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QUANTUM DIMENSIONS AND FUSION RULES FOR PARAFERMION VERTEX OPERATOR ALGEBRAS

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Abstract. The quantum dimensions and the fusion rules for the parafermion vertex operator algebra associated to the irreducible highest weight module for the affine Kac-Moody algebra $A_1^{(1)}$ of level k are determined.

1. Introduction

Parafermion vertex operator algebra is the commutant of the Heisenberg vertex operator algebra in the affine vertex operator algebra. It comes from a special kind of coset construction [23]. Precisely speaking, let $L_{\widehat{\mathfrak{g}}}(k,0)$ be the level k integrable highest weight module with weight zero for affine Kac-Moody algebra $\widehat{\mathfrak{g}}$ associated to a finite-dimensional simple Lie algebra $\mathfrak{g}$. Then $L_{\widehat{\mathfrak{g}}}(k,0)$ contains the Heisenberg vertex operator subalgebra generated by the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. The commutant $K(\mathfrak{g},k)$ of the Heisenberg vertex operator subalgebra in $L_{\widehat{\mathfrak{g}}}(k,0)$ is the parafermion vertex operator algebra. The structure and representation theory of parafermion vertex operator algebras has been widely studied since 2009 (see [14], [15], [17], [18], [4]). In particular, [17] and [18] show that the role of parafermion vertex operator algebra $K(sl_2,k)$ in the parafermion vertex algebra $K(\mathfrak{g},k)$ is similar to the role of 3-dimensional simple Lie algebra sl_2 played in Kac-Moody Lie algebras. We denote $K(sl_2,k)$ by K_0 in this paper. Moreover, it was proved that K_0 coincides with a certain W-algebra in [14] and [15]. Later in [4], the C_2-cofiniteness of parafermion vertex operator algebra $K(\mathfrak{g},k)$ has been established by proving the C_2-cofiniteness of parafermion vertex operator algebra K_0, and irreducible modules for parafermion vertex operator algebra K_0 were also classified therein. Recently, the rationality of K_0 was established [5] by identifying the parafermion vertex operator algebra K_0 with certain W-algebra [3]. Also see [16] for the rationality of parafermion vertex operator algebra $K(\mathfrak{g},k)$ for any $\mathfrak{g}$ and classification of irreducible modules.

The notion of quantum dimensions of modules for vertex operator algebras was introduced in [8]. It was proved therein for rational and C_2-cofinite vertex operator algebras that quantum dimensions do exist. In this paper, we first determine the

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quantum dimensions for the parafermion vertex operator algebra K_0. Then by using the important formula obtained in [8] which shows that quantum dimensions are multiplicative under tensor product, we give the fusion rules for the parafermion vertex operator algebra K_0. The quantum dimensions of irreducible modules and the fusion rules for any $K(\mathfrak{g}, k)$ were determined recently in [2].

The paper is organized as follows. In Section 2, we recall some results about parafermion vertex operator algebra K_0. In Section 3, after reviewing the notion and properties of quantum dimensions of modules for vertex operator algebras, we give the quantum dimensions of parafermion vertex operator algebra K_0. In the final section, we obtained the fusion rules of parafermion vertex operator algebra K_0 by using the results of quantum dimensions of parafermion vertex operator algebra K_0.

2. Preliminary

In this section, we recall from [14], [15] and [4] some basic results on the parafermion vertex operator algebra associated to the irreducible highest weight module for the affine Kac-Moody algebra $A_1^{(1)}$ of level k with $k \geq 2$ being an integer. First we recall the notion of the parafermion vertex operator algebra.

We are working in the setting of [14]. Let $\{h, e, f\}$ be a standard Chevalley basis of sl_2 with brackets $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$. Let $\widehat{sl}_2 = sl_2 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$ be the affine Lie algebra associated to sl_2. Let $k \geq 2$ be an integer and let

$$V(k, 0) = V_{\widehat{sl}_2}(k, 0) = \mathrm{Ind}_{sl_2 \otimes \mathbb{C}[t] \oplus \mathbb{C}K}^{\widehat{sl}_2} \mathbb{C}$$

be an induced $\widehat{sl}_2$-module such that on $\mathbf{1} = 1$, $sl_2 \otimes \mathbb{C}[t]$ acts as 0 and K acts as k. We denote by $a(n)$ the operator on $V(k, 0)$ corresponding to the action of $a \otimes t^n$. Then

$$[a(m), b(n)] = [a, b](m + n) + m\langle a, b\rangle \delta_{m+n,0} k \tag{2.1}$$

for $a, b \in sl_2$ and $m, n \in \mathbb{Z}$. It is well known [22] that there is a vertex operator algebra structure on $V(k, 0)$ and it has a unique maximal ideal $\mathcal{J}$, which is generated by a weight $k+1$ vector $e(-1)^{k+1}\mathbf{1}$ [25]. The quotient algebra $L(k, 0) = V(k, 0)/\mathcal{J}$ is the simple vertex operator algebra associated to an affine Lie algebra $\widehat{sl}_2$ of type $A_1^{(1)}$ with level k. The subspace $V_{\widehat{\mathfrak{h}}}(k, 0)$ of $V(k, 0)$ spanned by $h(-i_1) \cdots h(-i_p)\mathbf{1}$ for $i_1 \geq \cdots \geq i_p \geq 1$ and $p \geq 0$ is a vertex operator subalgebra of $V(k, 0)$ associated to the Heisenberg algebra. The parafermion vertex operator algebra K_0 is defined as the commutant of $V_{\widehat{\mathfrak{h}}}(k, 0)$ in $L(k, 0)$, that is,

$$K_0 = \{v \in L(k, 0) \mid h(m)v = 0 \text{ for } m \geq 0\}.$$

It was proved that K_0 is a simple vertex operator algebra and the irreducible K_0-modules $M^{i,j}$ for $0 \leq i \leq k, 0 \leq j \leq k-1$ were constructed in [14]. Note that $K_0 = M^{0,0}$. It was also proved that $M^{i,j} \cong M^{k-i,k-i+j}$ as K_0-module in [14, Theorem 4.4] and moreover, Theorem 8.2 in [4] showed that the $\frac{k(k+1)}{2}$ irreducible K_0-modules $M^{i,j}$ for $0 \leq i \leq k, 0 \leq j \leq i-1$ constructed in [14] form a complete set of isomorphism classes of irreducible K_0-module. Moreover, K_0 is C_2-cofinite [4] and rational [16].

Recall from [14] that $L = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_k$ with $\langle \alpha_p, \alpha_q \rangle = 2\delta_{pq}$. $V_L = M(1) \otimes \mathbb{C}[L]$ is the lattice vertex operator algebra associated with the lattice L. Let $\gamma = \alpha_1 + \cdots + \alpha_k$. Thus $\langle \gamma, \gamma \rangle = 2k$. It is well known that the vertex operator algebra

associated with a positive definite even lattice is rational [7]. And any irreducible module for the lattice vertex operator algebra $V_{\mathbb{Z}\gamma}$ is isomorphic to one of $V_{\mathbb{Z}\gamma+n\gamma/2k}$, $0 \leq n \leq 2k-1$ [7]. Let $L(k,i)$ for $0 \leq i \leq k$ be the irreducible modules for the rational vertex operator algebra $L(k,0)$. The following result was due to [14].

Lemma 2.1. *$L(k,i) = \bigoplus_{j=0}^{k-1} V_{\mathbb{Z}\gamma+(i-2j)\gamma/2k} \otimes M^{i,j}$ as $V_{\mathbb{Z}\gamma}$-modules.*

3. Quantum dimensions for irreducible K_0-modules

In this section, we first recall the notion and some basic facts about quantum dimension from [8]. Then we determine the quantum dimensions of the irreducible K_0-modules.

Let $(V, Y, 1, \omega)$ be a vertex operator algebra (see [21], [27]). We define weak module, module and admissible module following [10, 11]. Let $W\{z\}$ denote the space of W-valued formal series in arbitrary complex powers of z for a vector space W.

Definition 3.1. A *weak V-module* M is a vector space with a linear map

$$Y_M : V \to (\mathrm{End}M)\{z\}$$
$$v \mapsto Y_M(v,z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1} \quad (v_n \in \mathrm{End}M),$$

which satisfies the following conditions for $u, v \in V, w \in M$:

$$u_n w = 0 \text{ for } n \gg 0,$$
$$Y_M(\mathbf{1}, z) = Id_M,$$
$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_M(u,z_1) Y_M(v,z_2) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y_M(v,z_2) Y_M(u,z_1)$$
$$= \delta\left(\frac{z_1 - z_0}{z_2}\right) Y_M(Y(u,z_0)v, z_2),$$

where $\delta(z) = \sum_{n\in\mathbb{Z}} z^n$.

Recall that the canonical central element C of the Virasoro algebra acts on V as a scalar $c \in \mathbb{C}$, called the central charge of V. We remark that the component operators of $Y_M(\omega, z) = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2}$ still satisfy the Virasoro algebra relation on M [10] with the same central charge c.

Definition 3.2. A *V-module* is a weak V-module M which carries a $\mathbb{C}$-grading $M = \bigoplus_{\lambda\in\mathbb{C}} M_\lambda$, where $M_\lambda = \{w \in M | L(0)w = \lambda w\}$. Moreover, we require that $\dim M_\lambda$ is finite and for fixed λ, $M_{\lambda+n} = 0$ for all small enough integers n.

Definition 3.3. An *admissible V-module* $M = \bigoplus_{n\in\mathbb{Z}_+} M(n)$ is a $\mathbb{Z}_+$-graded weak module such that $u_m M(n) \subset M(\mathrm{wt}\, u - m - 1 + n)$ for homogeneous $u \in V$ and $m, n \in \mathbb{Z}$.

Definition 3.4. (1) A vertex operator algebra V is called *rational* if the admissible module category is semisimple.

(2) V is called C_2-cofinite if $\dim V/C_2(V) < \infty$ where $C_2(V)$ is a subspace of V spanned by $u_{-2}v$ for $u, v \in V$.

(3) V is of CFT type if $V = \bigoplus_{n\geq 0} V_n$ and $\dim V_0 = 1$.

The following lemma about rational vertex operator algebras is well known [12].

Lemma 3.5. *If V is rational and M is an irreducible admissible V-module, then*

(1) M is a V-module and there exists a $\lambda \in \mathbb{C}$ such that $M = \bigoplus_{n\in\mathbb{Z}_+} M_{\lambda+n}$ where $M_\lambda \neq 0$. And λ is called the conformal weight of M.

(2) There are only finitely many irreducible admissible V-modules up to isomorphism.

Let $M = \bigoplus_{\lambda\in\mathbb{C}} M_\lambda$ be a V-module. Set $M' = \bigoplus_{\lambda\in\mathbb{C}} M_\lambda^*$, the restricted dual of M, where $M_\lambda^* = \mathrm{Hom}_{\mathbb{C}}(M_\lambda, \mathbb{C})$. It was proved in [20] that $M^{'}$ is naturally a V-module where the vertex operator $Y_{M'}(v, z)$ is defined for $v \in V$ via

$$\langle Y_{M'}(v, z)f, u\rangle = \langle f, Y_M(e^{zL(1)}(-z^{-2})^{L(0)}v, z^{-1})u\rangle,$$

where $\langle f, w\rangle = f(w)$ is the natural pairing $M' \times M \to \mathbb{C}$. The V-module $M^{'}$ is called the *contragredient module* of M. A V-module M is called *self-dual* if M and $M^{'}$ are isomorphic V-modules. The following result was proved in [26].

Lemma 3.6. *Let V be a simple vertex operator algebra such that $L(1)V_{(1)} \neq V_{(0)}$. Then V is self-dual.*

Remark 3.7. Note that the weight one subspace of K_0 is zero. By using lemma 3.6, parafermion vertex operator algebra K_0 is obviously self-dual.

Now we recall from [20] the notions of intertwining operators and fusion rules.

Definition 3.8. Let $(V,\ Y)$ be a vertex operator algebra and let $(W^1,\ Y^1)$, $(W^2,\ Y^2)$ and $(W^3,\ Y^3)$ be V-modules. An *intertwining operator* of type $\begin{pmatrix} W^3 \\ W^1\ W^2 \end{pmatrix}$ is a linear map

$$I(\cdot,\ z):\ W^1 \to \mathrm{Hom}(W^2,\ W^3)\{z\}$$

$$w^1 \to I(w^1,\ z) = \sum_{n\in\mathbb{C}} w_n^1 z^{-n-1},$$

satisfying for $v \in V$, $w^i \in W^i$ with $i = 1, 2$ and $\lambda \in \mathbb{C}$:

(1) $w^1_{\lambda+n}w^2 = 0$ for $n \in \mathbb{Z}$ sufficiently large.

(2) $I(L(-1)w^1,\ z) = \frac{d}{dz}I(w^1,\ z)$.

(3) The Jacobi identity holds:

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)Y^3(v,\ z_1)I(w^1,\ z_2) - z_0^{-1}\delta\left(\frac{-z_2 + z_1}{z_0}\right)I(w^1,\ z_2)Y^2(v,\ z_1)$$

$$= z_2^{-1}\left(\frac{z_1 - z_0}{z_2}\right)I(Y^1(w^1,\ z_0)v,\ z_2).$$

The space of all intertwining operators of type $\begin{pmatrix} W^3 \\ W^1\ W^2 \end{pmatrix}$ is denoted by

$$I_V\begin{pmatrix} W^3 \\ W^1\ W^2 \end{pmatrix}.$$

Let $N_{W^1,\ W^2}^{W^3} = \dim I_V\begin{pmatrix} W^3 \\ W^1\ W^2 \end{pmatrix}$. These integers $N_{W^1,\ W^2}^{W^3}$ are usually called the *fusion rules*.

Definition 3.9. Let V be a vertex operator algebra, and W^1, W^2 be two V-modules. A module (W, I), where $I \in I_V \begin{pmatrix} & W & \\ W^1 & & W^2 \end{pmatrix}$, is called a *tensor product* (or fusion product) of W^1 and W^2 if for any V-module M and $\mathcal{Y} \in I_V \begin{pmatrix} & M & \\ W^1 & & W^2 \end{pmatrix}$, there is a unique V-module homomorphism $f : W \to M$, such that $\mathcal{Y} = f \circ I$. As usual, we denote (W, I) by $W^1 \boxtimes_V W^2$.

Remark 3.10. It is well known that if V is rational, then for any two irreducible V-modules W^1 and W^2, the fusion product $W^1 \boxtimes_V W^2$ exists and

$$W^1 \boxtimes_V W^2 = \sum_W N_{W^1,\ W^2}^W W,$$

where W runs over the set of equivalence classes of irreducible V-modules.

Now we recall some notions about quantum dimensions.

Definition 3.11. Let $M = \bigoplus_{n \in \mathbb{Z}_+} M_{\lambda+n}$ be a V-module, the *formal character* of M is defined as

$$\mathrm{ch}_q M = \mathrm{tr}_M q^{L(0)-c/24} = q^{\lambda - c/24} \sum_{n \in \mathbb{Z}_+} (\dim M_{\lambda+n})\, q^n,$$

where c is the central charge of the vertex operator algebra V and λ is the conformal weight of M.

We now assume that the vertex operator algebra V is rational and C_2-cofinite. It is proved [29] that $\mathrm{ch}_q M$ converges to a holomorphic function in the domain $|q| < 1$. We denote the holomorphic function $\mathrm{ch}_q M$ by $Z_M(\tau)$. Here and below, τ is in the upper half plane $\mathbb{H}$ and $q = e^{2\pi i \tau}$.

Let $M^0, \cdots, M^d$ be the inequivalent irreducible V-modules with corresponding conformal weights λ_i and $M^0 \cong V$. It is proved in [29] that for any i

$$Z_{M^i}\left(-\frac{1}{\tau}\right) = \sum_{j=0}^{d} S_{i,j} Z_{M^j}(\tau).$$

The complex matrix $S = (S_{i,j})_{i,j=0}^d$ is called the *S-matrix*.

The following definition of quantum dimension was introduced in [8].

Definition 3.12. Let V be a vertex operator algebra and M a V-module such that $Z_V(\tau)$ and $Z_M(\tau)$ exist. The quantum dimension of M over V is defined as

$$\mathrm{qdim}_V M = \lim_{y \to 0} \frac{Z_M(iy)}{Z_V(iy)},$$

where y is real and positive.

The following result was obtained in [8, Lemma 4.2].

Lemma 3.13. *Let V be a simple, rational and C_2-cofinite vertex operator algebra of CFT type with $V \cong V'$. Let M^i for $0 \le i \le d$ be the inequivalent irreducible V-modules with corresponding conformal weights λ_i and $M^0 \cong V$. Assume $\lambda_0 = 0$ and $\lambda_i > 0$ $\forall i \neq 0$. Then $\mathrm{qdim}_V M^i = \frac{S_{i,0}}{S_{0,0}}$.*

We now also assume that V is of CFT type with $V \cong V'$, the conformal weights λ_i of M^i are positive for all $i > 0$. From Remark 3.7 and statements in Section 2, the parafermion vertex operator algebra K_0 satisfies all the assumptions.

The following result shows that the quantum dimensions are multiplicative under tensor product [8] .

Proposition 3.14. *Let V and M_i for $0 \leq i \leq d$ be as in Lemma 3.13. Then*

$$qdim_V \left(M^i \boxtimes M^j\right) = qdim_V M^i \cdot qdim_V M^j$$

for i, $j = 0, \cdots, d$.

Before giving the main result of this section, we recall the following character formula of irreducible K_0-modules $M^{i,j}$ which is given in [6, 24]:

$$\mathrm{ch} M^{i,j} = \eta(\tau) c^i_{i-2j}(\tau),$$

where $\eta(\tau) = q^{-1/24} \prod_{n\geq 1}(1-q^n)$ is the Dedekind η-function and $c^i_{i-2j}(\tau)$ are the string functions [25]. Note that k, l and m in [24] are k, i and $i-2j$, respectively in our notation.

Theorem 3.15. *The quantum dimensions for all irreducible K_0-modules $M^{m,n}$ are*

$$qdim_{K_0} M^{m,n} = \frac{\sin \frac{\pi(m+1)}{k+2}}{\sin \frac{\pi}{k+2}}$$

for $0 \leq m \leq k$, $0 \leq n \leq k-1$, where $M^{m,n}$ are the irreducible modules of K_0 constructed in [14].

Proof. Let $M^{m,n}(\tau)$ denote the character of $M^{m,n}$ for $0 \leq m \leq k,\ 0 \leq n \leq k-1$. The S-modular transformation of characters has the following form [24], [25]:

$$M^{m,n}(-\frac{1}{\tau}) = \sum_{m',n'} S^{m',n'}_{m,n} M^{m',n'}(\tau),$$

where $S^{m',n'}_{m,n} = (k(k+2))^{-\frac{1}{2}} \exp \frac{i\pi(m-2n)(m'-2n')}{k} \sin \frac{\pi(m+1)(m'+1)}{k+2}$. From [14] and [4], we see that K_0 has $\frac{k(k+2)}{2}$ irreducible modules $M^{m,n}$ with the conformal weights

$$\lambda_{m,n} = \frac{1}{2k(k+2)}(k(m-2n) - (m-2n)^2 + 2kn(m-n+1))$$

for $0 \leq m \leq k,\ 0 \leq n \leq m-1$. It is easy to check that $\lambda_{k,0} = 0$ and $\lambda_{m,n} > 0$ for $(m,n) \neq (k,0)$. Thus by using Lemma 3.13, we have

$$\mathrm{qdim}_{K_0} M^{m,n} = \frac{S^{0,0}_{m,n}}{S^{0,0}_{0,0}} = \frac{\sin \frac{\pi(m+1)}{k+2}}{\sin \frac{\pi}{k+2}}.$$

□

4. Fusion rule for irreducible K_0-modules

In this section, we give the fusion rules for irreducible K_0-modules. First we fix some notation. Let W^1, W^2, W^3 be irreducible K_0-modules. In the following, we use $I\begin{pmatrix} W^3 \\ W^1\, W^2 \end{pmatrix}$ to denote the space $I_{K_0}\begin{pmatrix} W^3 \\ W^1\, W^2 \end{pmatrix}$ of all intertwining

operators of type $\begin{pmatrix} W^3 \\ W^1\, W^2 \end{pmatrix}$, and use $W^1 \boxtimes W^2$ to denote the fusion product $W^1 \boxtimes_{K_0} W^2$ for simplicity.

We recall the fusion rules for the affine vertex operator algebra of type $A_1^{(1)}$ [28] for later use.

Lemma 4.1.

$$L(k,i) \boxtimes_{L(k,0)} L(k,j) = \sum_l L(k,l),$$

where $|i-j| \le l \le i+j,\ i+j+l \in 2\mathbb{Z},\ i+j+l \le 2k$.

Theorem 4.2. *The fusion rule for the irreducible modules of parafermion vertex operator algebra* K_0 *is as follows:*

$$M^{i,i'} \boxtimes M^{j,j'} = \sum_l M^{l,\overline{\frac{1}{2}(2i'-i+2j'-j+l)}}, \tag{4.1}$$

where $\overline{a}$ *means the residue of the integer* a *modulo* k, $0 \le i,j \le k, 0 \le i', j' \le k-1$, $|i-j| \le l \le i+j,\ i+j+l \in 2\mathbb{Z},\ i+j+l \le 2k$. *Moreover, with fixed* i, i', j, j', $M^{l,\overline{\frac{1}{2}(2i'-i+2j'-j+l)}}$ *for* $|i-j| \le l \le i+j,\ i+j+l \in 2\mathbb{Z},\ i+j+l \le 2k$ *are inequivalent irreducible modules.*

Proof. We take $V = L(k,0), U = V_{\mathbb{Z}\gamma} \otimes K_0$ in Proposition 2.9 of [1], from Lemma 2.1, we see that

$$\dim I_V \begin{pmatrix} L(k,l) \\ L(k,i)\, L(k,j) \end{pmatrix} \le \dim I_U \begin{pmatrix} L(k,l) \\ V_{\mathbb{Z}\gamma+\frac{(i-2i')\gamma}{2k}} \otimes M^{i,i'}\ V_{\mathbb{Z}\gamma+\frac{(j-2j')\gamma}{2k}} \otimes M^{j,j'} \end{pmatrix}$$

for $0 \le i,j,l \le k, 0 \le i', j' \le k-1$. Note that $L(k,l) = \bigoplus_{l'=0}^{k-1} V_{\mathbb{Z}\gamma+(l-2l')\gamma/2k} \otimes M^{l,l'}$, thus we have

$$\dim I_U \begin{pmatrix} L(k,l) \\ V_{\mathbb{Z}\gamma+\frac{(i-2i')\gamma}{2k}} \otimes M^{i,i'}\ V_{\mathbb{Z}\gamma+\frac{(j-2j')\gamma}{2k}} \otimes M^{j,j'} \end{pmatrix} = \sum_{l'=0}^{k-1} D((i,i'),(j,j'),(l,l')),$$

where

$$D((i,i'),(j,j'),(l,l')) = \dim I_U \begin{pmatrix} V_{\mathbb{Z}\gamma+\frac{(l-2l')\gamma}{2k}} \otimes M^{l,l'} \\ V_{\mathbb{Z}\gamma+\frac{(i-2i')\gamma}{2k}} \otimes M^{i,i'}\ V_{\mathbb{Z}\gamma+\frac{(j-2j')\gamma}{2k}} \otimes M^{j,j'} \end{pmatrix}.$$

Using Theorem 2.10 of [1], we have

$$D((i,i'),(j,j'),(l,l')) = \dim I_{V_{\mathbb{Z}\gamma}} \begin{pmatrix} V_{\mathbb{Z}\gamma+\frac{(l-2l')\gamma}{2k}} \\ V_{\mathbb{Z}\gamma+\frac{(i-2i')\gamma}{2k}}\ V_{\mathbb{Z}\gamma+\frac{(j-2j')\gamma}{2k}} \end{pmatrix} \cdot \dim I_{K_0} \begin{pmatrix} M^{l,l'} \\ M^{i,i'}\ M^{j,j'} \end{pmatrix}.$$

Recall from [9] that the fusion rule for lattice vertex operator algebras is given by

$$V_{\mathbb{Z}\gamma+\lambda} \boxtimes_{V_{\mathbb{Z}\gamma}} V_{\mathbb{Z}\gamma+\mu} = V_{\mathbb{Z}\gamma+\lambda+\mu}$$

for $\lambda, \mu \in (\mathbb{Z}\gamma)^\circ$, where $(\mathbb{Z}\gamma)^\circ$ is the dual lattice of $\mathbb{Z}\gamma$. Using this together with the fusion rule for the affine vertex operator algebra given in Lemma 4.1, we see that if $l' \neq \frac{l-i+2i'-j+2j'}{2}$, then

$$D((i,i'),(j,j'),(l,l')) = 0.$$

Thus for any l satisfying $|i-j| \leq l \leq i+j,\ i+j+l \in 2\mathbb{Z},\ i+j+l \leq 2k$, we have

$$\dim I_{K_0}\begin{pmatrix} M^{l,\overline{\frac{1}{2}(2i'-i+2j'-j+l)}} \\ M^{i,i'}\ M^{j,j'} \end{pmatrix} \geq 1.$$

In the following, we use the quantum dimension of the irreducible modules over K_0 to prove that

$$\dim I_{K_0}\begin{pmatrix} M^{l,\overline{\frac{1}{2}(2i'-i+2j'-j+l)}} \\ M^{i,i'}\ M^{j,j'} \end{pmatrix} = 1.$$

From Proposition 3.14, we know

$$\mathrm{qdim}_{K_0}\left(M^{i,i'} \boxtimes M^{j,j'}\right) = \mathrm{qdim}_{K_0} M^{i,i'} \cdot \mathrm{qdim}_{K_0} M^{j,j'}.$$

We also have the fusion product

$$M^{i,i'} \boxtimes M^{j,j'} = \sum_{0\leq l\leq k, 0\leq l'\leq l-1} N^{(l,l')}_{(i,i'),(j,j')} M^{l,l'}.$$

Theorem 3.15 shows that

$$\mathrm{qdim}_{K_0} M^{i,i'} = \frac{\sin\frac{\pi(i+1)}{k+2}}{\sin\frac{\pi}{k+2}}.$$

Thus we only need to prove that the following identity holds:

$$\frac{\sin\frac{\pi(i+1)}{k+2}\cdot\sin\frac{\pi(j+1)}{k+2}}{\sin\frac{\pi}{k+2}\cdot\sin\frac{\pi}{k+2}} = \sum_l \frac{\sin\frac{\pi(l+1)}{k+2}}{\sin\frac{\pi}{k+2}}$$

for $0 \leq i,j \leq k, 0 \leq i', j' \leq k-1$, $|i-j| \leq l \leq i+j,\ i+j+l \in 2\mathbb{Z},\ i+j+l \leq 2k$. This identity is equivalent to the following identity:

$$\cos\frac{\pi(i+j+2)}{k+2} - \cos\frac{\pi(i-j)}{k+2} = \sum_l (\cos\frac{\pi(l+2)}{k+2} - \cos\frac{\pi l}{k+2}). \tag{4.2}$$

We note that if $i+j < k$, then $l_{\min} = i-j$, $l_{\max} = i+j$, thus (4.2) holds. If $i+j > k$, $l_{\min} = i-j$, $l_{\max} = i+j-2n$ for some n satisfying that $i+j-2n+i+j = 2k$, that is, $l_{\max} = 2k-i-j$, thus (4.2) also holds.

Now we prove that the modules $M^{l,\overline{\frac{1}{2}(2i'-i+2j'-j+l)}}$ appearing in the sum (4.1) are not isomorphic to each other. If $M^{l,\overline{\frac{1}{2}(2i'-i+2j'-j+l)}} \cong M^{l_1,\overline{\frac{1}{2}(2i'-i+2j'-j+l_1)}}$ for some l, l_1 satisfying $|i-j| \leq l, l_1 \leq i+j,\ i+j+l \in 2\mathbb{Z},\ i+j+l \leq 2k, i+j+l_1 \in 2\mathbb{Z},\ i+j+l_1 \leq 2k$. Since we know that $M^{i,j} \cong M^{k-i,k-i+j}$ as K_0-module for $0 \leq i \leq k, 0 \leq j \leq k-1$ and the $\frac{k(k+1)}{2}$ irreducible K_0-modules $M^{i,j}$ for $0 \leq i \leq k, 0 \leq j \leq i-1$ exhaust all the isomorphism classes of irreducible K_0-modules, it follows that $l_1 = k-l$ and

$$\overline{k-l+\frac{1}{2}(2i'-i+2j'-j+k-l)} = \overline{\frac{1}{2}(2i'-i+2j'-j+k-l)},$$

which is impossible by a direct calculation. This proves the assertion. □

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