

## SEMIALGEBRAIC GRAPHS HAVING COUNTABLE LIST-CHROMATIC NUMBERS

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ABSTRACT. For  $n \geq 1$  and a countable, nonempty set  $D$  of positive reals, the  $D$ -distance graph  $\mathbf{X}_n(D)$  is the graph on Euclidean  $n$ -space  $\mathbb{R}^n$  in which two points form an edge exactly when the distance between them is in  $D$ . Each of these graphs is  $\sigma$ -algebraic. Komjáth characterized those  $\mathbf{X}_n(D)$  having a countable list-chromatic number, easily implying a different, but essentially equivalent, noncontainment characterization. It is proved here that this noncontainment characterization extends to all  $\sigma$ -algebraic graphs. We obtain, in addition, similar noncontainment characterizations for those  $\sigma$ -semialgebraic graphs and those semialgebraic graphs having countable list-chromatic numbers.

### 1. INTRODUCTION

This introductory section is divided into four subsections. The first of these contains the basic graph-theoretic definitions used in this paper, the second gives the definitions of notions related to semialgebraicity, the third has some of the background material including Komjáth's result mentioned in the abstract, and then in the final subsection we state the main results of this paper.

**1.1. Graphs and their coloring parameters.** A *graph*  $G$  is a pair  $(V, E)$ , where  $V$  is its set of vertices,  $E$  is its set of edges, and  $E \subseteq V^2$  is a symmetric, irreflexive binary relation on  $V$ . (Graphs can be infinite; indeed, almost all graphs considered here are infinite.) A *coloring* of  $G$  is a function  $\varphi : V \rightarrow C$ . A coloring  $\varphi$  is *proper* if  $\varphi(x) \neq \varphi(y)$  whenever  $\langle x, y \rangle \in E$ . The *chromatic number*  $\chi(G)$  is the least cardinal number  $\kappa$  for which there is a proper coloring  $\varphi : V \rightarrow C$  such that  $|C| = \kappa$ .

A function  $\Lambda$  on  $V$  is a *listing* of the graph  $G = (V, E)$  if  $\Lambda(x)$  is a set for each  $x \in V$ . A coloring  $\varphi$  of  $G$  is a  $\Lambda$ -*coloring* if  $\varphi(x) \in \Lambda(x)$  for each  $x \in V$ . If  $\kappa$  is a cardinal, then a listing  $\Lambda$  is a  $\kappa$ -*listing* if  $|\Lambda(x)| = \kappa$  for each  $x \in V$ . The *list-chromatic number*  $\chi_\ell(G)$  is the least cardinal number  $\kappa$  such that for each  $\kappa$ -listing  $\Lambda$  of  $G$ , there is a proper  $\Lambda$ -coloring of  $G$ .

The *coloring number*  $\text{Col}(G)$  of the graph  $G = (V, E)$  is the least cardinal  $\kappa$  for which there is a well-ordering  $\prec$  of  $V$  such that for every  $x \in V$ ,  $|\{y \in V : \langle x, y \rangle \in E \text{ and } y \prec x\}| < \kappa$ . The relatively easy proofs of the inequalities

$$\chi(G) \leq \chi_\ell(G) \leq \text{Col}(G) \leq |V|$$

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relating these three graph parameters can be found in [9, Lemma 3]. Other relations of these quantities can also be found in [9].

For cardinal numbers  $\kappa, \lambda$ , we let  $K(\kappa, \lambda)$  be the complete bipartite graph with parts having cardinality  $\kappa$  and  $\lambda$ . (The set of vertices of  $K(\kappa, \lambda)$  is  $X \cup Y$ , where  $|X| = \kappa$ ,  $|Y| = \lambda$  and  $X \cap Y = \emptyset$ , and the set of edges is  $(X \times Y) \cup (Y \times X)$ .) Obviously,  $\chi(K(\kappa, \lambda)) \leq 2$ . A simple exercise is to show that if  $m < \omega$  and  $\lambda \geq \binom{m^2}{m}$ , then  $\chi_\ell(K(m, \lambda)) = m + 1$ . It is easy to see that  $\chi_\ell(K(\aleph_0, 2^{\aleph_0})) = \text{Col}(K(\aleph_0, 2^{\aleph_0})) = \aleph_1$  and  $\text{Col}(K(2^{\aleph_0}, 2^{\aleph_0})) = 2^{\aleph_0}$ .

Suppose that  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  are graphs. When we say that  $G_1 = (V_1, E_1)$  is a *subgraph* of  $G_2 = (V_2, E_2)$ , we mean that  $V_1 \subseteq V_2$  and  $E_1 \subseteq E_2$ . If  $V_1 \subseteq V_2$  and  $E_1 = E_2 \cap V_1^2$ , then  $G_1$  is an *induced subgraph* of  $G_2$ , in which case  $G_1$  is the subgraph of  $G_2$  *induced* by  $V_1$ . If  $V_1 = V_2$  and  $E_1 \subseteq E_2$ , then  $G_1$  is a *spanning* subgraph of  $G_2$ . We say that  $G_2$  *contains* a  $G_1$  if  $G_2$  has a subgraph isomorphic to  $G_1$ . Obviously, if  $G_2$  contains a  $G_1$ , then  $\chi(G_1) \leq \chi(G_2)$ ,  $\chi_\ell(G_1) \leq \chi_\ell(G_2)$  and  $\text{Col}(G_1) \leq \text{Col}(G_2)$ .

Given a graph  $G = (V, E)$ , we let  $N(x) = \{y \in V : \langle x, y \rangle \in E\}$  whenever  $x \in V$  and  $N(X) = \bigcap_{x \in X} N(x)$  whenever  $X \subseteq V$ .

**1.2. Semialgebraic graphs.** Let  $\tilde{\mathbb{R}} = (\mathbb{R}, +, \cdot, 0, 1, \leq)$  be the ordered field of the reals. A set  $X$  is *semialgebraic* if there is  $n < \omega$  such that  $X \subseteq \mathbb{R}^n$  and  $X$  is definable in  $\tilde{\mathbb{R}}$  by a first-order formula in which parameters from  $\mathbb{R}$  are allowed. If  $D \subseteq \mathbb{R}$  and all the parameters in the definition of  $X$  are in  $D$ , then  $X$  is *D-definable*. Thus,  $X$  is semialgebraic iff it is  $\mathbb{R}$ -definable. A set  $X \subseteq \mathbb{R}^n$  is *algebraic* if it is the set of zeroes of some polynomial over  $\mathbb{R}$ . A graph  $G = (V, E)$  is *semialgebraic* if both  $V$  and  $E$  are semialgebraic, and it is *algebraic* if both  $V$  and  $E$  are algebraic. Every finite graph  $(V, E)$  with  $V \subseteq \mathbb{R}^n$  is algebraic.

We weaken the notions of algebraicity and semialgebraicity by saying that a set  $X \subseteq \mathbb{R}^n$  is  $\sigma$ -*algebraic* if  $X$  is the union of countably many algebraic sets and that it is  $\sigma$ -*semialgebraic* if it is the union of countably many semialgebraic sets. A graph  $G = (V, E)$  is  $\sigma$ -*algebraic* [or  $\sigma$ -*semialgebraic*] if both  $V$  and  $E$  are  $\sigma$ -algebraic [or  $\sigma$ -semialgebraic].

A familiar example of an algebraic graph is the *unit-distance graph*  $\mathbf{X}_2(\{1\})$ , whose set of vertices is the Euclidean plane  $\mathbb{R}^2$  and whose edges are those ordered pairs of points at a distance 1 from each other. The unit-distance graph has been generalized to the *D-distance graphs*  $\mathbf{X}_n(D)$ , where  $\emptyset \neq D \subseteq \mathbb{R}_+$  (where  $\mathbb{R}_+$  is the set of positive reals). The vertices of  $\mathbf{X}_n(D)$  are the points in  $\mathbb{R}^n$  and the edges are those pairs  $\langle x, y \rangle$  such that the distance  $\|x - y\|$  between them is in  $D$ . If  $D$  is finite, then  $\mathbf{X}_n(D)$  is algebraic; if  $D$  is countable, then  $\mathbf{X}_n(D)$  is  $\sigma$ -algebraic.

When considering a  $\sigma$ -semialgebraic graph  $G = (V, E)$ , where  $V \subseteq \mathbb{R}^n$ , we might replace it with the graph  $G' = (\mathbb{R}^n, E)$  since  $\chi_\ell(G) = \chi_\ell(G')$  and  $\text{Col}(G) = \text{Col}(G')$ .

**1.3. The motivating background.** The well-known Hadwiger-Nelson problem (see [12, Chapter 3]) is to determine  $\chi(\mathbf{X}_2(\{1\}))$ . Very little, other than the relatively simple bounds of  $4 \leq \chi(\mathbf{X}_2(\{1\})) \leq 7$ , is known about the exact value of  $\chi(\mathbf{X}_2(\{1\}))$ . Johnson [6] first suggested the problem of determining  $\chi_\ell(\mathbf{X}_2(\{1\}))$ . Jensen & Toft [4] observed that  $\chi_\ell(\mathbf{X}_2(\{1\})) \geq \aleph_0$  since the list-chromatic number of the  $d$ -dimensional cube  $\{0, 1\}^d$  (which is a  $d$ -regular bipartite graph that is isomorphic to an induced subgraph of  $\mathbf{X}_2(\{1\})$ ) becomes arbitrarily large as  $d$  goes to infinity (following a deep result of Alon & Tarsi [1]). They asked in [4], and

again in [5], for the actual value of  $\chi_\ell(\mathbf{X}_2(\{1\}))$ . It was then proved in [10] that  $\chi_\ell(\mathbf{X}_2(\{1\})) = \aleph_0$ ; however, as later pointed out by Komjáth [8], this could have easily been concluded from a much earlier result of Erdős & Hajnal [2]. Komjáth then proceeded to consider the  $D$ -distance graphs  $\mathbf{X}_n(D)$ . If  $D$  is countable, then  $\chi(\mathbf{X}_n(D)) \leq \aleph_0$ , as first proved by Komjáth [7]. In [8], he reports the following theorem (which is mostly a compilation of earlier results) characterizing those  $\mathbf{X}_n(D)$ , with  $D$  countable, having a countable list-chromatic number.

**Theorem 1.1** (Komjáth [8]). *If  $1 \leq n < \omega$  and  $D \subseteq \mathbb{R}_+$  is nonempty and countable, then the following are equivalent:*

- (1)  $\chi_\ell(\mathbf{X}_n(D)) \leq \aleph_0$ ;
- (2)  $\text{Col}(\mathbf{X}_n(D)) \leq \aleph_0$ ;
- (3) either  $n \leq 2$  or else  $n = 3$  and  $\inf D_0 = 0$  for all infinite  $D_0 \subseteq D$ .

Of course, (2)  $\implies$  (1) is trivial. Of the remaining implications, (1)  $\implies$  (3)  $\implies$  (2), the proof of the first one is easier and is accomplished by showing that  $\mathbf{X}_4(D)$  contains a  $K(2^{\aleph_0}, 2^{\aleph_0})$ , and if  $D$  is infinite and  $\inf(D) > 0$ , then  $\mathbf{X}_3(D)$  contains a  $K(\aleph_0, 2^{\aleph_0})$ . Incidentally, it is easy to see that no  $\mathbf{X}_n(D)$  as in (3) contains a  $K(\aleph_0, 2^{\aleph_0})$ . Thus, we get that Theorem 1.1 has the following corollary.

**Corollary 1.2.** *If  $1 \leq n < \omega$  and  $D \subseteq \mathbb{R}_+$  is countable, then the following are equivalent:*

- (1)  $\chi_\ell(\mathbf{X}_n(D)) \leq \aleph_0$ ;
- (2)  $\text{Col}(\mathbf{X}_n(D)) \leq \aleph_0$ ;
- (3)  $\mathbf{X}_n(D)$  does not contain a  $K(\aleph_0, 2^{\aleph_0})$ .

Conversely, Theorem 1.1 easily follows from Corollary 1.2. Therefore, it is fair to say that Theorem 1.1 and Corollary 1.2 are equivalent. Corollary 1.2, being a noncontainment characterization, lends itself more readily to generalizations than does Theorem 1.1.

**1.4. Main results.** Each graph  $\mathbf{X}_n(D)$  with  $D$  being countable is  $\sigma$ -algebraic. One goal of this paper is to prove the following Theorem 1.3, which generalizes Corollary 1.2 from the class of  $\mathbf{X}_n(D)$  with  $D$  countable to the class of  $\sigma$ -algebraic graphs. Observe that there is a  $\sigma$ -algebraic graph that is isomorphic to  $K(\aleph_0, 2^{\aleph_0})$ . For example, the complete bipartite graph with parts  $\omega \times \{0\}$  and  $\mathbb{R} \times \{1\}$  is such a graph.

**Theorem 1.3.** *If  $G$  is a  $\sigma$ -algebraic graph, then the following are equivalent:*

- (1)  $\chi_\ell(G) \leq \aleph_0$ ;
- (2)  $\text{Col}(G) \leq \aleph_0$ ;
- (3)  $G$  does not contain a  $K(\aleph_0, 2^{\aleph_0})$ .

We will also get similar characterizations for the classes of semialgebraic graphs and  $\sigma$ -semialgebraic graphs in Theorems 1.5 and 1.4, respectively. In each case, we get a noncontainment characterization of those graphs in the class having countable list-chromatic numbers and countable coloring numbers. Sometimes the term *obligatory* is used (such as in [3]) for a graph that is necessarily contained in every graph (in some class) that has uncountable chromatic (or list-chromatic or coloring) number. For our characterizations, we obtain obligatory graphs that contain all the obligatory graphs for the class under consideration.

We will define in §3 a  $\sigma$ -semialgebraic bipartite graph that we call the *Cantor graph*. The following theorem tells us that both the list-chromatic number and the coloring number of the Cantor graph are uncountable; in fact, both are  $\aleph_1$ .

**Theorem 1.4.** *If  $G$  is a  $\sigma$ -semialgebraic graph, then the following are equivalent:*

- (1)  $\chi_\ell(G) \leq \aleph_0$ ;
- (2)  $\text{Col}(G) \leq \aleph_0$ ;
- (3)  $G$  does not contain a Cantor graph.

Our last result is noncontainment characterization for semialgebraic graphs. Notice that the complete bipartite graph with parts  $\mathbb{R} \times \{0\}$  and  $\mathbb{R} \times \{1\}$  is an algebraic graph isomorphic to  $K(2^{\aleph_0}, 2^{\aleph_0})$

**Theorem 1.5.** *If  $G$  is a semialgebraic graph, then the following are equivalent:*

- (1)  $\chi_\ell(G) \leq \aleph_0$ ;
- (2)  $\text{Col}(G) \leq \aleph_0$ ;
- (3) there is  $m < \omega$  such that  $G$  does not contain a  $K(m, m)$ ;
- (4)  $G$  does not contain a  $K(2^{\aleph_0}, 2^{\aleph_0})$ .

In the next three sections, we will prove Theorems 1.5, 1.4 and 1.3, respectively.

## 2. PROVING THEOREM 1.5 ON SEMIALGEBRAIC GRAPHS

For an infinite cardinal  $\kappa$ , let  $H(\kappa)$  be a bipartite graph  $(A \cup B, E)$ , whose two parts are  $A = \omega$  and  $B$ , where  $B$  is partitioned into sets  $B_0, B_1, B_2, \dots$ , each of cardinality  $\kappa$ , and if  $a \in A$  and  $b \in B$ , then  $\langle a, b \rangle \in E$  iff  $b \in B_i$  for some  $i \geq a$ . Although we won't be needing it, an easy exercise is to show that for every infinite  $\kappa$ ,  $\chi_\ell(H(\kappa)) = \text{Col}(H(\kappa)) = \aleph_0$ .

If  $\kappa \geq \aleph_0$ , then  $K(\aleph_0, \kappa)$  contains an  $H(\kappa)$ . Also,  $H(\kappa)$  contains a  $K(m, \kappa)$  for each  $m < \omega$ . For countable  $D$ ,  $\mathbf{X}_2(D)$  does not contain a  $K(2, \aleph_1)$ , and if  $\inf D_0 = 0$  for all infinite  $D_0 \subseteq D$ , then there is  $m < \omega$  such that  $\mathbf{X}_3(D)$  does not contain a  $K(m, \aleph_1)$ . Therefore, (3)  $\implies$  (2) of Theorem 1.1 is a consequence of the following theorem, which is suggested by [2, Cor. 5.6].

**Theorem 2.1** (after Erdős & Hajnal [2]). *If  $G$  is a graph that does not contain an  $H(\aleph_1)$ , then  $\text{Col}(G) \leq \aleph_0$ .*

Before presenting the proof of Theorem 2.1, we make some definitions and then prove a lemma.

Suppose that  $G = (V, E)$  is a graph. We will say (in this section only) that  $A \subseteq V$  is *closed* if whenever  $X \subseteq A$  is finite and  $N(X)$  is countable, then  $N(X) \subseteq A$ . For any  $X \subseteq V$ , there is a unique smallest closed  $A$  such that  $X \subseteq A \subseteq V$ ; moreover,  $|A| \leq |X| + \aleph_0$ . The union of an increasing sequence of closed subsets is closed.

**Lemma 2.2.** *Suppose that  $G = (V, E)$  is a graph that does not contain an  $H(\aleph_1)$ . If  $A \subseteq V$  is closed and  $b \in V \setminus A$ , then  $N(b) \cap A$  is finite.*

*Proof.* Assume, for a contradiction, that  $A \subseteq V$  and  $b \in V \setminus A$  are such that  $A$  is closed and  $N(b) \cap A$  is infinite. Let  $a_0, a_1, a_2, \dots$  be infinitely many distinct members of  $N(b) \cap A$ . For each finite  $S \subseteq N(b) \cap A$ , the set  $N(S)$  is uncountable as otherwise  $b \in A$ . Let  $B_m = N(\{a_0, a_1, \dots, a_m\})$  so that  $B_0 \supseteq B_1 \supseteq B_2 \supseteq \dots$  and each  $B_m$  is uncountable. Clearly, the subgraph induced by  $\{a_0, a_1, a_2, \dots\} \cup B_0$  contains an  $H(\aleph_1)$ , a contradiction that proves the lemma. □

*Proof of Theorem 2.1.* The proof that Lemma 2.2 implies Theorem 2.1 is a standard argument by induction on the cardinalities of graphs used to prove upper bounds on the coloring number. We present it here since it will be used again in the proof of Theorem 1.4 in §3.

For each infinite cardinal  $\lambda$  we will prove:

- (\*) If  $G = (V, E)$  is a graph such that  $|V| \leq \lambda$  and  $G$  does not contain an  $H(\aleph_1)$ , then  $\text{Col}(G) \leq \aleph_0$ .

The coloring number of every countable graph is at most  $\aleph_0$ , so (\*) holds when  $\lambda = \aleph_0$ .

To proceed by induction, suppose that  $\kappa > \aleph_0$  and that (\*) holds whenever  $\aleph_0 \leq \lambda < \kappa$ . We will show that (\*) holds for  $\lambda = \kappa$ .

For brevity, we will say that  $\prec$  is *proper* for the graph  $G = (V, E)$  if  $\prec$  is a well-ordering of  $V$  and  $\{y \in V : \langle x, y \rangle \in E \text{ and } y \prec x\}$  is finite for every  $x \in V$ .

Consider a graph  $G = (V, E)$  such that  $|V| = \kappa$  and  $G$  does not contain an  $H(\aleph_1)$ . Let  $\langle V_\alpha : \alpha < \kappa \rangle$  be an increasing sequence of closed sets such that:

- $V = \bigcup_{\alpha < \kappa} V_\alpha$ ;
- $V_\alpha = \bigcup_{\beta < \alpha} V_\beta$  whenever  $\alpha < \kappa$  is a limit ordinal;
- $|V_\alpha| < \kappa$  for all  $\alpha < \kappa$ .

Let  $G_\alpha$  be the subgraph of  $G$  induced by  $V_\alpha$ .

We will obtain by transfinite recursion a sequence  $\langle \prec_\alpha : \alpha < \kappa \rangle$  such that whenever  $\alpha < \beta < \kappa$ , then  $\prec_\alpha$  is proper for  $G_\alpha$ , and whenever  $a, b \in V_\alpha, c \in V_\beta \setminus V_\alpha$  and  $a \prec_\alpha b$ , then  $a \prec_\beta b \prec_\beta c$ . Having such a sequence, we then let  $\prec = \bigcup_{\alpha < \kappa} \prec_\alpha$ , which will be proper for  $G$  and, therefore, demonstrate that  $\text{Col}(G) \leq \aleph_0$ .

Let  $\prec_0$  be proper for  $G_0$ , the existence of which is guaranteed by the inductive hypothesis.

If  $\alpha < \kappa$  is a limit ordinal, then let  $\prec_\alpha = \bigcup_{\beta < \alpha} \prec_\beta$ .

It remains to obtain  $\prec_{\alpha+1}$  assuming we already have  $\prec_\alpha$ . By the inductive hypothesis, let  $\prec'$  be proper for  $G_{\alpha+1}$ . Then let  $\prec_{\alpha+1}$  be such that whenever  $a, b \in V_{\alpha+1}$ , then  $a \prec_{\alpha+1} b$  iff one of the following holds:

- $a, b \in V_\alpha$  and  $a \prec_\alpha b$ ;
- $a \in V_\alpha$  and  $b \in V_{\alpha+1} \setminus V_\alpha$ ;
- $a, b \in V_{\alpha+1} \setminus V_\alpha$  and  $a \prec' b$ .

Lemma 2.2 implies that  $\prec_{\alpha+1}$  is proper for  $G_{\alpha+1}$ . Hence, (\*) is proved when  $\lambda = \kappa$ , so the theorem is also proved. □

*Proof of Theorem 1.5.* We will prove the circle of implications

$$(1) \implies (4) \implies (3) \implies (2) \implies (1).$$

(1)  $\implies$  (4): This holds since  $\chi_\ell(K(2^{\aleph_0}, 2^{\aleph_0})) > \aleph_0$ .

(4)  $\implies$  (3): This is a special case of [11, Lemma 4.2].

(3)  $\implies$  (2): By Theorem 2.1, since  $H(\aleph_1)$  contains every  $K(m, m)$ .

(2)  $\implies$  (1): This is true for any  $G$ . □

As mentioned in the previous proof, the implication (4)  $\implies$  (3) is a special case of [11, Lemma 4.2]. This lemma from [11] actually yields a stronger result that we state in its contrapositive form in the following corollary. Keep in mind that every infinite semialgebraic set has cardinality  $2^{\aleph_0}$ .

**Corollary 2.3.** *If  $G = (V, E)$  is a semialgebraic graph such that  $\chi_\ell(G) > \aleph_0$ , then there are infinite semialgebraic  $X, Y \subseteq V$  such that  $X \times Y \subseteq E$ .  $\square$*

Theorem 1.5 and Corollary 2.3 have implications concerning the decidability of the set of semialgebraic graphs having countable list-chromatic numbers (and also countable coloring numbers). Some definitions are required.

For convenience, we will say that  $\langle \theta(u), \psi(u, x) \rangle$  is an  $(m, n)$ -pair if  $m, n < \omega$ ,  $u$  is an  $m$ -tuple of variables,  $x$  is an  $n$ -tuple of variables, and both  $\theta(u)$  and  $\psi(u, x)$  are formulas in the language appropriate for ordered fields. Then,  $\langle \theta(u), \psi(u, x) \rangle$  is an  $(\omega, \omega)$ -pair if it is an  $(m, n)$ -pair or some  $m, n < \omega$ . We say that a set  $\mathcal{S}$  of semialgebraic sets is *decidably enumerable* if there is a computable set  $\Psi$  of  $(\omega, \omega)$ -pairs such that for any  $n < \omega$  and semialgebraic  $X \subseteq \mathbb{R}^n$ ,  $X \in \mathcal{S}$  iff there are  $m < \omega$ ,  $a \in \mathbb{R}^m$  and an  $(m, n)$ -pair  $\langle \theta(u), \psi(u, x) \rangle \in \Psi$  such that  $a$  is in the set defined by  $\theta(u)$  and  $\psi(a, x)$  defines  $X$ . A set  $\mathcal{S}$  of semialgebraic sets is *decidable* if both  $\mathcal{S}$  and its complement (i.e., the set of semialgebraic sets not in  $\mathcal{S}$ ) are decidably enumerable. A set  $\mathcal{G}$  of semialgebraic graphs is *decidable* [or *decidably enumerable*] if the set  $\{V \times E : (V, E) \in \mathcal{G}\}$  is. Notice that the set of all semialgebraic graphs is decidable.

Behind this definition of a decidable set of semialgebraic sets is Tarski’s famous theorem that  $\text{Th}(\widetilde{\mathbb{R}})$  is a decidable theory.

**Corollary 2.4.** *The set  $\mathcal{G}$  of all semialgebraic graphs  $G$  for which  $\chi_\ell(G) \leq \aleph_0$  is decidable.*

*Proof.* It follows from (1)  $\iff$  (3) of Theorem 1.5 that  $\mathcal{G}$  is decidably enumerable, and it follows from Corollary 2.3 that its complement is decidably enumerable.  $\square$

The set  $\mathcal{G}$  in the previous corollary is also the set of semialgebraic graphs  $G$  for which  $\text{Col}(G) \leq \aleph_0$ .

### 3. PROVING THEOREM 1.4 ON $\sigma$ -SEMIALGEBRAIC GRAPHS

Recall from §1.2 that a set  $X \subseteq \mathbb{R}^n$  is  $\sigma$ -semialgebraic iff  $X$  is the union of countably many semialgebraic sets and that a graph  $G = (V, E)$  is  $\sigma$ -semialgebraic iff both  $V$  and  $E$  are  $\sigma$ -semialgebraic. All graphs  $\mathbf{X}_n(D)$  with  $D$  countable, as in Theorem 1.1 and Corollary 1.2, are  $\sigma$ -semialgebraic.

Next, we define the Cantor graph. Let  $P \subseteq [0, 1]$  be Cantor’s middle third set, and let  $I_2, I_3, I_4, \dots$  be some natural one-to-one enumeration of all those closed subintervals of  $[0, 1]$  that are involved in the construction of  $P$ . The enumeration could be done so that

$$P = \bigcap_{i < \omega} \left( \bigcup \{I_j : 2^i < j \leq 2^{i+1}\} \right).$$

For example, let  $I_2 = [0, 1]$ ,  $I_3 = [0, 1/3]$ ,  $I_4 = [2/3, 1]$ ,  $I_5 = [0, 1/9]$ , etc. The *Cantor graph*, which we denote by  $C$ , is the bipartite graph having parts  $\{j < \omega : j \geq 2\}$  and  $[0, 1]$  such that whenever  $2 \leq j < \omega$  and  $x \in [0, 1]$ , then  $\langle j, x \rangle$  is an edge iff  $x \in I_j$ . The Cantor graph is  $\sigma$ -semialgebraic since its vertex set is the union of the semialgebraic sets  $[0, 1], \{2\}, \{3\}, \{4\}, \dots$  and its edge set is the union of the countably many semialgebraic sets  $(\{j\} \times I_j) \cup (I_j \times \{j\})$  for  $j \geq 2$ . The Cantor graph sits strictly between  $H(2^{\aleph_0})$  and  $K(\aleph_0, 2^{\aleph_0})$ . That is,  $K(\aleph_0, 2^{\aleph_0})$  contains a  $C$  (implying that  $\text{Col}(C) \leq \aleph_1$ ) and  $C$  contains an  $H(2^{\aleph_0})$ , while  $H(2^{\aleph_0})$  does not contain a  $C$  and  $C$  does not contain a  $K(\aleph_0, 2^{\aleph_0})$ .

On the other hand,  $\chi_\ell(C) \geq \aleph_1$  as can be seen by considering an  $\aleph_0$ -listing  $\Lambda$  of  $C$  such that:

- if  $2 \leq a < b < \omega$ , then  $\Lambda(a) \cap \Lambda(b) = \emptyset$ ;
- there is a bijection  $f : P \rightarrow \prod_{2 \leq a < \omega} \Lambda(a)$  such that if  $x \in P$ , then

$$\Lambda(x) = \{(f(x))(a) : \langle a, x \rangle \text{ is an edge of } C\}.$$

Let  $\varphi$  be a  $\Lambda$ -coloring of  $C$ . Let  $x \in P$  be such that  $f(x) = \varphi \upharpoonright \{a : 2 \leq a < \omega\}$ . Since  $\varphi(x) \in \Lambda(x)$ , there is  $a$  such that  $\langle a, x \rangle$  is an edge and  $\varphi(x) = f(x)(a) = \varphi(a)$ , so  $\varphi$  is not proper.

Thus,  $\chi_\ell(C) \geq \aleph_1$ , so that  $\chi_\ell(C) = \text{Col}(C) = \aleph_1$ .

We let  $C_0$  be the subgraph of  $C$  induced by  $\omega \cup P$ . This graph is not  $\sigma$ -semialgebraic, but it does contain a  $C$ .

*Proof of Theorem 1.4.* Let  $G = (V, E)$  be a  $\sigma$ -semialgebraic graph. We have just seen that (1)  $\implies$  (3) and (2)  $\implies$  (3). Obviously, (2)  $\implies$  (1). Thus, it remains to prove that (3)  $\implies$  (2).

Suppose that  $G$  does not contain a Cantor graph. We will prove that  $\text{Col}(G) \leq \aleph_0$ . The proof is just as the proof of Theorem 1.5 given in §2, except we will need another interpretation of a closed set and also another lemma to replace Lemma 2.2.

Let  $n < \omega$  be such that  $V \subseteq \mathbb{R}^n$ . Without loss of generality, we assume that  $V = \mathbb{R}^n$ . (We can do this since  $(\mathbb{R}^n, E)$  is also a  $\sigma$ -semialgebraic graph that does not contain a Cantor graph, and  $\text{Col}(\mathbb{R}^n, E) = \text{Col}(G)$ .) Since  $E$  is  $\sigma$ -semialgebraic,  $E = \bigcup_{i < \omega} E_i$ , where each  $E_i$  is semialgebraic and  $(\mathbb{R}^n, E_i)$  is a graph. By replacing  $E_i$  with  $E_i \setminus \bigcup_{j < i} E_j$ , we can assume that the  $E_i$ 's are pairwise disjoint.

If  $x \in \mathbb{R}^n$ , then we let  $N_i(x) = \{y \in \mathbb{R}^n : \langle x, y \rangle \in E_i\}$  for each  $i < \omega$ . Hence,  $N(x) = \bigcup_{i < \omega} N_i(x)$ .

Recall that  $X \subseteq \mathbb{R}^n$  is a variety iff  $X$  is algebraic and is not the union of two algebraic sets each of which is distinct from  $X$ . Now, suppose that  $X \subseteq \mathbb{R}^n$  is a variety and  $i < \omega$ . If  $a \in \mathbb{R}^n$  we define  $U_i(X, a)$  to be the relative interior of  $N_i(a) \cap X$  in  $X$ , and then define

$$A_i(X) = \{a \in \mathbb{R}^n : U_i(X, a) \neq \emptyset\}.$$

*Claim.* If  $X \subseteq \mathbb{R}^n$  is an infinite variety and  $i < \omega$ , then  $A_i(X)$  is finite.

We prove the claim by contradiction. Assume  $A_i(X)$  is infinite. Since  $A_i(X)$  is semialgebraic, there is an infinite  $T \subseteq A_i(X)$  and a nonempty relatively open  $U \subseteq X$  such that  $T \times U \subseteq E_i$ . [A sketch of one way to see this is the following. There are semialgebraic functions  $g : A_i(X) \rightarrow X$  and  $d : A_i(X) \rightarrow \mathbb{R}_+$  such that whenever  $a \in A_i(X)$ ,  $x \in X$  and  $\|x - g(a)\| < d(a)$ , then  $x \in U_i(X, a)$ . There is an infinite connected  $S \subseteq A_i(X)$  on which both  $g$  and  $d$  are continuous. Let  $a \in S$ , let  $W$  be a sufficiently small neighborhood of  $a$ , and then let  $T = W \cap S$  and  $U = \bigcap \{U_i(X, a) : a \in T\}$ .] Thus,  $G$  contains a  $K(\aleph_0, 2^{\aleph_0})$ , so it contains a Cantor graph. This contradiction proves the claim.

Given a variety  $X \subseteq \mathbb{R}^n$ , let

$$A(X) = \bigcup_{i < \omega} A_i(X).$$

A consequence of the claim is that  $A(X)$  is countable. For each  $a \in \mathbb{R}^n$ , let

$$U(X, a) = \bigcup_{i < \omega} U_i(X, a),$$

and then let

$$L(X) = \{x \in X : \{a \in \mathbb{R}^n : x \in U(X, a)\} \text{ is infinite}\}.$$

We now come to the new definition of a closed set. A subset of  $V = \mathbb{R}^n$  is *closed* if it is some  $F^n$ , where  $F$  is a real-closed subfield of  $\mathbb{R}$ , each  $E_i$  is  $F$ -definable, and whenever  $X \subseteq \mathbb{R}^n$  is an  $F$ -definable variety and  $L(X)$  is countable, then  $L(X) \subseteq F^n$ .

For any  $X \subseteq \mathbb{R}^n$ , there is a unique smallest closed  $F^n$  such that  $X \subseteq F^n \subseteq \mathbb{R}^n$ ; moreover,  $|F^n| = |X| + \aleph_0$ . The union of an increasing sequence of closed subsets of  $\mathbb{R}^n$  is closed.

**Lemma 3.1.** *If  $F^n \subseteq \mathbb{R}^n$  is closed and  $b \in \mathbb{R}^n \setminus F^n$ , then  $N(b) \cap F^n$  is finite.*

*Proof.* For a contradiction, suppose that  $b \in \mathbb{R}^n \setminus F^n$  is such that  $N(b) \cap F^n$  is infinite.

By the Hilbert Basis Theorem, let  $X \subseteq \mathbb{R}^n$  be the smallest  $F$ -definable variety such that  $b \in X$ . Since  $F$  is real-closed,  $X$  is infinite, so  $A(X)$  is countable. Also, each  $N(b) \cap F^n \subseteq A(X)$ , so  $b \in L(X)$ . Since  $F^n$  is closed and  $b \notin F^n$ , it must be that  $L(X)$  is uncountable. It is clear that  $L(X)$  is a Borel set. We will see that this entails a contradiction.

It will be shown that  $G$  contains a  $C_0$  and, therefore, also a Cantor graph. By recursion on the length of  $s$ , we choose, for each  $s \in \{0, 1\}^{<\omega}$ , an element  $a_s$  and a perfect Borel set  $B_s \subseteq L(X) \cap N(a_s)$  such that if  $s, t \in \{0, 1\}^{<\omega}$ , then:

- $B_s$  has diameter at most  $1/n$ , where  $n$  is the length of  $s$ ;
- if  $s \subseteq t$ , then  $B_s \supseteq B_t$ ;
- if  $B_s \cap B_t \neq \emptyset$ , then either  $s \subseteq t$  or  $t \subseteq s$ .

Let  $a_\emptyset \in A(X)$  be such that  $N(a_\emptyset) \cap L(X)$  is uncountable, and then let  $B_0 \subseteq N(a_\emptyset) \cap L(X)$  be a perfect Borel set. Suppose that we have  $a_s$  and  $B_s$ . Let  $a_{s_0}, a_{s_1} \in A(X)$  be distinct from each other and from all previously chosen  $a_t$ 's such that both  $N(a_{s_0}) \cap B_s$  and  $N(a_{s_1}) \cap B_s$  are uncountable. Then let  $B_{s_0} \subseteq N(a_{s_0}) \cap B_s$  and  $B_{s_1} \subseteq N(a_{s_1}) \cap B_s$  be sufficiently small, disjoint perfect Borel sets. For each  $t \in \{0, 1\}^\omega$ ,  $\bigcap_{i < \omega} B_{t|_i} = \{b_t\}$ . It is clear that the subgraph of  $G$  induced by  $\{a_s : s \in \{0, 1\}^{<\omega}\} \cup \{b_t : t \in \{0, 1\}^\omega\}$  has a spanning subgraph that is isomorphic to  $C_0$ . This contradiction completes the proof of the lemma.  $\square$

We now return to the proof of (3)  $\implies$  (2) of Theorem 1.4. We have the  $\sigma$ -semialgebraic graph  $G = (\mathbb{R}^n, E)$  that does not contain a Cantor graph, and we have the notion of a closed subset of  $\mathbb{R}^n$  that appears in Lemma 3.1.

For each infinite cardinal  $\lambda \leq 2^{\aleph_0}$  we will prove:

- (\*) *If  $F^n \subseteq \mathbb{R}^n$  is closed and  $|F| \leq \lambda$ , then the subgraph of  $G$  induced by  $F^n$  has countable coloring number.*

Then letting  $\lambda = 2^{\aleph_0}$  and  $F^n = \mathbb{R}^n$ , we will get that  $\text{Col}(G) \leq \aleph_0$ . The inductive proof of (\*) for all appropriate  $\lambda$  is identical to the proof of (\*) that occurs in the proof of Theorem 2.1, except that here we are using Lemma 3.1 instead of Lemma 2.2. We leave it to the reader to confirm that this can be done.  $\square$

#### 4. PROVING THEOREM 1.3 ON $\sigma$ -ALGEBRAIC GRAPHS

In this final section, we prove Theorem 1.3 by showing that it follows from Theorem 1.4.

*Proof of Theorem 1.3.* Let  $G = (V, E)$  be a  $\sigma$ -algebraic graph. Obviously, (2)  $\implies$  (1). Also, (1)  $\implies$  (3) since  $\chi_\ell(K(\aleph_0, 2^{\aleph_0})) = \aleph_1$ . Thus, it remains to prove that (3)  $\implies$  (2). By Theorem 1.4, it suffices to prove that if  $G$  contains a Cantor graph, then it contains a  $K(\aleph_0, 2^{\aleph_0})$ . Assume that  $G$  contains a Cantor graph.

Let  $n < \omega$  be such that  $V \subseteq \mathbb{R}^n$ . Without loss of generality, we assume that  $V = \mathbb{R}^n$ . (We can do this since  $(\mathbb{R}^n, E)$  is also a  $\sigma$ -algebraic graph that contains a Cantor graph and contains a  $K(\aleph_0, 2^{\aleph_0})$  iff  $G$  does.) Since  $E$  is  $\sigma$ -algebraic, let  $E = \bigcup_{i < \omega} E_i$ , where each  $E_i$  is algebraic.

If  $x \in \mathbb{R}^n$ , then we let  $N_i(x) = \{y \in \mathbb{R}^n : \langle x, y \rangle \in E_i\}$  for each  $i < \omega$ . Hence,  $N(x) = \bigcup_{i < \omega} N_i(x)$ .

Since  $G$  contains a Cantor graph, it also contains a  $C_0$ . Thus, there are disjoint sets  $A = \{a_s : s \in \{0, 1\}^{<\omega}\} \subseteq \mathbb{R}^n$  and  $B = \{b_t : t \in \{0, 1\}^\omega\} \subseteq \mathbb{R}^n$  such that whenever  $a_s \in A$ ,  $b_t \in B$  and  $s \subseteq t$ , then  $a_s \in N(b_t)$ . Moreover, the indexing of both  $A$  and  $B$  is one-to-one. We define three sequences by recursion:

- a sequence  $s \in \{0, 1\}^\omega$ ;
- a sequence  $\langle i_m : m < \omega \rangle$  of elements of  $\omega$ ;
- a decreasing sequence  $\langle B_m : m < \omega \rangle$  such that each  $B_m$  is an uncountable subset of  $N_{i_m}(a_{s|m}) \cap B$ .

To start, since  $B \subseteq N(a_\emptyset)$ , we let  $i_0 < \omega$  be such that  $N_{i_0}(a_\emptyset) \cap B$  is uncountable, and then let  $B_0 = N_{i_0}(a_\emptyset) \cap B$ .

Suppose that  $m < \omega$  and that  $s|m$ ,  $B_m$  and  $i_m$  have been defined. Since  $B_m$  is an uncountable subset of  $N(a_{s|m})$ , we can let  $s_m \in \{0, 1\}$  be such that  $N(a_{s|(m+1)}) \cap B_m$  is uncountable. Then, let  $i_{m+1} < \omega$  be such that  $N_{i_{m+1}}(a_{s|(m+1)}) \cap B_m$  is uncountable. Finally, let  $B_{m+1} = N_{i_{m+1}}(a_{s|(m+1)}) \cap B_m$ .

We have  $B_0 \supseteq B_1 \supseteq B_2 \supseteq \dots$ . Let  $X_m$  be the smallest algebraic set such that  $X_m \supseteq B_m$ . Each  $X_m \subseteq N_{i_m}(a_{s|m})$ . Then,  $X_0 \supseteq X_1 \supseteq X_2 \supseteq \dots$  is a decreasing sequence of algebraic sets, which, by the Hilbert Basis Theorem, is eventually constant at  $X$ . Then  $|X| = 2^{\aleph_0}$  since it is an infinite algebraic set. For each  $m < \omega$ ,

$$X \subseteq X_m \subseteq N_{i_m}(a_{s|m}) \subseteq N(a_{s|m}).$$

Thus,  $G$  contains a  $K(\aleph_0, 2^{\aleph_0})$ , the two parts being  $\{a_{s|m} : m < \omega\}$  and  $X$ .  $\square$

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