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THE KADEC-PEŁCZYŃSKI THEOREM IN L^p, $1 \leq p < 2$

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Abstract. By a classical result of Kadec and Pełczyński (1962), every normalized weakly null sequence in L^p, $p > 2$, contains a subsequence equivalent to the unit vector basis of ℓ^2 or to the unit vector basis of ℓ^p. In this paper we investigate the case $1 \leq p < 2$ and show that a necessary and sufficient condition for the first alternative in the Kadec-Pełczyński theorem is that the limit random measure μ of the sequence satisfies $\int_{\mathbb{R}} x^2 d\mu(x) \in L^{p/2}$.

1. Introduction

Call two sequences (x_n) and (y_n) in a Banach space $(B, \|\cdot\|)$ equivalent if there exists a constant $K > 0$ such that

$$K^{-1}\Big\|\sum_{i=1}^{n} a_i x_i\Big\| \leq \Big\|\sum_{i=1}^{n} a_i y_i\Big\| \leq K\Big\|\sum_{i=1}^{n} a_i x_i\Big\|$$

for every $n \geq 1$ and every $(a_1, \dots, a_n) \in \mathbb{R}^n$. By a classical theorem of Kadec and Pełczyński [11], any normalized weakly null sequence (x_n) in $L^p(0,1)$, $p > 2$, has a subsequence equivalent to the unit vector basis of ℓ^2 or to the unit vector basis of ℓ^p. In the case when $\{|x_n|^p, n \geq 1\}$ is uniformly integrable, the first alternative holds, while if the functions (x_n) have disjoint support, the second alternative holds trivially. The general case follows via a subsequence splitting argument as in [11].

The purpose of the present paper is to investigate the case $1 \leq p < 2$ and to give a necessary and sufficient condition for the first alternative in the Kadec-Pełczyński theorem. To formulate our result, we use probabilistic terminology. Let $1 \leq p < 2$ and let (X_n) be a sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$; assume that $\{|X_n|^p,\ n \geq 1\}$ is uniformly integrable and $X_n \to 0$ weakly in L^p. (This is meant as $\lim_{n\to\infty} \mathbb{E}(X_n Y) = 0$ for all $Y \in L^q$ where $1/p + 1/q = 1$. To avoid confusion with weak convergence of probability measures and distributions, the latter will be called convergence in distribution and denoted by $\xrightarrow{\mathcal{D}}$.) Using the terminology of [5], we call a sequence (X_n) of random variables *determining* if it has a limit distribution relative to any set A in the probability space with $P(A) > 0$, i.e., for any $A \subset \Omega$ with $P(A) > 0$ there exists a distribution function F_A such that

$$\lim_{n\to\infty} P(X_n \leq t \mid A) = F_A(t)$$

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for all continuity points t of F_A. Here $P(\cdot|A)$ denotes conditional probability given A. (This concept is the same as that of stable convergence, introduced in [15].) Since $\{|X_n|^p,\, n \geq 1\}$ is uniformly integrable, the sequence (X_n) is tight and thus by an extension of the Helly-Bray theorem (see e.g. [5]), it contains a determining subsequence. Hence in the sequel we can assume, without loss of generality, that the sequence (X_n) itself is determining. As is shown in [1], [5], for any determining sequence (X_n) there exists a random measure μ (i.e. a measurable map from $(\Omega, \mathcal{F}, P)$ to $(\mathcal{M}, \pi)$, where $\mathcal{M}$ is the set of probability measures on $\mathbb{R}$ and π is the Prohorov distance; see Section 3) such that for any A with $P(A) > 0$ and any continuity point t of F_A we have

$$F_A(t) = \mathbb{E}_A(\mu(-\infty, t]), \tag{1.1}$$

where $\mathbb{E}_A$ denotes conditional expectation given A. We call μ the *limit random measure* of (X_n). We will prove the following result.

Theorem 1.1. *Let $1 \leq p < 2$ and let (X_n) be a determining sequence of random variables such that $\|X_n\|_p = 1$ $(n = 1, 2, \ldots)$, $\{|X_n|^p,\, n \geq 1\}$ is uniformly integrable and $X_n \to 0$ weakly in L^p. Let μ be the limit random measure of (X_n). Then there exists a subsequence (X_{n_k}) equivalent to the unit vector basis of ℓ^2 if and only if*

$$\int_{-\infty}^{\infty} x^2 d\mu(x) \in L^{p/2}. \tag{1.2}$$

By assuming the uniform integrability of $|X_n|^p$, we exclude "spike" situations leading to a subsequence equivalent to the unit vector basis of ℓ^p as in the Kadec-Pelczyński theorem. It is easily seen that (1.2) (and in fact $\int_{-\infty}^{\infty} x^2 d\mu(x) < \infty$ a.s.) imply that for any $\delta > 0$ there exists a set $A \subset \Omega$ with $P(A) \geq 1 - \delta$ and a subsequence (X_{n_k}) such that

$$\sup_{k \geq 1} \int_A |X_{n_k}|^2 dP < \infty.$$

Thus the first alternative in the Kadec-Pełczyński theorem 'almost' implies bounded L^2 norms.

Call a sequence (X_n) of random variables in L^p *almost symmetric* if for any $\varepsilon > 0$ there exists a $K = K(\varepsilon)$ such that for any $k \geq 1$, any indices $j_1 > j_2 > \ldots > j_k \geq K$, any permutation $(\sigma(j_1), \ldots, \sigma(j_k))$ of $(j_1, \ldots, j_k)$ and any $(a_1, \ldots, a_k) \in \mathbb{R}^k$ we have

$$(1-\varepsilon)\|\sum_{i=1}^{k} a_i X_{j_i}\|_p \leq \|\sum_{i=1}^{k} a_i X_{\sigma(j_i)}\|_p \leq (1+\varepsilon)\|\sum_{i=1}^{k} a_i X_{j_i}\|_p\,.$$

Once in Theorem 1.1 we found a subsequence (X_{n_k}) equivalent to the unit vector basis of ℓ^2, a result of Guerre [9] implies the existence of a further subsequence (X_{m_k}) of (X_{n_k}) which is almost symmetric. Note that this conclusion also follows from the proof of Theorem 1.1. Guerre and Raynaud [10] also showed that for any $1 \leq p < q < 2$ there exists a sequence (X_n) in L^p, equivalent to the unit vector basis of ℓ^q, but not having an almost symmetric subsequence. No characterization for the existence of almost symmetric subsequences of (X_n) in terms of the limit random measure of (X_n) or related quantities is known.

2. Some lemmas

The necessity of the proof of Theorem 1.1 depends on a general structure theorem for lacunary sequences proved in [3] (see Theorem 2 of [3] and the definition preceding it); for the convenience of the reader we state it here as a lemma.

Lemma 2.1. *Let (X_n) be a determining sequence of r.v.'s and (ε_n) a positive numerical sequence tending to 0. Then, if the underlying probability space is rich enough, there exists a subsequence (X_{m_k}) and a sequence (Y_k) of discrete r.v.'s such that*

$$P\big(|X_{m_k} - Y_k| \ge \varepsilon_k\big) \le \varepsilon_k, \quad k = 1, 2, \dots, \tag{2.1}$$

and for each $k > 1$ the atoms of the σ-field $\sigma\{Y_1, \dots, Y_{k-1}\}$ can be divided into two classes Γ_1 and Γ_2 such that

(i) $\sum_{B \in \Gamma_1} P(B) \le \varepsilon_k$;

(ii) *For any $B \in \Gamma_2$ there exist P_B-independent r.v.'s $\{Z_j^{(B)}, j = k, k+1, \dots\}$ defined on B with common distribution function F_B such that*

$$P_B\big(|Y_j - Z_j^{(B)}| \ge \varepsilon_k\big) \le \varepsilon_k, \quad j = k, k+1, \dots. \tag{2.2}$$

Here F_B denotes the limit distribution of (X_n) relative to B and P_B denotes conditional probability given B.

Note that, instead of relation (2.1), in Theorem 2 of [3] the conclusion is $\sum_{k=1}^{\infty} |X_{m_k} - Y_k| < \infty$ a.s., but after a further thinning, (2.1) will also hold. The phrase "the underlying probability space is rich enough" is meant in Lemma 2.1 in the sense that on the underlying space there exists a sequence of independent r.v.'s, uniformly distributed over $(0, 1)$ and also independent of the sequence (X_n). Clearly, this condition can be guaranteed by a suitable enlargement of the probability space not changing the distribution of (X_n) and μ and thus this assumption can be assumed without loss of generality.

Lemma 2.1 means that every tight sequence of r.v.'s has a subsequence which can be closely approximated by an exchangeable sequence having a very simple structure, namely which is i.i.d. on each set of a suitable partition of the probability space. This fact is an "effective" form of the general subsequence principle of Aldous [1] (for a related result see Berkes and Rosenthal [5]) and reduces the studied problem to the i.i.d. case which will be handled by the classical concentration technique of Lévy [12], as improved by Esseen [7].

Lemma 2.2. *Let $X_1, X_2, \dots, X_n$ be i.i.d. random variables with distribution function F and put $S_n = X_1 + \dots + X_n$. Then for any $t > 0$ we have*

$$P\left(\left|S_n\right| \le t\right) \le A\frac{t}{\sqrt{n}}\left[\int\limits_{|x|<t} x^2 dF(x) - 2\left(\int\limits_{|x|<t} x dF(x)\right)^2\right]^{-1/2} \tag{2.3}$$

provided the difference on the right-hand side is positive and $\int_{|x|<t} dF(x) \ge 1/2$. Here A is an absolute constant.

Proof. Let F^* denote the distribution function obtained from F by symmetrization. The left-hand side of (2.3) is clearly bounded by $Q_{S_n}(2t)$, where $Q_{S_n}(\lambda) = \sup_x P(x \le S_n \le x + \lambda)$ is the concentration function of S_n. By a well-known

concentration function inequality of Esseen (see [7], formula (3.3)) we have

$$Q_{S_n}(\lambda) \le An^{-1/2}\left(\lambda^{-2}\int_{|x|<\lambda} x^2 dF^*(x) + \int_{|x|\ge\lambda} dF^*(x)\right)^{-1/2}$$

$$\le A\lambda\, n^{-1/2}\left(\int_{|x|<\lambda} x^2 dF^*(x)\right)^{-1/2} \tag{2.4}$$

for any $\lambda > 0$, where A is an absolute constant. Thus the left-hand side of (2.3) is bounded by the last expression in (2.4) with $\lambda = 2t$ and thus to prove (2.3) it suffices to show that $\int_{|x|<t} dF(x) \ge 1/2$ implies

$$\int\limits_{|x|<2t} x^2 dF^*(x) \ge \int\limits_{|x|<t} x^2 dF(x) - 2\Big(\int\limits_{|x|<t} x dF(x)\Big)^2. \tag{2.5}$$

Let ξ and η be independent r.v.'s with distribution function F and set

$$C = \{|\xi - \eta| < 2t\}, \quad D = \{|\xi| < t, |\eta| < t\}.$$

Then

$$\int\limits_{|x|<2t} x^2 dF^*(x) = \int\limits_C (\xi-\eta)^2 dP \ge \int\limits_D (\xi-\eta)^2 dP$$

$$= 2\int\limits_{|\xi|<t} \xi^2 dP \cdot P(|\eta| < t) - 2\Big(\int\limits_{|\xi|<t} \xi dP\Big)^2 \ge \int\limits_{|\xi|<t} \xi^2 dP - 2\Big(\int\limits_{|\xi|<t} \xi dP\Big)^2$$

since $P(|\eta| < t) \ge 1/2$. Thus (2.5) is valid.

Lemma 2.3. *Let (X_n) be a determining sequence of r.v.'s with limit random distribution function $F_\bullet$. Then for any set $A \subset \Omega$ with $P(A) > 0$ we have*

$$\mathbb{E}_A\Big(\int\limits_{-\infty}^{+\infty} x^2 dF_\bullet(x)\Big) = \int\limits_{-\infty}^{+\infty} x^2 dF_A(x) \tag{2.6}$$

in the sense that if one side is finite, then the other side is also finite and the two sides are equal. The statement remains valid if in (2.6) *we replace the intervals of integration by $(-t, t)$, provided t and $-t$ are continuity points of F_Ω.*

We used here the notation $F_\bullet$ to distinguish it from the ordinary limit distribution function of (X_n).

Proof. Assume that t and $-t$ are continuity points of F_Ω. As observed in [5, p. 482], t and $-t$ are continuity points of $F_\bullet$ with probability 1 (and hence also for F_A for any $A \subset \Omega$ with $P(A) > 0$) and thus almost surely

$$\int_{|x|<t} x^2 dF_\bullet(x) = -\left[x^2(1 - F_\bullet(x) + F_\bullet(-x))\right]_{-t}^{t} + \int_{|x|<t} (1 - F_\bullet(x) + F_\bullet(-x))2x dx$$

as it is seen by splitting the integral on the left-hand side into subintegrals over $(-t, 0)$ and $(0, t)$ (the integral over $\{0\}$ clearly equals 0) and using integration by parts. The same formula holds with $F_\bullet$ replaced by F_A. Integrating the last relation over $A \subset \Omega$ and using (1.1) and Fubini's theorem, we get the validity of (2.6) over $(-t, t)$. Letting $t \to \infty$ we get (2.6) over $(-\infty, \infty)$.

For the following lemma (which is the key tool for the proof of the sufficiency part of Theorem 1.1) we need some definitions. Given probability measures ν_n, ν on the Borel sets of a separable metric space (S, d) we say that $\nu_n \xrightarrow{\mathcal{D}} \nu$ if

$$\int_S f(x)\nu_n(dx) \longrightarrow \int_S f(x)\nu(dx) \quad \text{as} \quad n \to \infty \tag{2.7}$$

for every bounded, real valued continuous function f on S. (For equivalent definitions and properties of this convergence see [4]). (2.7) is clearly equivalent to

$$Ef(Z_n) \longrightarrow Ef(Z) \tag{2.8}$$

where Z_n, Z are r.v.'s valued in (S, d) (i.e., measurable maps from some probability space to (S, d)) with distribution ν_n, ν. A class $\mathcal{G}$ of real valued functions on S is called *locally equicontinuous* if for every $\varepsilon > 0$ and $x \in S$ there is a $\delta = \delta(\varepsilon, x) > 0$ such that $y \in S$, $d(x, y) \le \delta$ imply $|f(x) - f(y)| \le \varepsilon$ for every $f \in \mathcal{G}$.

Lemma 2.4 (Ranga Rao [14])**.** *Let (S, d) be a separable metric space and ν, ν_n $(n = 1, 2, \ldots)$ probability measures on the Borel sets of (S, d) such that $\nu_n \xrightarrow{\mathcal{D}} \nu$. Let $\mathcal{G}$ be a class of real valued functions on (S, d) such that*

(a) $\mathcal{G}$ is locally equicontinuous.

(b) There exists a continuous function $g \ge 0$ on S such that $|f(x)| \le g(x)$ for all $f \in \mathcal{G}$ and $x \in S$ and

$$\int_S g(x)\nu_n(dx) \longrightarrow \int_S g(x)\nu(dx) \ (< \infty) \quad \textit{as} \quad n \to \infty. \tag{2.9}$$

Then

$$\int_S f(x)\nu_n(dx) \longrightarrow \int_S f(x)\nu(dx) \quad \textit{as} \quad n \to \infty \tag{2.10}$$

uniformly in $f \in \mathcal{G}$.

3. Proof of Theorem 1.1

Let $(\Omega, \mathcal{F}, P)$ be the probability space of the X_n's and $\mathbf{X} = (X_1, X_2, \ldots)$; let further μ be the limit random measure of (X_n). Let (Y_n) be a sequence of r.v.'s on $(\Omega, \mathcal{F}, P)$ such that, given $\mathbf{X}$ and μ, the r.v.'s $Y_1, Y_2, \ldots$ are conditionally i.i.d. with distribution μ, i.e.,

$$P(Y_1 \in A_1, \ldots, Y_k \in A_k | \mathbf{X}, \mu) = \prod_{i=1}^{k} P(Y_i \in A_i | \mathbf{X}, \mu) \quad \text{a.s.} \tag{3.1}$$

$$P(Y_j \in A | \mathbf{X}, \mu) = \mu(A) \quad \text{a.s.} \tag{3.2}$$

for any j, k and Borel sets $A, A_1, \ldots, A_k$ on the real line. Such a sequence (Y_n) always exists after a suitable enlargement of the probability space (in fact (Y_n) exists on $(\Omega, \mathcal{F}, P)$ if $(\Omega, \mathcal{F}, P)$ is atomless over $\sigma(\mathbf{X}, \mu)$; see the vector-valued version of Theorem (1.5) of [5]; see also the remark preceding Theorem (1.3) in [5, p. 479]) or, alternatively, the sequence (X_n) can be redefined, without changing its distribution, on a standard sequence space over which (Y_n) can be defined; see [1, p. 72]. Clearly, (Y_n) is an exchangeable sequence; we call it the *limit exchangeable sequence* of (X_n).

It is not hard to see (cf. [1], [5]) that there exists a subsequence (X_{n_k}) such that for every $k \geq 1$ we have

$$(X_{n_{j_1}}, \dots, X_{n_{j_k}}) \xrightarrow{\mathcal{D}} (Y_1, \dots, Y_k) \text{ if } j_1 < \cdots < j_k \ , \ j_1 \to \infty. \tag{3.3}$$

Note that the existence of a subsequence (X_{n_k}) and exchangeable (Y_k) satisfying (3.3) was first proved by Dacunha-Castelle and Krivine [6] via ultrafilter techniques. The limit exchangeable sequence, as defined above, also has the following simple property, proved in [1, Lemma 12].

Lemma 3.1. *For every $\sigma(\mathbf{X})$-measurable r.v. Z and any $j \geq 1$ we have*

$$(X_n, Z) \xrightarrow{\mathcal{D}} (Y_j, Z).$$

As before, let $\mathcal{M}$ denote the set of all probability measures on $\mathbb{R}$ and let π be the Prohorov metric on $\mathcal{M}$ defined by

$$\pi(\nu, \lambda) = \inf\big\{\varepsilon > 0 : \nu(A) \leq \lambda(A^\varepsilon) + \varepsilon \ \text{ and } \\ \lambda(A) \leq \nu(A^\varepsilon) + \varepsilon \ \text{ for all Borel sets } \ A \subset \mathbb{R}\big\}.$$

Here

$$A^\varepsilon = \{x \in \mathbb{R} : |x - y| < \varepsilon \text{ for some } y \in A\}$$

denotes the open ε-neighborhood of A. Let

$$S = \left\{\nu \in \mathcal{M} : \int x d\nu(x) = 0 \ , \ \int x^2 d\nu(x) < +\infty\right\}. \tag{3.4}$$

Since $\int_{-\infty}^{\infty} x^2 d\mu(x) < \infty$ a.s. (which follows from (1.2)) and $\int_{-\infty}^{\infty} x d\mu(x) = 0$ a.s. by $X_n \to 0$ weakly, we have

$$P\big\{\mu \in S\big\} = 1. \tag{3.5}$$

Following Aldous [1] we define another metric d on S by

$$d(\nu, \lambda) = \left(\int_0^1 \big(F_\nu^{-1}(x) - F_\lambda^{-1}(x)\big)^2 dx\right)^{1/2} \tag{3.6}$$

where F_ν and F_λ are the distribution functions of ν and λ, respectively, and F^{-1} is defined by

$$F^{-1}(x) = \inf\big\{t : \ F(t) \geq x\big\}, \qquad 0 < x < 1,$$

for any distribution function F. The right side of (3.6) equals $\|F_\nu^{-1}(\eta) - F_\lambda^{-1}(\eta)\|_2$ where η is a random variable uniformly distributed in $(0, 1)$. Since $F_\nu^{-1}(\eta)$ and $F_\lambda^{-1}(\eta)$ are r.v.'s with distribution ν and λ, respectively (and thus square integrable), it follows that d is a metric on S. It is easily seen (cf. [1, p. 80] and relation (5.15) in [1, p. 74]) that d is separable and generates the same Borel σ-field as π. By the definition of d we have, letting 0 denote the zero distribution,

$$\mathbb{E}d(\mu, 0)^p = \mathbb{E}(\mathrm{Var}(\mu))^{p/2} = \mathbb{E}\left(\int_{-\infty}^{\infty} x^2 d\mu(x)\right)^{p/2} < \infty \tag{3.7}$$

by our assumption (1.2). The following lemma expresses the crucial equicontinuity property of d.

Lemma 3.2. *Let*

$$\psi(a_1, \dots, a_n) = \Big\| \sum_{i=1}^n a_i Y_i \Big\|_p. \tag{3.8}$$

Then we have

$$\left| \|t + \sum_{k=1}^n a_k \xi_k^{(\nu)}\|_p - \|t + \sum_{k=1}^n a_k \xi_k^{(\lambda)}\|_p \right| \le K d(\nu, \lambda) \psi(a_1, \dots, a_n) \tag{3.9}$$

for some constant $K > 0$, every $n \ge 1$, $\nu, \lambda \in S$, real numbers $t, a_1, \dots, a_n$ and i.i.d. sequences $(\xi_n^{(\nu)}), (\xi_n^{(\lambda)})$ with respective distributions ν and λ.

Relation (3.9) means that the class of functions $\{f_{t,\mathbf{a},n}\}$ defined by

$$f_{t,\mathbf{a},n}(\nu) = \psi(\mathbf{a})^{-1} \Big\| t + \sum_{k=1}^n a_k \xi_k^{(\nu)} \Big\|_p, \qquad \mathbf{a} = (a_1, \dots, a_n) \ne \mathbf{0} \tag{3.10}$$

(where the variable is ν and $t, \mathbf{a}, n$ are parameters), is equicontinuous. In the context of unconditional convergence of lacunary series, the importance of such equicontinuity conditions was discovered by Aldous [1]. A similar condition in terms of the compactness of the 1-conic class belonging to the type determined by (X_n) was given by Krivine and Maurey (see Proposition 3 in Guerre [9]). The proof of our results is, however, purely probabilistic and we will not use types.

Proof of Lemma 3.2. We start with recalling the well-known fact that if (ξ_n) is an i.i.d. sequence with $E\xi_n = 0$, $E\xi_n^2 < +\infty$, then

$$C\|\xi\|_1 \left(\sum_{i=1}^k a_i^2 \right)^{1/2} \le \Big\| \sum_{i=1}^k a_i \xi_i \Big\|_p \le \|\xi\|_2 \left(\sum_{i=1}^k a_i^2 \right)^{1/2} \tag{3.11}$$

for any $1 \le p < 2$ and any $(a_1, \dots, a_n) \in \mathbb{R}^n$, where $C > 0$ is an absolute constant. Since the L^p norm of $\sum_{i=1}^k a_i \xi_i$ in (3.11) cannot exceed the L^2 norm, the upper bound in (3.11) is obvious, while the lower bound is classical; see [13]. Since $\mathbb{E}\left|\sum_{i=1}^n a_i Y_i\right|^p$ can be obtained by integrating $\mathbb{E}\left|\sum_{i=1}^n a_i \xi_i^{(\omega)}\right|^p$ over Ω with respect to $dP(\omega)$ where for each $\omega \in \Omega$ the $\xi_i^{(\omega)}$ are i.i.d. with distribution $\mu(\omega)$, relation (3.11) implies that

$$A \left(\sum_{i=1}^k a_i^2 \right)^{1/2} \le \Big\| \sum_{i=1}^k a_i Y_i \Big\|_p \le B \left(\sum_{i=1}^k a_i^2 \right)^{1/2} \tag{3.12}$$

where

$$A = C \left[\mathbb{E} \left(\int_{-\infty}^{\infty} |x| d\mu(x) \right)^p \right]^{1/p}, \qquad B = \left[\mathbb{E} \left(\int_{-\infty}^{\infty} x^2 d\mu(x) \right)^{p/2} \right]^{1/p}.$$

By (1.2) and since the assumptions of Theorem 1.1 imply that μ is not concentrated at zero a.s., we have $0 < A \le B < \infty$.

Turning to the proof of (3.9), note that the L_p norms on the left-hand side depend on the sequences $(\xi_n^{(\nu)}), (\xi_n^{(\lambda)})$ only through their distributions ν, λ, but not the actual choice of these i.i.d. sequences and thus it suffices to verify (3.9) for any specific construction. Let (η_n) be a sequence of independent r.v.'s, uniformly distributed over $(0, 1)$. Then $\xi_n^{(\nu)} = F_\nu^{-1}(\eta_n)$ and $\xi_n^{(\lambda)} = F_\lambda^{-1}(\eta_n)$ are i.i.d. sequences

with distribution ν and λ, respectively. Using these sequences in (3.9), the left-hand side is at most $\|\sum_{i=1}^n a_i(\xi_i^{(\nu)} - \xi_i^{(\lambda)})\|_p$ and since $\xi_n^{(\nu)} - \xi_n^{(\lambda)} = F_\nu^{-1}(\eta_n) - F_\lambda^{-1}(\eta_n)$ is also an i.i.d. sequence with mean 0 and variance $d(\nu,\lambda)^2$, using (3.11) and the first relation of (3.12) we get that the left-hand side of (3.9) is at most $Kd(\nu,\lambda)\psi(a_1,\dots,a_n)$ with some constant $K > 0$. This completes the proof of Lemma 3.2.

With the equicontinuity statement of Lemma 3.2 at hand, we can prove the sufficiency part of Theorem 1.1 with a selection procedure similar to [2]. Assume that (X_n) satisfies (1.2) and fix $0 < \varepsilon \le 1/2$. We shall construct a sequence $n_1 < n_2 < \cdots$ of integers such that

$$(1-\varepsilon)\psi(a_1,\dots,a_k) \le \Big\|\sum_{i=1}^k a_i X_{n_i}\Big\|_p \le (1+\varepsilon)\psi(a_1,\dots,a_k) \tag{3.13}$$

for every $k \ge 1$ and $(a_1,\dots,a_k) \in \mathbb{R}^k$. In view of (3.12), this will imply that (X_{n_k}) is equivalent to the unit vector basis of ℓ^2, but it actually shows more, namely that under the assumptions of Theorem 1.1 there is a subsequence $(1+\varepsilon)$-equivalent to the limit exchangeable sequence and hence $(1+\varepsilon)$-symmetric.

To construct n_1 we set

$$Q(\mathbf{a},n,\ell) = |a_1X_n + a_2Y_2 + \cdots + a_\ell Y_\ell|^p,$$
$$R(\mathbf{a},\ell) = |a_1Y_1 + a_2Y_2 + \cdots + a_\ell Y_\ell|^p$$

for every $n \ge 1$, $\ell \ge 2$ and $\mathbf{a} = (a_1,\dots,a_\ell) \in \mathbb{R}^\ell$. We show that

$$E\left\{\frac{Q(\mathbf{a},n,\ell)}{\psi(\mathbf{a})^p}\right\} \longrightarrow E\left\{\frac{R(\mathbf{a},\ell)}{\psi(\mathbf{a})^p}\right\} \quad \text{as } n\to\infty \quad \text{uniformly in } \mathbf{a},\ell. \tag{3.14}$$

(The right side of (3.14) equals 1.) To this end we recall that, given $\mathbf{X}$ and μ, the r.v.'s $Y_1, Y_2, \dots$ are conditionally i.i.d. with common conditional distribution μ and thus, given $\mathbf{X}, \mu$ and Y_1, the r.v.'s $Y_2, Y_3, \dots$ are conditionally i.i.d. with distribution μ. Thus

$$E\big(Q(\mathbf{a},n,\ell)|\mathbf{X},\mu\big) = g^{\mathbf{a},\ell}(X_n,\mu) \tag{3.15}$$

and

$$E\big(R(\mathbf{a},\ell)|\mathbf{X},\mu,Y_1\big) = g^{\mathbf{a},\ell}(Y_1,\mu) \tag{3.16}$$

where

$$g^{\mathbf{a},\ell}(t,\nu) = E\Big|a_1t + \sum_{i=2}^\ell a_i\xi_i^{(\nu)}\Big|^p \qquad (t\in\mathbb{R}^1\ ,\ \nu\in S)$$

and $(\xi_n^{(\nu)})$ is an i.i.d. sequence with distribution ν. Integrating (3.15) and (3.16) we get

$$E\big(Q(\mathbf{a},n,\ell)\big) = Eg^{\mathbf{a},\ell}(X_n,\mu), \tag{3.17}$$

$$E\big(R(\mathbf{a},\ell)\big) = Eg^{\mathbf{a},\ell}(Y_1,\mu) \tag{3.18}$$

and thus (3.14) is equivalent to

$$E\frac{g^{\mathbf{a},\ell}(X_n,\mu)}{\psi(\mathbf{a})^p} \longrightarrow E\frac{g^{\mathbf{a},\ell}(Y_1,\mu)}{\psi(\mathbf{a})^p} \quad \text{as } n\to\infty\ , \quad \text{uniformly in } \mathbf{a},\ell. \tag{3.19}$$

We shall derive (3.19) from Lemma 2.4 and Lemma 3.1. As we have seen above, there exists a separable metric d on S, generating the same σ-field as the Prohorov metric π, such that (3.9) holds. But then the limit random measure μ, which is a random variable taking values in (S, π) (i.e., a measurable map from the underlying probability space to $(S, \mathcal{B}_\pi)$ where $\mathcal{B}_\pi$ denotes the Borel σ-field in S generated by π) can be also regarded as a random variable taking values in (S, d). Also, μ is clearly $\sigma(\mathbf{X})$ measurable and thus $(X_n, \mu) \xrightarrow{\mathcal{D}} (Y_1, \mu)$ by Lemma 3.1. Hence, (3.19) will follow from Lemma 2.4 (note the equivalence of (2.7) and (2.8)) if we show that the class of functions

$$\left\{ \frac{g^{\mathbf{a},\ell}(t,\nu)}{\psi(\mathbf{a})^p} \right\} \tag{3.20}$$

defined on the product metric space $(\mathbb{R}^1 \times S\ ,\ \lambda^1 \times d)$ (λ^1 denotes the ordinary distance on $\mathbb{R}^1$) satisfies conditions (a),(b) of Lemma 2.4. Observe now that

$$\psi(a_1, \dots, a_n) \ge \psi(a_1^*, \dots, a_n^*) \tag{3.21}$$

where a_i^* equals either a_i or 0. (In case (Y_n) is an i.i.d. sequence with mean 0, (3.21) follows from Jensen's inequality (see e.g. [8, p. 153]) and the fact that, for any $H \subset \{1, 2, \dots, n\}$, the conditional expectation of $\sum_{i=1}^n a_i Y_i$ given $\sigma\{Y_j\ ,\ j \in H\}$ is $\sum_{i \in H} a_i Y_i$. Since (Y_n) is a mixture of i.i.d. sequences with mean 0, (3.21) holds in general.) In particular,

$$\psi(a_1, \dots, a_n) \ge \psi(0, a_2, \dots, a_n) \tag{3.22}$$

and

$$\psi(a_1, \dots, a_n) \ge \psi(a_1, 0, \dots, 0) = \text{const}\ \cdot |a_1| \tag{3.23}$$

and thus using (3.9) we get for any $\nu \in S$, $t \in \mathbb{R}^1$ and $\mathbf{a} = (a_1, \dots, a_\ell) \in \mathbb{R}^\ell$,

$$\begin{aligned} \Big\| a_1 t + \sum_{i=2}^{\ell} a_i \xi_i^{(\nu)} \Big\|_p &\le |a_1 t| + \Big\| \sum_{i=2}^{\ell} a_i \xi_i^{(\nu)} \Big\|_p \\ &\le |a_1 t| + \psi(a_2, \dots, a_\ell) d(\nu, 0) \le \text{const} \cdot \psi(\mathbf{a})|t| + \psi(\mathbf{a}) d(\nu, 0) \\ &\le \ \text{const}\ \psi(\mathbf{a}) \big(|t| + d(\nu, 0)\big). \end{aligned} \tag{3.24}$$

Hence using (3.9), (3.22), (3.23) and the inequality $|x^p - y^p| \le |x-y| \cdot p \cdot (x^{p-1} + y^{p-1})$ $(x > 0,\ y > 0)$ we get for any real t, t' and $\nu, \nu' \in S$

$$\begin{aligned} &\big| g^{\mathbf{a},\ell}(t,\nu) - g^{\mathbf{a},\ell}(t',\nu') \big| = \Big| \big\| a_1 t + \sum_{i=2}^{\ell} a_i \xi_i^{(\nu)} \big\|_p^p - \big\| a_1 t' + \sum_{i=2}^{\ell} a_i \xi_i^{(\nu')} \big\|_p^p \Big| \\ &\le \Big| \big\| a_1 t + \sum_{i=2}^{\ell} a_i \xi_i^{(\nu)} \big\|_p \\ &\qquad - \big\| a_1 t' + \sum_{i=2}^{\ell} a_i \xi_i^{(\nu')} \big\|_p \Big| \ \text{const}\ p \psi(\mathbf{a})^{p-1} \big(|t| + |t'| + d(\nu, 0) + d(\nu', 0) \big)^{p-1} \\ &\le \text{const} \big(|a_1|\ |t - t'| + \psi(a_2, \dots, a_n) d(\nu, \nu') \big) p \psi(\mathbf{a})^{p-1} \big(|t| + |t'| + d(\nu, 0) + d(\nu', 0) \big)^{p-1} \\ &\le \text{const} \big(|t - t'| + d(\nu, \nu') \big) p \psi(\mathbf{a})^p \big(|t| + |t'| + d(\nu, 0) + d(\nu', 0) \big)^{p-1} \\ &\le \text{const} \big(|t - t'| + d(\nu, \nu') \big) p \psi(\mathbf{a})^p \big(2|t| + 2d(\nu, 0) + |t - t'| + d(\nu, \nu') \big)^{p-1}. \end{aligned}$$

Given t, ν and $\varepsilon > 0$, there exists a $\delta = \delta(t, \nu, \varepsilon) > 0$ such that the last expression is $\leq \varepsilon\psi(\mathbf{a})^p$ provided $|t - t'| + d(\nu, \nu') \leq \delta$ and thus the class (3.20) is locally equicontinuous on the product metric space $(\mathbb{R}^1 \times S\ ,\ \lambda^1 \times d)$. On the other hand, (3.24) shows that the function in (3.20) is bounded by $\text{const}(|t| + d(\nu, 0))^p \leq \text{const } 2^p(|t|^p + d(\nu, 0)^p)$. Now, using $(X_n, \mu) \xrightarrow{\mathcal{D}} (Y_1, \mu)$, the uniform integrability of $|X_n|^p$ and $Ed(\mu, 0)^p < +\infty$ (see (3.7)) we get

$$E\big(|X_n|^p + d(\mu, 0)^p\big) \longrightarrow E\big(|Y_1|^p + d(\mu, 0)^p\big).$$

Thus the class (3.20) satisfies also condition (b) of Lemma 2.4. We thus proved relation (3.19) and thus also (3.14) whence it follows (note again that the right side of (3.14) equals 1) that

$$\begin{aligned} &\psi(\mathbf{a})^{-1}\|a_1X_n + a_2Y_2 + \cdots + a_\ell Y_\ell\|_p \\ &\qquad\longrightarrow \psi(\mathbf{a})^{-1}\|a_1Y_1 + a_2Y_2 + \cdots + a_\ell Y_\ell\|_p \quad \text{as} \quad n \to \infty \end{aligned} \tag{3.25}$$

unformly in $\ell, \mathbf{a}$. Hence we can choose n_1 so large that

$$\big|\ \|a_1X_{n_1} + a_2Y_2 + \cdots + a_\ell Y_\ell\|_p - \|a_1Y_1 + a_2Y_2 + \cdots + a_\ell Y_\ell\|_p\big| \leq \frac{\varepsilon}{2}\psi(a_1, \ldots, a_\ell)$$

for every $\ell, \mathbf{a}$. This completes the first induction step.

Assume now that $n_1, \ldots, n_{k-1}$ have already been chosen. Exactly in the same way as we proved (3.25), it follows that for $\ell > k$

$$\begin{aligned} &\psi(\mathbf{a})^{-1}\|a_1X_{n_1} + \cdots + a_{k-1}X_{n_{k-1}} + a_kX_n + a_{k+1}Y_{k+1} + \cdots + a_\ell Y_\ell\|_p \\ &\quad\longrightarrow \psi(\mathbf{a})^{-1}\|a_1X_{n_1} + \cdots + a_{k-1}X_{n_{k-1}} + a_kY_k + \cdots + a_\ell Y_\ell\|_p \quad \text{as} \quad n \to \infty \end{aligned}$$

uniformly in $\mathbf{a}$ and ℓ. Hence we can choose n_k so large that $n_k > n_{k-1}$ and

$$\begin{aligned} \big|\ &\|a_1X_{n_1} + \cdots + a_{k-1}X_{n_{k-1}} + a_kX_{n_k} + a_{k+1}Y_{k+1} + \cdots + a_\ell Y_\ell\|_p \\ &\quad - \|a_1X_{n_1} + \cdots + a_{k-1}X_{n_{k-1}} + a_kY_k + \cdots + a_\ell Y_\ell\|_p\big| \leq \frac{\varepsilon}{2^k}\psi(a_1, \ldots, a_\ell) \end{aligned}$$

for every $(a_1, \ldots, a_\ell) \in \mathbb{R}^\ell$ and $\ell > k$. This completes the k-th induction step; the so constructed sequence (n_k) obviously satisfies

$$\big|\ \|a_1X_{n_1} + \cdots + a_\ell X_{n_\ell}\|_p - \|a_1Y_1 + \cdots + a_\ell Y_\ell\|_p\big| \leq \varepsilon\psi(a_1, \ldots, a_\ell)$$

for every $\ell \geq 1$ and $(a_1, \ldots, a_\ell) \in \mathbb{R}^\ell$. The last relation is equivalent to (3.13) and thus the sufficiency of (1.2) in Theorem 1.1 is proved.

We now turn to the proof of necessity of (1.2) in Theorem 1.1. Assume that (X_n) is equivalent to the unit vector basis of ℓ^2; then for any increasing sequence (m_k) of integers we have

$$\left\|\frac{1}{\sqrt{N}}\sum_{k=1}^N X_{m_k}\right\|_p = O(1)$$

and thus by the Markov inequality we have for any $A \subset \Omega$ with $P(A) > 0$,

$$P_A\left\{\left|\frac{1}{\sqrt{N}}\sum_{k=1}^N X_{m_k}\right| \geq T\right\} \leq 1/2 \qquad \text{for } T \geq T_0,\ N \geq 1 \tag{3.26}$$

where T_0 depends on A and the sequence (X_n). We show first that

$$\int_{-\infty}^{\infty} x^2 d\mu(x) < \infty \qquad \text{a.s.} \tag{3.27}$$

Let $F_\bullet(x)$ denote the random distribution function corresponding to μ and assume indirectly that there exists a set $B \subset \Omega$ with $P(B) > 0$ such that

$$\lim_{t\to\infty} \int_{|x|<t} x^2 dF_\bullet(x) = +\infty \quad \text{on } B. \tag{3.28}$$

By Egorov's theorem there exists a set $B^* \subset B$ with $P(B^*) \ge P(B)/2$ such that on B^* (3.28) holds uniformly, i.e., there exists a positive, nondecreasing, nonrandom function $K_t \to +\infty$ such that

$$\int_{|x|<t} x^2 dF_\bullet(x) \ge K_t \quad \text{on} \quad B^*. \tag{3.29}$$

Also,

$$\int_{|x|\ge t} dF_\bullet(x) \longrightarrow 0 \quad \text{a.s. as } t \to \infty \tag{3.30}$$

and thus we can choose a set $B^{**} \subset B^*$ with $P(B^{**}) \ge P(B^*)/2$ such that on B^{**} relation (3.30) holds uniformly, i.e., there exists a positive, nonincreasing, nonrandom function $\tilde{\varepsilon}_t \to 0$ such that

$$\int_{|x|\ge t} dF_\bullet(x) \le \tilde{\varepsilon}_t \quad \text{on} \quad B^{**}. \tag{3.31}$$

We show that there exists a subsequence (X_{m_k}) of (X_n) such that (3.26) fails for $A = B^{**}$. Since our argument will involve the sequence (X_n) only on the set B^{**} and on B^{**} (X_n) satisfies the assumptions of Theorem 1.1 with the same μ and with $\|X_n\|_p = 1$ replaced by $\|X_n\|_p = O(1)$ (which is all we need for the rest of the proof), in the sequel we can assume, without loss of generality, that $B^{**} = \Omega$. That is, we may assume that (3.29), (3.31) hold on the whole probability space.

Let C be an arbitrary set in the probability space with $P(C) > 0$. Integrating (3.29), (3.31) on C and using (1.1) and Lemma 2.3 we get

$$\int_{|x|<t} x^2 dF_C(x) \ge K_t, \qquad \int_{|x|\ge t} dF_C(x) \le \tilde{\varepsilon}_t \quad \text{for } t, -t \in H \tag{3.32}$$

where H denotes the set of continuity points of F_C. Since the integrals in (3.32) are monotone functions of t and $\mathbb{R} \setminus H$ is countable, (3.32) remains valid with $K_{t/2}$ resp. $\tilde{\varepsilon}_{t/2}$ if we drop the assumption $t, -t \in H$. Thus, keeping the original notation, in the sequel we can assume that (3.32) holds for all $t > 0$. Choose now t_0 so large that $\tilde{\varepsilon}_{t_0} \le 1/16$ and then choose $t_1 > t_0$ so large that

$$K_t^{1/2} \ge 4t_0 \quad \text{for} \quad t \ge t_1.$$

Then for $t \ge t_1$ we have, using (3.32),

$$\begin{aligned}
\left| \int_{|x|<t} x dF_C(x) \right| &\le t_0 + \int\limits_{t_0 \le |x| < t} |x| dF_C(x) \\
&\le t_0 + \Bigg(\int\limits_{|x|\ge t_0} dF_C(x) \Bigg)^{1/2} \Bigg(\int\limits_{|x|<t} x^2 dF_C(x) \Bigg)^{1/2} \\
&\le t_0 + \frac{1}{4} \Bigg(\int\limits_{|x|<t} x^2 dF_C(x) \Bigg)^{1/2} \le \frac{1}{2} \Bigg(\int\limits_{|x|<t} x^2 dF_C(x) \Bigg)^{1/2}
\end{aligned}$$

and thus for any $C \subset \Omega$ with $P(C) > 0$ we have

$$\int_{|x|<t} x^2 dF_C(x) - 2\Big(\int_{|x|<t} x dF_C(x)\Big)^2 \geq \frac{1}{2}K_t, \qquad t \geq t_1. \tag{3.33}$$

Let now (ε_n) tend to 0 so rapidly that

$$\sum_{j=a_k+1}^{\infty} \varepsilon_j \leq 2^{-k}. \tag{3.34}$$

Let $a_k = [\log k + 1]$ $(k = 1, 2, \ldots)$. By Lemma 2.1 there exists a subsequence (X_{m_k}) and a sequence (Y_k) of discrete r.v.'s such that (2.1) holds and for each $k \geq 1$ the atoms of the finite σ-field $\sigma\{Y_1, \ldots, Y_{a_k}\}$ can be divided into two classes Γ_1 and Γ_2 such that

$$\sum_{B \in \Gamma_1} P(B) \leq \varepsilon_{a_k+1} \leq 2^{-k} \tag{3.35}$$

and for each $B \in \Gamma_2$ there exist P_B-independent r.v.'s $Z_{a_k+1}^{(B)}, \ldots, Z_k^{(B)}$ defined on B with common distribution F_B such that

$$P_B\big(|Y_j - Z_j^{(B)}| \geq 2^{-k}\big) \leq 2^{-k} \quad (j = a_k + 1, \ldots, k). \tag{3.36}$$

Set

$$S_{a_k,k}^{(B)} = \sum_{j=a_k+1}^{k} Z_j^{(B)}, \qquad B \in \Gamma_2,$$
$$\overline{S}_{a_k,k} = \sum_{B \in \Gamma_2} S_{a_k,k}^{(B)} I(B),$$

where $I(B)$ denotes the indicator function of B. By (3.36) and $k2^{-k} \leq 1$,

$$P_B\Big(\Big|\sum_{j=a_k+1}^{k} Y_j - \sum_{j=a_k+1}^{k} Z_j^{(B)}\Big| \geq 1\Big) \leq k2^{-k}, \qquad B \in \Gamma_2,$$

and thus using (3.35) we get

$$P\Big(\Big|\sum_{j=a_k+1}^{k} Y_j - \overline{S}_{a_k,k}\Big| \geq 1\Big) \leq (k+1)2^{-k}. \tag{3.37}$$

Since $\|X_n\|_1 = O(1)$, by the triangular inequality and the Markov inequality we have

$$P\Big(\Big|\sum_{j=1}^{a_k} X_{m_j}\Big| \geq a_k k^{1/4}\Big) \leq \sum_{j=1}^{a_k} P\big(|X_{m_j}| \geq k^{1/4}\big)$$
$$\leq \text{const}\,(\log k + 1)k^{-1/4} =: \delta_k$$

which, together with (3.37), (2.1) and (3.34), yields

$$P\left(\left|\sum_{j=1}^{k} X_{m_j} - \overline{S}_{a_k,k}\right| \ge 3a_k k^{1/4}\right) \le P\left(\left|\sum_{j=1}^{a_k} X_{m_j}\right| \ge a_k k^{1/4}\right)$$
$$+ P\left(\sum_{j=a_k+1}^{k} |X_{m_j} - Y_j| \ge 1\right) + P\left(\left|\sum_{j=a_k+1}^{k} Y_j - \overline{S}_{a_k,k}\right| \ge 1\right)$$
$$\le \delta_k + (k+2)2^{-k}. \tag{3.38}$$

Applying Lemma 2.2 to the i.i.d. sequence $\{Z_j^{(B)}, a_k + 1 \le j \le k\}$ and using (3.33) with $C = B$, $a_k \le k/2$ and the monotonicity of K_t we get for any $T \ge 2$,

$$P_B\left(\left|\frac{S_{a_k,k}^{(B)}}{\sqrt{k}}\right| \le T\right) \le P_B\left(\left|\frac{S_{a_k,k}^{(B)}}{\sqrt{k-a_k}}\right| \le 2T\right) \le \text{const} \cdot 2TK_{2T\sqrt{k-a_k}}^{-1/2}$$
$$\le \text{const} \cdot TK_{2T\sqrt{k/2}}^{-1/2}$$

where the constants are absolute. Thus using (3.35) it follows that

$$P\left(\left|\frac{\overline{S}_{a_k,k}}{\sqrt{k}}\right| \le T\right) \le \text{const} \cdot TK_{\sqrt{k/2}}^{-1/2} + 2^{-k}. \tag{3.39}$$

Using (3.38), (3.39) and $a_k \le \log k + 1$ it follows that

$$P\left(\left|\frac{1}{\sqrt{k}}\sum_{j=1}^{k} X_{m_j}\right| \le T\right) \le P\left(\left|\frac{\overline{S}_{a_k,k}}{\sqrt{k}}\right| \le T + 3a_k k^{-1/4}\right) + (k+2)2^{-k} + \delta_k$$
$$\le \text{const}\left(T + 3a_k k^{-1/4}\right) K_{\sqrt{k/2}}^{-1/2} + (k+2)2^{-k} + \delta_k \longrightarrow 0 \qquad \text{as } k \to \infty$$

for any fixed $T \ge 2$ which clearly contradicts (3.26) with $A = \Omega$. This completes the proof of (3.27).

Since $X_n \longrightarrow 0$ weakly in L^p, we have $\int_{-\infty}^{\infty} x d\mu(x) = 0$ a.s., and thus the random measure μ has mean zero and finite variance with probability one. Thus the subsequence principle, specialized to the central limit theorem (see e.g. [3, Theorem 3] and the remark following it) there exists a subsequence (X_{n_k}) such that

$$N^{-1/2}\sum_{k=1}^{N} X_{n_k} \xrightarrow{\mathcal{D}} Y\zeta \tag{3.40}$$

where $Y = \left(\int_{-\infty}^{\infty} x^2 d\mu(x)\right)^{1/2}$, ζ is a standard normal variable and Y and ζ are independent. Note that (3.40) holds in distribution, but by a well-known result of Skorokhod (see e.g. [4], p. 70) there exist r.v.'s W_N, W $(N = 1, 2, \ldots)$ such that W_N has the same distribution as $N^{-1/2}\sum_{k=1}^{N} X_{n_k}$ in (3.40), W has the same distribution as $Y\zeta$ and $W_N \longrightarrow W$ a.s. Thus (3.40) and Fatou's lemma imply

$$\|Y\zeta\|_p \le \liminf_{N\to\infty} \|N^{-1/2}\sum_{k=1}^{N} X_{n_k}\|_p < \infty \tag{3.41}$$

where the second inequality follows from the equivalence of (X_n) to the unit vector basis of ℓ^2, assumed at the beginning of the proof. Since Y and ζ are independent,

(3.41) implies $E|Y|^p < \infty$, i.e., (1.2) holds, completing the proof of the converse part of Theorem 1.1.

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