$ yields that $\phi(L) = 0$. Hence $\phi(L) \neq 0$ implies that $\dim L/\phi(L) \leq 2$ and thus that L is nilpotent. So suppose that $\phi(L) = 0$ and $\dim L = 4$. Then $A + Fb$ is a maximal subalgebra for every $b \in B$, and so Fa is a 3-maximal subalgebra, and hence an ideal, of L for every $a \in A$, contradicting the minimality of A. Thus $\dim L \leq 3$.

So, finally, suppose that case (v) of Theorem 2.5 holds. We have that $\phi(L) = 0$ as in the paragraph above. Also, if $\dim A_i > 1$ $(i = 1, 2)$ there is a proper subalgebra C of A_i which is a 3-maximal subalgebra, and hence an ideal, of L. It follows that $\dim A_i = 1$ for $i = 1, 2$ and $\dim L = 3$.

Conversely, suppose that (i) or (ii) hold. If (ii) holds, then every 3-maximal is 0 and thus an ideal of L, so suppose that (i) holds. Let J be a 3-maximal subalgebra of L. Then J is a maximal subalgebra of a 2-maximal subalgebra K of L and $M^2 = \phi(M) = \phi(K) \subseteq J$ for every maximal subalgebra M containing K. It follows that J is an ideal of M. But now $\dim L/J = 3$ and there are two maximal subalgebras M_1 and M_2 of L containing J with $L = M_1 + M_2$. Since J is an ideal of M_1 and M_2, it is an ideal of L. □

Example 3.1. Let H_3 be the seven-dimensional Heisenberg Lie algebra over F with basis $e_1, e_2, e_3, e_4, e_5, e_6, e_7$ and multiplication $[e_1, e_2] = [e_3, e_4] = [e_5, e_6] = e_7$. Then every 2-maximal subalgebra K of H_3 has dimension five. But there are no abelian subalgebras of H_3 of dimension greater than four (see [6, page 710]). Hence every such subalgebra K has $\phi(K) = Fe_7$, and every maximal subalgebra M containing K has $\phi(M) = Fe_7$. So this algebra is of type (i) in the above theorem. It is also easy to see that every algebra of type (ii) from Theorem 2.3 falls into this same class.

Theorem 3.3. *Let L be a non-solvable Lie algebra over a field F. Then every 3-maximal subalgebra of L is an ideal of L if and only if one of the following holds:*

(i) *L is simple, all 2-maximal subalgebras of L are at most one-dimensional and at least one of them has dimension one;*
(ii) *$L/Z(L)$ is a simple algebra, all of whose maximal subalgebras are one-dimensional, $Z(L) = \phi(L)$ and $\dim Z(L) = 0$ or 1;*
(iii) *$L = S \dot{+} Fx$ where S is a simple ideal of L and all maximal subalgebras of S are one-dimensional.*

Proof. Suppose that every 3-maximal subalgebra of L is an ideal of L. Clearly, if L is simple, then every 2-maximal subalgebra has dimension at most one, by Lemma 3.1, and so satisfies (i) or (ii). So let N be a maximal ideal of L. Suppose first that $\dim L/N = 1$, so $L = N \dot{+} Fx$, say. Clearly N has more than one maximal subalgebra, since otherwise it is one-dimensional and L is solvable. If N has a maximal subalgebra K_1 that is an ideal of L, then $N = K_1 + K_2$, where K_2 is another maximal subalgebra of L, and both K_1 and K_2 are nilpotent. But then N, and hence L, is solvable. It follows from Lemma 3.1 that every maximal subalgebra of N is one-dimensional. Let I be a non-trivial ideal of N. Then $\dim N/C_N(I) \leq 1$. But this implies that $\dim N = 2$ and L is solvable again. It follows that N is simple with all maximal subalgebras one-dimensional. Hence, L is as in case (iii).

So suppose now that L/N is simple. Then all 2-maximal subalgebras of L/N have dimension at most one. Suppose first that L/N has a one-dimensional 2-maximal subalgebra A/N. Then $\dim A = 1$, by Lemma 3.1, and so $N = 0$ and we have case (i) again. So suppose now that all maximal subalgebras of L/N are one-dimensional. Then N is nilpotent and if K has codimension one in N, K is an ideal of L. Moreover, $K + Fs$ is a 2-maximal subalgebra of L for every $s \notin N$. It follows from Lemma 3.1 that $K = 0$. Hence $\dim N = 1$. But now $\dim L/C_L(N) \leq 1$, which implies that $N = Z(L)$. If $Z(L) = \phi(L)$ we have case (ii). If $Z(L) \neq \phi(L)$, then we have a special case of (iii).

The converse is straightforward. □

Note that $sl(2)$ is an example of an algebra of type (i) above. For type (iii) we could take the direct sum of a three-dimensional non-split simple Lie algebra and a one-dimensional ideal. However, we know of no example of an algebra of type (ii) with $\dim Z(L) = 1$. They cannot exist over a perfect field of characteristic zero or $p > 3$, as then $L/Z(L)$ is three-dimensional simple, by [16, Theorem 3.4]. But then $L = \phi(L) \oplus L^2 = L^2$, by [18, Lemma 1.4], which is a contradiction. There are, of course, such algebras without the restriction on the maximal subalgebras: namely, $sl(n)$ over a field of characteristic p where $n \equiv -1 (mod\, p)$, $n > 2$.

Corollary 3.4. *Let L be a non-solvable Lie algebra over an algebraically closed field F of characteristic different from $2, 3$. Then every 3-maximal subalgebra of L is an ideal of L if and only if $L \cong sl(2)$.*

Proof. Suppose that every 3-maximal subalgebra of L is an ideal of L. Then every 2-maximal subalgebra of L is nilpotent, so $L/\phi(L) \cong sl(2)$, by Theorem 2.4. But $\phi(L) = 0$ by Theorem 3.3. The converse is clear. □

Next we give a characterisation of those Lie algebras in which every 3-maximal subalgebra is a subideal.

Theorem 3.5. *Let L be a solvable Lie algebra over a field F. Then every 3-maximal subalgebra of L is a subideal of L if and only if one of the following occurs:*

(i) *L is nilpotent;*
(ii) *$L = N \dot{+} Fb$ where N is the nilradical, $\dim N^2 = 1$, $ad\, b$ acts irreducibly on N/N^2, and $N^2 + Fb$ is abelian;*
(iii) *$\bar{L} = \bar{A} \dot{+} F\bar{b}$, where $\bar{A}$ is the unique minimal abelian ideal of $\bar{L}$, $\phi(L)^2 = 0$ and $\phi(L)$ is an irreducible Fb-module;*
(iv) *$L = A \dot{+} (Fb_1 + Fb_2)$, where A is a minimal abelian ideal of L, and $B = Fb_1 + Fb_2$ is a subalgebra of L; or*
(v) *$L = (A_1 \oplus A_2) \dot{+} Fb$, where A_1 and A_2 are minimal abelian ideals of L.*

Proof. Suppose that every 3-maximal subalgebra of L is a subideal of L. Then L is as described in Theorem 2.5. We consider each of the cases in turn. In case (i) every subalgebra of L is a subideal of L. Suppose that case (ii) holds, so $L = N \dot{+} Fb$ where N/N^2 is a faithful irreducible Fb-module and $N^2 + Fb$ is nilpotent. Let C be an ideal of $N^2 + Fb$ of codimension one in N^2. Then $C + Fb$ is a 2-maximal subalgebra of L. Suppose that $C \neq 0$. Then $b \in D$ where $D \subset C + Fb$ is a 3-maximal subalgebra of L. Since D is a nilpotent subideal of L, there is a $k \in \mathbb{N}$ such that $N(\operatorname{ad} D)^k = 0$. Since $b \in D$ and N/N^2 is faithful, this is impossible. Hence $D = 0$ and $\dim N^2 = 1$.

Suppose that (iii) holds. Then $\phi(L)/\phi(L)^2$ is a faithful irreducible Fb-module, and so $\phi(L)^2 + Fb$ is a 2-maximal subalgebra of L. If $\phi(L)^2 \neq 0$, then $b \in D$ where $D \subset \phi(L)^2 + Fb$ is a 3-maximal subalgebra of L. But this yields a contradiction as in the preceding paragraph.

Suppose next that (iv) holds. Then we can choose b_1, b_2 so that $[\bar{b_1}, \bar{b_2}] = \lambda \bar{b_2}$ where $\lambda = 0, 1$. Then $[\bar{A}, \bar{b_2}]$ is an ideal of $\bar{L}$ and so is equal to $\bar{A}$, since, otherwise,

$\bar{b_2} \in C_{\bar{L}}(\bar{A}) = \bar{A}$. Now $\phi(L)+Fb_2$ is a 2-maximal subalgebra of L. Thus, if $\phi(L) \neq 0$ we have that b_2 belongs to a 3-maximal subalgebra of L, giving a contradiction again.

Finally, suppose that (v) holds. Then $\phi(L) + Fb$ is a 2-maximal subalgebra of L and we conclude that $\phi(L) = 0$ as above.

Conversely, if any of these cases are satisfied then every 3-maximal subalgebra of L is inside the nilradical of L, and hence is a subideal of L. □

Algebras of each of the types given in the theorem above exist, as the following examples show.

Example 3.2. Let L be the four-dimensional Diamond Lie algebra over $\mathbb{R}$ with basis e_1, e_2, e_3, e_4 and non-zero products

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = -e_2, \quad [e_2, e_3] = e_4.$$

Then $N = \mathbb{R}e_2 + \mathbb{R}e_3 + \mathbb{R}e_4$ is the nilradical, $N^2 = \mathbb{R}e_4$, $ad\, e_1$ acts irreducibly on $\mathbb{R}e_2 + \mathbb{R}e_3$, and $N^2 + \mathbb{R}e_1 = \mathbb{R}e_4 + \mathbb{R}e_1$ is abelian, so L is of type (ii) above.

Example 3.3. Let L be the three-dimensional Lie algebra over $\mathbb{R}$ with basis e_1, e_2, e_3 and non-zero products

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = -e_2.$$

Then $A = \mathbb{R}e_2 + \mathbb{R}e_3$ is the unique minimal ideal of L and $\phi(L) = 0$, so L is of type (iii) above.

Example 3.4. Let L be the four-dimensional Lie algebra over $\mathbb{R}$ with basis e_1, e_2, e_3, e_4 and non-zero products

$$[e_1, e_3] = e_4, \quad [e_1, e_4] = -e_3.$$

Then $A = \mathbb{R}e_3 + \mathbb{R}e_4$ is a minimal abelian ideal of L and $B = \mathbb{R}e_1 + \mathbb{R}e_2$ is a subalgebra of L with $L = A \dot{+} B$, so L is of type (iv) above.

Example 3.5. Let L be the four-dimensional Lie algebra over $\mathbb{R}$ with basis e_1, e_2, e_3, e_4 and non-zero products

$$[e_1, e_2] = e_2, \quad [e_1, e_3] = e_4, \quad [e_1, e_4] = -e_3.$$

Then $A_1 = \mathbb{R}e_2$ and $A_2 = \mathbb{R}e_3 + \mathbb{R}e_4$ are minimal abelian ideals and $B = \mathbb{R}e_1 + \mathbb{R}e_2$ is a subalgebra of L with $L = (A_1 \oplus A_2) \dot{+} \mathbb{R}e_1$, so L is of type (v) above.

Proposition 3.6. *Let L be a non-solvable Lie algebra over an algebraically closed field F of characteristic different from $2, 3$. Then every 3-maximal subalgebra of L is a subideal of L if and only if $L/\phi(L) \cong sl(2)$.*

Proof. Suppose that every 3-maximal subalgebra of L is a subideal of L. Then every 2-maximal subalgebra of L is nilpotent, so $L/\phi(L) \cong sl(2)$, by Theorem 2.4. Conversely, if $L/\phi(L) \cong sl(2)$, then every 3-maximal subalgebra of L is contained in $\phi(L)$, which is nilpotent, and so they are all subideals of L. □

4. N-MAXIMAL SUBALGEBRAS

The following result was proved by Schenkman in [10] for fields of characteristic zero, and can be extended to cover a large number of cases in characteristic p by using a result of Maksimenko from [8].

Lemma 4.1. *Let I be a nilpotent subideal of a Lie algebra L over a field F. If F has characteristic zero, or has characteristic p and L has no subideal with nilpotency class greater than or equal to $p-1$, then $I \subseteq N$, where N is the nilradical of L.*

Proof. If F has characteristic zero this is [10, Lemma 4]. For the characteristic p case we follow Schenkman's proof. Let I be a nilpotent subideal of L and suppose that $I = I_0 < I_1 < \ldots < I_n = L$ is a chain of subalgebras of L with I_j an ideal of I_{j+1} for $j = 0, \ldots, n-1$. Let N_j be the nilradical of I_j and let $x_j \in I_j$. Then $I \subseteq N_1$, since I is a nilpotent ideal of I_1. Also $[I_j, x_{j+1}] \subseteq I_j$, and so $\operatorname{ad} x_{j+1}$ defines a derivation of I_j for each $j = 0, \ldots, n-1$. Moreover, N_j is a subideal of L and so has nilpotency class less than $p-1$. It follows from [8, Corollary 1] that $[N_j, x_{j+1}] \subseteq N_j$, and hence that N_j is an ideal of I_{j+1}. But then $N_j \subseteq N_{j+1}$, and $I \subseteq N_1 \subseteq N_2 \subseteq \ldots \subseteq N_n = N$, as claimed. □

We will refer to the characteristic p condition in the above result as F having characteristic *big enough*.

Lemma 4.2. *Let L be a Lie algebra over a field F. Consider the following two conditions:*

(i) *every n-maximal subalgebra of L is contained in N; and*
(ii) *every n-maximal subalgebra of L is a subideal of L.*

Then (i) implies (ii) and, if F has characteristic zero or big enough, (ii) implies (i).

Proof. (i) $\Rightarrow$ (ii): It is clear that any subideal of N is a subideal of L.

(ii) $\Rightarrow$ (i): Let I be an n-maximal subalgebra of L and suppose that it is a subideal of L. Then, under the extra hypothesis, it is a nilpotent subideal of L, by Lemma 2.2 and so is contained in N, by Lemma 4.1. □

Clearly, if L is solvable, then a necessary condition for Lemma 4.2 (i) to hold is that $\dim L/N \leq n$, since there is a chain of subalgebras of length n from N to L. However, this condition is not sufficient, in general, as is clear from previous results and the next. Recall that a Lie algebra L is called *supersolvable* if there is a chain of subalgebras

$$0 = L_0 \subset L_1 \subset \ldots \subset L_n = L$$

of L, where L_i is an i-dimensional ideal of L

Theorem 4.3. *Let L be a supersolvable Lie algebra over a field F of characteristic zero or big enough. Then every n-maximal subalgebra of L is a subideal of L if and only if either*

(i) *L is nilpotent; or*
(ii) $\dim L \leq n$.

Proof. Suppose that every n-maximal subalgebra of L is a subideal of L, but that L is not nilpotent, and let N be the nilradical of L. Let

$$0 = A_0 < A_1 < \ldots < A_k = N < \ldots < A_r = L$$

be a chief series for L through N. Then each chief factor is one-dimensional since L is supersolvable and so $r = \dim L$. Let $x \in A_r \setminus A_{r-1}$. Then

$$Fx < A_1 + Fx < \ldots < A_{r-1} + Fx = L$$

is a maximal chain of subalgebras of L, and Fx is an $(r-1)$-maximal subalgebra of L. If $r > n$ it follows that x belongs to an n-maximal subalgebra of L. Since $x \notin N$ this contradicts Lemma 4.2. $\square$

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