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CENTRAL SUBALGEBRAS OF THE CENTRALIZER OF A NILPOTENT ELEMENT

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Abstract. Let G be a connected, semisimple algebraic group over a field k whose characteristic is *very good* for G. In a canonical manner, one associates to a nilpotent element $X \in \mathrm{Lie}(G)$ a parabolic subgroup P – in characteristic zero, P may be described using an $\mathfrak{sl}_2$-triple containing X; in general, P is the "instability parabolic" for X as in geometric invariant theory.

In this setting, we are concerned with the center $Z(C)$ of the centralizer C of X in G. Choose a Levi factor L of P, and write d for the dimension of the center $Z(L)$. Finally, assume that the nilpotent element X is *even*. In this case, we can *deform* $\mathrm{Lie}(L)$ to $\mathrm{Lie}(C)$, and our deformation produces a d-dimensional subalgebra of $\mathrm{Lie}(Z(C))$. Since $Z(C)$ is a smooth group scheme, it follows that $\dim Z(C) \geq d = \dim Z(L)$.

In fact, Lawther and Testerman have proved that $\dim Z(C) = \dim Z(L)$. Despite only yielding a partial result, the interest in the method found in the present work is that it avoids the extensive case-checking carried out by Lawther and Testerman.

1. Introduction

Let G be a connected and reductive group over the infinite field k, and suppose that G is *standard* in the sense spelled out in §4 below. In the special case of a semisimple group, G is standard if and only if the characteristic of k is *very good* for G.

Now let $\mathfrak{g}$ denote the Lie algebra of G, and let $X \in \mathfrak{g}$ denote a nilpotent element. We are concerned here with the centralizer $C = C_G(X)$ of X, and the center $Z(C)$ of C. We first recall that the group schemes C and $Z(C)$ are *smooth*, i.e. both may be viewed as linear algebraic groups over k. See Proposition 4.2 for the smoothness of C and Proposition 6.1 for the smoothness of $Z(C)$.

Now choose a cocharacter ϕ associated to X; see §5 and Theorem 5.1. Then ϕ determines a parabolic subgroup $P = P(\phi)$ of G, together with a Levi factor $L = C_G(\mathrm{im}(\phi))$ of P.

Suppose that the nilpotent element X is *even* – i.e. that all weights of the image of ϕ on $\mathrm{Lie}(G)$ are even integers. In that case, the equality $\dim L = \dim C$ holds; see Proposition 5.4 below. For various reasons, one might hope to somehow view the (non-reductive) group C as a "deformation" of the reductive group L, and perhaps relate features of C and L.

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In the present paper, this point of view is partially achieved. More precisely, we view $\mathrm{Lie}(C)$ as a deformation of $\mathrm{Lie}(L)$; using rather general deformation results found in §§2 and 3, we give a proof of the following theorem.

Theorem A. *Suppose that X is an even nilpotent element. With notation as above, we have* $\dim Z(C) \geq \dim Z(L)$.

In their memoir [LT11], Lawther and Testerman also studied the centers $Z(C)$ and $Z(L)$. In fact, in *loc. cit.*, Lawther and Testerman already proved:

Theorem B ([LT11])**.** *Assume that G is semisimple, that k is algebraically closed, and that the characteristic of k is* good *for G. Suppose that X is an even nilpotent element. With notation as above,* $\dim Z(C) = \dim Z(L)$.

In contrast to the proof of Theorem A given here, the proof given by Lawther and Testerman of Theorem B depends on extensive analysis of cases using the Bala-Carter classification of nilpotent orbits and the Cartan-Killing enumeration of simple algebraic groups.

We formulate some generalities about standard reductive groups in §4, and we recall important facts about nilpotent elements for these groups in §5. In §6 we verify that a description of the center of a nilpotent centralizer given by Lawther and Testerman in [LT11] for semisimple groups in very good characteristic remains valid for standard reductive groups; see Theorem 6.4.

Finally, the proof of Theorem A is given in §7, using the deformation results of §3. Note that we have chosen to work over an arbitrary field k, even though one could prove Theorem A for a reductive group G over k by proving it after extending scalars to an algebraic closure of k; it seems to us useful to work as we have done, since the methods and intermediate results may prove useful for other applications.

One reason for interest in the results of this paper is to shed further light on the *structure* of the centralizer of a unipotent element. As Lawther and Testerman explain in the introduction to [LT11]:

> Our interest in $Z(C_G(u))$ is motivated by the desire to embed u in a connected abelian unipotent subgroup of G satisfying certain uniqueness properties.

Of course, here the unipotent element u corresponds to the nilpotent element X via a Springer isomorphism (Remark 4.4(iv)); thus the center of $C_G(u) = C_G(X)$ is a natural place to look for such an abelian subgroup. It seems interesting and suggestive to us that infinitesimally – at least when X is even – this center can be viewed as arising by deformation from the center of a Levi factor of the parabolic subgroup determined by X.

2. Deforming a kernel over a Dedekind variety

Let A be an integral domain which is *Dedekind*. Recall for example that a principal ideal domain is Dedekind. For a maximal ideal $\mathfrak{m}$ of A, write $k(\mathfrak{m}) = A/\mathfrak{m}$ for the residue field. For an A-module M, write $M(\mathfrak{m}) = M \otimes_A k(\mathfrak{m})$. Write $K = \mathrm{Frac}(A)$ for the field of fractions of A. We first recall some well-known facts; see e.g. [Bou89, VII.4.10 and II.5.3].

Proposition 2.1. *Let M be a finitely generated A-module.*

(a) $M = M_0 \oplus M_{\mathrm{tor}}$ where M_0 is a projective A-module and M_{tor} is the torsion submodule. In particular, M is projective if and only if it is torsion free.

(b) If M is projective, then $\dim_{k(\mathfrak{m})} M(\mathfrak{m}) = \dim_K M \otimes_A K$ *for each maximal ideal* $\mathfrak{m}$ *of A, and this common quantity is written* $\mathrm{Rank}_A(M)$.

Let $\phi : M \to N$ be a homomorphism between finitely generated projective A-modules M and N, and write

$$P = \ker \phi, \qquad Q = \operatorname{coker} \phi, \qquad Q = Q_0 \oplus Q_{\mathrm{tor}} \text{ as in Proposition 2.1(a).}$$

For each maximal ideal $\mathfrak{m}$ of A, write $\phi(\mathfrak{m}) : M(\mathfrak{m}) \to N(\mathfrak{m})$ for the homomorphism obtained from ϕ by base-change.

We first record the following:

Proposition 2.2. *The A-module M/P is torsion free.*

Proof. Indeed, M/P is isomorphic to a submodule of N. □

Proposition 2.3. *Let $\mathfrak{m}$ be a maximal ideal of A.*
(a) $\mathrm{Tor}_1^A(Q, k(\mathfrak{m})) \simeq Q_{\mathrm{tor}} \otimes_A k(\mathfrak{m})$.
(b) If ϕ is injective and if $Q_{\mathrm{tor}} \otimes_A k(\mathfrak{m}) = 0$, then $\phi(\mathfrak{m})$ is injective.

Proof. Write $m = \mathrm{Rank}_A(M)$, $p = \mathrm{Rank}_A(P)$, $n = \mathrm{Rank}_A(N)$ and $q = \mathrm{Rank}_A(Q_0)$. There is an exact sequence

$$0 \to P \to M \xrightarrow{\phi} N \to Q \to 0.$$

Since K is a flat A-module, the sequence remains exact after extending scalars to K; thus we see that $m - p = n - q$.

Since the finitely generated A-module M/P is isomorphic to a submodule of N, it is torsion free and thus it is projective by Proposition 2.1; in particular, $\mathrm{Tor}_1^A(M/P, k(\mathfrak{m})) = 0$. Thus the sequence

$$(\flat) \quad 0 \to \mathrm{Tor}_1^A(Q, k(\mathfrak{m})) \to (M/P) \otimes_A k(\mathfrak{m}) \xrightarrow{\overline{\phi}(\mathfrak{m})} N \otimes_A k(\mathfrak{m}) \to Q \otimes_A k(\mathfrak{m}) \to 0$$

is exact, where $\overline{\phi}$ is the mapping induced by ϕ. Now $(\flat)$ implies that

$$m - p - \dim_{k(\mathfrak{m})} \mathrm{Tor}_1^A(Q, k(\mathfrak{m})) = n - q - \dim_{k(\mathfrak{m})} Q_{\mathrm{tor}} \otimes_A k(\mathfrak{m}).$$

It follows that

$$\dim_{k(\mathfrak{m})} Q_{\mathrm{tor}} \otimes_A k(\mathfrak{m}) = \dim_{k(\mathfrak{m})} \mathrm{Tor}_1^A(Q, k(\mathfrak{m})),$$

so that the $k(\mathfrak{m})$ vector spaces Q_{tor} and $\mathrm{Tor}_1^A(Q, k(\mathfrak{m}))$ are (non-canonically) isomorphic, proving (a). Now (b) follows from (a) together with $(\flat)$. □

We record the following consequence of Proposition 2.3(b).

Proposition 2.4. *If X is a finitely generated projective A-module, if $Y \subset X$ is a submodule, and if X/Y is torsion free, then for each maximal ideal $\mathfrak{m} \subset A$, the natural mapping from $Y(\mathfrak{m})$ to $X(\mathfrak{m})$ is injective; in particular, we may view $Y(\mathfrak{m})$ as a subspace of $X(\mathfrak{m})$.*

Proof. Take $Y = M$ and $X = N$. According to Proposition 2.1, Y is projective. After applying Proposition 2.3(b) to the inclusion mapping $Y = M \to X = N$, the result follows at once. □

Returning to the previous notation, note in particular that we view $P(\mathfrak{m})$ as a subspace of $M(\mathfrak{m})$.

Proposition 2.5. *For a maximal ideal $\mathfrak{m}$ of A, we have:*

$$P(\mathfrak{m}) \subset \ker(\phi(\mathfrak{m})),$$

and equality holds if and only if $Q_{\mathrm{tor}} \otimes_A k(\mathfrak{m}) = 0$. In particular, equality holds for all but finitely many maximal ideals $\mathfrak{m}$ of A.

Proof. If $\iota : P \to M$ denotes the inclusion mapping, then $0 = (\phi \circ \iota)(\mathfrak{m}) = \phi(\mathfrak{m}) \circ \iota(\mathfrak{m})$ so that indeed $P(\mathfrak{m}) \subset \ker \phi(\mathfrak{m})$.

It follows from ($\flat$) (in the proof of Proposition 2.3) together with Proposition 2.3(a) that

$$\ker(\overline{\phi}(\mathfrak{m})) \simeq Q_{\mathrm{tor}} \otimes_A k(\mathfrak{m}).$$

Now the first assertion of the proposition follows since $\ker(\overline{\phi}(\mathfrak{m})) \simeq \ker(\phi(\mathfrak{m}))/P(\mathfrak{m})$. Moreover, Q_{tor} is a finitely generated torsion A-module; since A is Dedekind, all prime ideals containing the (non-zero) annihilator of Q_{tor} are maximal, so Q_{tor} has finite length, e.g. by [Eis95, Cor. 2.17]. Since $k(\mathfrak{n}_1) \otimes_A k(\mathfrak{n}_2) = 0$ for maximal ideals $\mathfrak{n}_1 \neq \mathfrak{n}_2$ of A, it follows that $Q_{\mathrm{tor}} \otimes_A k(\mathfrak{m}) = 0$ for all but finitely many $\mathfrak{m}$. The remaining assertion is immediate. □

Remark 2.6. With the notation of Proposition 2.5, suppose one knows in advance that the quantity

$$\dim_{k(\mathfrak{m})} \ker(\phi(\mathfrak{m}))$$

takes a constant value d for an infinite collection Γ of maximal ideals $\mathfrak{m}$ of A. Under this assumption, Proposition 2.5 and Proposition 2.1(b) together imply that $d = \mathrm{Rank}_A(P)$, and in particular the equality

$$\ker(\phi(\mathfrak{m})) = P(\mathfrak{m})$$

holds for every $\mathfrak{m}$ in the collection Γ.

We now formulate some consequences of Proposition 2.5. For the first such result, let M be a finitely generated and projective A-module, and let $H_1, H_2 \subset M$ be A-submodules for which the quotients M/H_i are torsion free.

Proposition 2.7. *(a) The A-module $M/(H_1 \cap H_2)$ is torsion free.*

(b) For each maximal ideal $\mathfrak{m} \subset A$, the dimension of the $k(\mathfrak{m})$-vector subspace $(H_1 \cap H_2)(\mathfrak{m})$ of $M(\mathfrak{m})$ is a constant, independent of the choice of $\mathfrak{m}$.

(c) For each maximal ideal $\mathfrak{m}$ of A, we have

$$(\heartsuit) \qquad (H_1 \cap H_2)(\mathfrak{m}) \subset H_1(\mathfrak{m}) \cap H_2(\mathfrak{m})$$

and equality holds provided that $(M/(H_1 + H_2))_{\mathrm{tor}} \otimes_A k(\mathfrak{m}) = 0$. In particular, equality holds in ($\heartsuit$) for all but finitely many $\mathfrak{m}$.

(d) Suppose that there is an infinite collection Γ of maximal ideals of A together with a non-negative integer d for which

$$\dim_{k(\mathfrak{m})} H_1(\mathfrak{m}) \cap H_2(\mathfrak{m}) = d$$

whenever $\mathfrak{m}$ belongs to Γ. Then d is equal to the rank of $H_1 \cap H_2$, and equality holds in ($\heartsuit$) for any $\mathfrak{m}$ in the collection Γ.

Remark 2.8. Before giving the proof, we point out that by Proposition 2.4, $H_i(\mathfrak{m})$ may be viewed as a subspace of $M(\mathfrak{m})$ for $i = 1, 2$. Moreover, once we establish (a) – i.e. the assertion that $M/(H_1 \cap H_2)$ is torsion free – the same result shows that $(H_1 \cap H_2)(\mathfrak{m})$ may be viewed as a subspace of $M(\mathfrak{m})$ as well.

Proof. First observe that the sequence

$$0 \to H_1 \cap H_2 \to M \xrightarrow{\psi} M/H_1 \oplus M/H_2 \xrightarrow{\gamma} M/(H_1 + H_2) \to 0$$

is exact, where ψ and γ are given by

$$\psi(x) = (x + H_1, x + H_2) \qquad \text{and} \qquad \gamma(x + H_1, y + H_2) = x - y + (H_1 + H_2).$$

It follows from Proposition 2.2 that the A-module $M/(H_1 \cap H_2)$ is torsion free; this proves (a). Now (b) follows from Proposition 2.1; in fact, $\dim_{k(\mathfrak{m})}(H_1 \cap H_2)(\mathfrak{m})$ coincides with $\mathrm{Rank}_A(H_1 \cap H_2)$.

For each maximal ideal $\mathfrak{m}$ we have $\ker \psi(\mathfrak{m}) = H_1(\mathfrak{m}) \cap H_2(\mathfrak{m})$. Since

$$\mathrm{coker}(\psi) \simeq M/(H_1 + H_2),$$

(c) now follows from Proposition 2.5. Finally, (d) follows from (c) as in Remark 2.6. □

The second required consequence of Proposition 2.5 gives a result on the center of a Lie algebra over A. Recall that a Lie algebra L over a commutative ring B is a B-module L together with a B-bilinear mapping $[-,-] : L \times L \to L$ satisfying the usual axioms for a Lie bracket.

Let $Z \subset L$ be the center of L; i.e

$$Z = \{E \in L \mid [F, E] = 0 \text{ for all } F \in L\}.$$

Proposition 2.9. *Let L be a Lie algebra over the Dedekind domain A which is finitely generated and projective as an A-module. Then the center Z of L is an A-Lie subalgebra such that*

(a) The A-module L/Z is torsion free.
(b) The dimension of the $k(\mathfrak{m})$-subalgebra $Z(\mathfrak{m}) \subset L(\mathfrak{m})$ is a constant, independent of the maximal ideal $\mathfrak{m} \subset A$.
(c) For each maximal ideal $\mathfrak{m}$ of A,

$$(\spadesuit) \qquad Z(\mathfrak{m}) \subset \mathfrak{z}(L(\mathfrak{m})),$$

where $\mathfrak{z}(L(\mathfrak{m}))$ denotes the center $\mathfrak{z}(L(\mathfrak{m}))$ of the $k(\mathfrak{m})$-Lie algebra $L(\mathfrak{m})$. Moreover, equality holds in $(\spadesuit)$ for all but finitely many maximal ideals $\mathfrak{m}$ of A.
(d) Suppose that there is an infinite collection Γ of maximal ideals of A and a non-negative integer d for which

$$\dim_{k(\mathfrak{m})} \mathfrak{z}(L(\mathfrak{m})) = d$$

whenever $\mathfrak{m}$ is in Γ. Then $d = \mathrm{Rank}_A(Z)$ and equality holds in $(\spadesuit)$ for all $\mathfrak{m}$ in Γ.

Proof. Choose generators $X_1, \ldots, X_\ell$ for L viewed as an A-module, and consider the mapping

$$\Phi : L \to L \times \cdots \times L \qquad (\ell \text{ factors})$$

given by $\Phi(Y) = ([Y, X_1], \ldots, [Y, X_\ell])$. Since the X_i generate L as an A-module, we have $Z = \ker \Phi$. For any maximal ideal $\mathfrak{m} \subset A$, the images of the X_i generate $L(\mathfrak{m})$. Thus $\ker \Phi(\mathfrak{m})$ is precisely the center $\mathfrak{z}(L(\mathfrak{m}))$.

Now (a) follows from Proposition 2.2, and (b) follows from Proposition 2.1; in fact, $\dim_{k(\mathfrak{m})} Z(\mathfrak{m})$ coincides with $\mathrm{Rank}_A Z$.

Assertion (c) follows from Proposition 2.5, and finally (d) follows from (c) as in Remark 2.6. □

3. Deformation application

In this section, A denotes a Dedekind domain, and we now impose the additional condition that A is also a finitely generated k-algebra for some field k. Moreover, we suppose that the corresponding affine variety $\mathcal{V} = \mathrm{Spec}(A)$ has infinitely many k-points: in other words, we suppose that there are infinitely many maximal ideals $\mathfrak{m} \subset A$ for which $A/\mathfrak{m}$ identifies with k.

We are going to *fix* a preferred k-point $t_0 \in \mathrm{Spec}(A)(k)$. If M is an A-module and if $t \in \mathrm{Spec}(A)(k)$, write $\mathfrak{m}_t$ for t "viewed as a maximal ideal of A", and write $M(t)$ for the k-vector space $M(\mathfrak{m}_t)$.

In fact, we are mainly interested in the following two possibilities for A:

- $A = k[T]$, in which case $\mathcal{V} = \mathrm{Spec}(A)$ is the affine line, and $\mathcal{V}(k) = k$.
- $A = k[T, T^{-1}]$, in which case $\mathcal{V} = \mathrm{Spec}(A)$ is the punctured affine line, and $\mathcal{V}(k) = k^\times$.

Of course, in both these cases the set of points $\mathcal{V}(k)$ is infinite if and only if the field k is infinite.

We now fix some further notation. Consider finite dimensional Lie algebras over k

$$\mathfrak{h} \subset \mathfrak{b}$$

and form the A-Lie algebra

$$L = \mathfrak{b} \otimes_k A.$$

In particular, L is a free A-module of finite rank, and for any $t \in \mathcal{V}(k)$, the algebra $L(t)$ may be canonically identified with $\mathfrak{b}$.

We now denote by σ one of the following two objects:

(S1) An element $\sigma \in \mathfrak{b} \otimes_k A = L$, which we view as a morphism of varieties[1] $\sigma : \mathcal{V} = \mathrm{Spec}(A) \to \mathfrak{b}$.

(S2) An element $\sigma \in J(A)$ for some linear algebraic k-subgroup $J \subset \mathrm{GL}(\mathfrak{b})$ which operates on $\mathfrak{b}$ by Lie algebra automorphisms. In this case, we view σ as a morphism $\mathcal{V} \to J$.

Remark 3.1. Let $t \in \mathcal{V}(k)$. In case (S1), the element $\sigma(t) \in \mathfrak{b}$ is just the image of σ in $\mathfrak{b} = L(t) = L/\mathfrak{m}_t L$. In case (S2), the element $\sigma(t) \in J(k)$ is just the image of σ under the natural group homomorphism $J(A) \to J(A/\mathfrak{m}_t) = J(k)$.

For $t \in \mathcal{V}(k)$, we can speak of the *centralizer* $\mathfrak{c}_\mathfrak{b}(\sigma(t))$:

$$\mathfrak{c}_\mathfrak{b}(\sigma(t)) = \{Y \in \mathfrak{b} \mid [Y, \sigma(t)] = 0\} \quad \text{in case (S1)}$$

and

$$\mathfrak{c}_\mathfrak{b}(\sigma(t)) = \{Y \in \mathfrak{b} \mid \sigma(t)Y = Y\} \quad \text{in case (S2)}.$$

Now let us fix $t_0 \in \mathcal{V}(k)$, and write $X = \sigma(t_0)$. Thus $X \in \mathfrak{b}$ in case (S1), while $X \in J(k)$ in case (S2).

We are going to formulate conditions under which we may view the centralizer $\mathfrak{c}_\mathfrak{b}(X)$ as a "deformation" of the centralizers $\mathfrak{c}_\mathfrak{b}(\sigma(t))$ for $t \in \mathcal{V}(k) - \{t_0\}$. Under some further assumptions, we will then use this deformation to study the intersection $\mathfrak{h} \cap \mathfrak{z}(\mathfrak{c}_\mathfrak{b}(X))$ of $\mathfrak{h}$ with the center $\mathfrak{z}(\mathfrak{c}_\mathfrak{b}(X))$ of the centralizer of X.

To formulate our assumptions, we require a finite subset

$$\Theta \subset \mathcal{V}(k)$$

[1] Abusing notation somewhat, we write $\mathfrak{b}$ both for the affine variety over k determined by $\mathfrak{b}$, and for the set of k-points of that variety.

with $t_0 \notin \Theta$. Here are the conditions of interest:

(A1) The dimension of $\mathfrak{c}_{\mathfrak{b}}(\sigma(t))$ is given by a constant which is independent of $t \in \mathcal{V}(k) - \Theta$.

(A2) The dimension of the center $\mathfrak{z}(\mathfrak{c}_{\mathfrak{b}}(\sigma(t)))$ is constant for $t \in \mathcal{V}(k) - (\Theta \cup \{t_0\})$.

(A3) The center $\mathfrak{z}(\mathfrak{c}_{\mathfrak{b}}(\sigma(t)))$ is contained in $\mathfrak{h}$ for all $t \in \mathcal{V}(k) - (\Theta \cup \{t_0\})$.

Consider the A-Lie algebra C which is the centralizer in L of σ – i.e.

$$C = \{E \in L \mid [E, \sigma] = 0\} \text{ in case (S1)} \quad \text{and} \quad C = \{E \in L \mid \sigma E = E\} \text{ in case (S2)}.$$

Write Z for the center of the A-Lie algebra C.

Since L is a free A-module of finite rank, and since A is Noetherian, an A-submodule M of L is torsion free and finitely generated; since A is Dedekind, M is projective as an A-module by Proposition 2.1. It follows that both C and Z are projective A-modules, and in particular, for each $t \in \mathcal{V}(k)$ we have

$$\dim_{k(t)} C(t) = \operatorname{Rank}_A C \quad \text{and} \quad \dim_{k(t)} Z(t) = \operatorname{Rank}_A Z.$$

This shows that the $k(t)$ dimension of $C(t)$ is constant as a function of t, and likewise the $k(t)$ dimension of $Z(t)$ is constant as a function of t.

Proposition 3.2. *(a) If assumption (A1) holds, then $C(t) = \mathfrak{c}_{\mathfrak{b}}(\sigma(t))$ for all $t \in \mathcal{V}(k) - \Theta$.*

(b) Assume that (A1) and (A2) hold. Then

$$Z(t) = \mathfrak{z}(\mathfrak{c}_{\mathfrak{b}}(\sigma(t))) \quad \textit{for all } t \in \mathcal{V}(k) - (\Theta \cup \{t_0\}),$$

and

$$Z(t_0) \subset \mathfrak{z}(\mathfrak{c}_{\mathfrak{b}}(X)).$$

(c) Assume that (A1), (A2) and (A3) all hold. Then

$$\dim_k \mathfrak{z}(\mathfrak{c}_{\mathfrak{b}}(X)) \cap \mathfrak{h} \geq \dim_k \mathfrak{z}(\mathfrak{c}_{\mathfrak{b}}(\sigma(t))) \qquad \textit{for any } t \in \mathcal{V}(k) - (\Theta \cup \{t_0\}).$$

Proof. The subalgebra C is the kernel of the A-module mapping $\Psi : L \to L$ given by

$$\Psi(E) = [E, \sigma] \quad \text{in case (S1), and by} \quad \Psi(E) = \sigma E - E \quad \text{in case (S2)}.$$

Moreover, for $t \in \mathcal{V}(k)$, the centralizer $\mathfrak{c}_{\mathfrak{b}}(\sigma(t))$ is the kernel of the k-linear mapping $\Psi(t) : \mathfrak{b} \to \mathfrak{b}$ given by

$$(E \mapsto [E, \sigma(t)]) \quad \text{in case (S1), and by} \quad (E \mapsto \sigma(t)E - E) \quad \text{in case (S2)}.$$

Since assumption (A1) holds, Remark 2.6 and Proposition 2.5 yield (a).

We have observed that Z is a projective A-module. Moreover, C/Z is also projective, by Proposition 2.9. Using Proposition 2.4, it follows for $t \in \mathcal{V}(k)$ that $Z(t)$ may be identified with a subalgebra of $\mathfrak{b} = C(t)$.

Now, Proposition 2.9 shows for all $t \in \mathcal{V}(k)$ that

$$(\sharp) \qquad Z(t) \subset \mathfrak{z}(C(t)),$$

and that equality holds for all but finitely many points $t \in \mathcal{V}(k)$.

Now, for $t \in \mathcal{V}(k) - \Theta$, (a) yields $C(t) = \mathfrak{c}_{\mathfrak{b}}(\sigma(t))$. Combined with assumption (A2), this equality shows that there is a non-negative integer d for which

$$d = \dim_k \mathfrak{z}(C(t)) = \dim_k \mathfrak{z}(\mathfrak{c}_{\mathfrak{b}}(\sigma(t)))$$

for any $t \in \mathcal{V}(k) - (\Theta \cup \{t_0\})$. Now ($\sharp$) together with Proposition 2.9(d) implies that

$$(\sharp\sharp) \qquad Z(t) = \mathfrak{z}(C(t)) = \mathfrak{z}(\mathfrak{c}_{\mathfrak{b}}(\sigma(t)))$$

for each $t \in \mathcal{V}(k) - (\Theta \cup \{t_0\})$, and the first assertion in (b) follows. For the remaining assertion, first note since $t_0 \notin \Theta$ that $C(t) = \mathfrak{c}_{\mathfrak{b}}(\sigma(t_0)) = \mathfrak{c}_{\mathfrak{b}}(X)$. Now the result follows from ($\sharp$).

For (c), we apply Proposition 2.7 to the subalgebras Z and $H = \mathfrak{h} \otimes_k A$ of L. According to that result, the quotient $C/(Z \cap H)$ is torsion free, so that by Proposition 2.4 we may view $(Z \cap H)(t)$ as a subspace of $C(t)$. Now Proposition 2.7 shows for $t \in \mathcal{V}(k)$ that

$$(\diamondsuit) \qquad (Z \cap H)(t) \subset Z(t) \cap H(t) = Z(t) \cap \mathfrak{h},$$

and that equality holds for all but finitely many $t \in \mathcal{V}(k)$.

In particular when $t = t_0$, we know by (b) that $Z(t_0) \subset \mathfrak{z}(\mathfrak{c}_{\mathfrak{b}}(X))$ so that ($\diamondsuit$) yields

$$(\diamondsuit\diamondsuit) \qquad (Z \cap H)(t_0) \subset \mathfrak{z}(\mathfrak{c}_{\mathfrak{b}}(X)).$$

For $t \in \mathcal{V} - (\Theta \cup \{t_0\})$, by assumption (A3) the center of $\mathfrak{c}_{\mathfrak{b}}(\sigma(t))$ is contained in $\mathfrak{h}$. Thus for these t, ($\diamondsuit$) and ($\sharp\sharp$) together show that

$$(\diamondsuit\diamondsuit\diamondsuit) \qquad (Z \cap H)(t) \subset Z(t) \cap \mathfrak{h} = \mathfrak{z}(\mathfrak{c}_{\mathfrak{b}}(\sigma(t))) \cap \mathfrak{h} = \mathfrak{z}(\mathfrak{c}_{\mathfrak{b}}(\sigma(t))).$$

Write $e = \mathrm{Rank}_A(Z \cap H)$. Now ($\diamondsuit\diamondsuit\diamondsuit$) implies that $\dim_k Z(t) \cap H(t) = d$ is constant for $t \in \mathcal{V} - (\Theta \cup \{t_0\})$; thus Proposition 2.7(d) shows that

$$e = \dim_k Z(t) \cap H(t) = \dim_k \mathfrak{z}(\mathfrak{c}_{\mathfrak{b}}(\sigma(t))) = d$$

for $t \in \mathcal{V} - (\Theta \cup \{t_0\})$.

On the other hand, ($\diamondsuit\diamondsuit$) shows that

$$e \leq \dim_k \mathfrak{z}(\mathfrak{c}_{\mathfrak{b}}(X)),$$

and (c) now follows. □

4. Standard reductive groups

Fix a ground field k of characteristic $p \geq 0$. We consider the class $\mathcal{C}$ of all reductive linear algebraic groups over k satisfying the following properties:

(S1) $\mathcal{C}$ contains all simple k-groups in *very good* characteristic.
(S2) $\mathcal{C}$ contains all k-tori.
(S3) If G_1 and G_2 are in $\mathcal{C}$, then $G_1 \times G_2$ is in $\mathcal{C}$.
(S4) If G is in $\mathcal{C}$ and H is a reductive k-group, and if there is a separable isogeny between G and H, then H is in $\mathcal{C}$.
(S5) If G is in $\mathcal{C}$ and $D \subset G$ is a diagonalizable subgroup scheme, then $C_G(D)^o$ is in $\mathcal{C}$.
(S6) If G is in $\mathcal{C}$ and $G \simeq H \times T$ for a linear k-group H and a k-torus T, then H is in $\mathcal{C}$.

We say that the groups in $\mathcal{C}$ are the *standard* reductive groups over k.

Remark 4.1. (a) Herpel [Her13] has introduced a class of reductive groups – those for which the characteristic of k is "pretty good"; see [Her13, 2.11]. Consider for a reductive group G a splitting field L for G, and a maximal split torus T of G_L. If the characteristic p of k is positive, it is pretty good for G provided

that the group $X/\mathbf{Z}\psi$ has no p-torsion for each subset $\psi \subset R$ of the system of roots $R \subset X = X^*(T)$.

Since the conditions (S1)-(S6) are compatible with base extension, results described by Herpel in [Her13, §5] show that the characteristic is pretty good for any reductive group in the class $\mathcal{C}$ above; see [Her13, Remark 5.4]. On the other hand, if k is algebraically closed, it is proved in [Her13] that the class $\mathcal{C}$ coincides with the class of reductive groups in pretty good characteristic. It does not seem to be clear whether this coincidence remains valid over k.

(b) The semisimple groups G which are standard – i.e. which are in $\mathcal{C}$ – are precisely those for which the characteristic is *very good*. In particular, SL_n is in $\mathcal{C}$ if and only if $n \not\equiv 0 \pmod p$. On the other hand, GL_n is in $\mathcal{C}$ for any n (argue as in [McN05, Remark 3]).

Proposition 4.2. *Let G be a standard reductive group over k, and let $X \in \mathrm{Lie}(G)(k)$ and $g \in G(k)$. Then $C_G(X)$ and $C_G(g)$ are smooth subgroups.*

Proof. Argue as in [McN05, Prop. 5]. □

Remark 4.3. Let G be reductive over k and suppose that the characteristic of k is pretty good for G. Let H be a closed subgroup scheme of G and consider the centralizer subgroup scheme $C = C_G(H) \subset G$. Herpel showed – see [Her13, Theorem 3.3] – that C is smooth over k. This can be used to recover the conclusion of Proposition 4.2 for G.

Remark 4.4. Consider the following classes of reductive groups studied in the indicated citations:[2]

CLASS OF REDUCTIVE GROUPS	CITATION
groups satisfying the "standard hypotheses"	[JN04]
strongly standard	[McN05]
strongly standard	[MT07]
D-standard	[McN08], [MT09]
T-standard	[McN08]

In each case, the reader will easily check that the indicated class of reductive groups is a subclass of $\mathcal{C}$.

Here is a further (partial) list of properties which hold for a reductive group G in $\mathcal{C}$ – together with a citation for a proof in the case of a "strongly standard" or D-standard group in one of the above citations; the indicated proof generalizes *mutatis mutandum* to all G in $\mathcal{C}$:

(i) [MT07, Prop. 12] – If L is a Levi factor of a parabolic subgroup of G, then L is in $\mathcal{C}$.
(ii) [MT07, Prop. 12] – The Lie algebra $\mathrm{Lie}(G)$ has a non-degenerate G-invariant bilinear form defined over k.
(iii) [MT09, (3.4.2)] – The center ζ_G of G is smooth over k.
(iv) [McN05, Prop. 9] or [Her13, Cor. 5.5] – There is a Springer isomorphism for G – i.e. a G-equivariant isomorphism defined over k between the unipotent variety in G and the nilpotent variety in $\mathrm{Lie}(G)$.

[2]Note that the notion of "strongly standard" in [MT07] is slightly more general than that used in [McN05].

In the spirit of the preceding remark, it seems reasonable to expect that the results of [McN05], [McN08], [MT07], [MT09] – which were formulated for the "strongly standard", D-standard, or T-standard groups (see the above table) – in fact hold for all groups in $\mathcal{C}$; of course, a reader wishing to use results for this apparently larger class of groups should carefully check this assertion.

5. Standard groups and nilpotent elements

Let k be an infinite field. Suppose that G is a standard reductive group over k, as in §4, with Lie algebra $\mathfrak{g}$. Choose a G-equivariant isomorphism (a *Springer isomorphism*) $\varepsilon : \mathcal{N} \to \mathcal{U}$ where $\mathcal{N} \subset \mathfrak{g}$ is the nilpotent variety and $\mathcal{U} \subset G$ is the unipotent variety; see Remark 4.4(iv).

Let $X \in \mathfrak{g}$ be a non-zero nilpotent element, and write $u = \varepsilon(X) \in G(k)$ for the unipotent element corresponding to X via the chosen Springer isomorphism. If $C = C_G(X)$ denotes the centralizer of X in G, then of course also $C = C_G(u)$. Write N for the stabilizer in G of the point in the projective space $\mathbf{P}(\mathfrak{g})$ which "is" the line through X; then N is a k-defined smooth subgroup of G [MT07, Prop. 15].

The following theorem essentially recapitulates an important part of Premet's 2003 proof of the Bala-Carter Theorem which avoids the case-checking required in previous proofs; see [Pre03].

Theorem 5.1. *For each maximal torus S of N, there is a unique cocharacter $\phi : \mathbf{G}_m \to S$ such that*

(a) $X \in \mathfrak{g}(\phi; 2)$; i.e. X has weight two for the adjoint action of the torus which is the image of ϕ, and
(b) the image of ϕ is contained in the derived group of $L = C_G(S)$.
(c) Write $C = C_G(X)$ and $M = C_G(X) \cap C_G(\mathrm{image}(\phi))$. Then C is contained in the parabolic subgroup $P(\phi)$ determined by ϕ, and M is a Levi factor of C; i.e. $C = R \cdot M$ is a semidirect product, where $R = R_u(C)$ is the unipotent radical.

Proof. In the "geometric case" – when k is algebraically closed – the theorem is essentially a consequence of Premet's proof of the Bala-Carter Theorem. Working over the ground field k, (a) and (b) follow from [McN04, Thm. 26], and (c) is [McN04, Cor. 20 and Cor. 29]. □

Following [JN04], one says that the cocharacter ϕ of Theorem 5.1 is *associated with* the nilpotent element X.

Lemma 5.2. *Let H be a linear algebraic group over k, and let $S \subset H$ be a subtorus. There is a k-defined dense open subset $\mathcal{U} \subset S$ for which $C_H^0(S) = C_H^0(x)$ – and in particular $\mathfrak{h}^S = \mathfrak{h}^x$ – for each $x \in \mathcal{U}(k)$.*

Proof. This lemma is essentially contained in [Bor91, Cor. 9.5(2)], but we give some details for clarity. If $C_H^0(S) = H^0$ the result is trivial, so suppose $C_H^0(S) \neq H^0$. Recall that if C is either the centralizer of a semisimple element of G or the centralizer of a subtorus of G, then C is a smooth subgroup scheme of G [Bor91, 9.4(1)]. In particular, $\mathfrak{c}_{\mathfrak{h}}(S) = \mathfrak{h}^S \subsetneq \mathfrak{h}$, where $\mathfrak{h}$ is the Lie algebra of H.

Let k_{sep} be a separable closure of k, and consider the set $X^* = X^*(S_{k_{\mathrm{sep}}})$ of cocharacters of $S_{k_{\mathrm{sep}}}$. For any linear representation V of S, it follows from [Spr 98, 13.1.1] that there is a set of non-zero weights $\Delta_V \subset X^* - \{0\}$ for which

$$V \otimes_k k_{\mathrm{sep}} = \bigoplus_{\lambda \in \Delta_V \cup \{0\}} (V \otimes_k k_{\mathrm{sep}})_\lambda,$$

where $(V \otimes_k k_{\mathrm{sep}})_\lambda$ denotes the λ-weight space; since the representation V is defined over k, the Galois group $\Gamma = \mathrm{Gal}(k_{\mathrm{sep}}/k)$ acts on Δ_V.

We apply these considerations to the adjoint action of S on $\mathfrak{h}$. Thus the set of non-zero weights $\Delta = \Delta_{\mathfrak{h}}$ for the action of $S_{k_{\mathrm{sep}}}$ on $\mathfrak{h} \otimes_k k_{\mathrm{sep}}$ is Γ-stable. Since $\mathfrak{h}^S \neq \mathfrak{h}$, the set Δ is non-empty. Now, $\Delta \subset X^* \subset k_{\mathrm{sep}}[S]$, and the non-zero regular function $F = \prod_{\lambda \in \Delta} \lambda \in k_{\mathrm{sep}}[S]$ is stable under the action of $\mathrm{Gal}(k_{\mathrm{sep}}/k)$. Thus $F \in k[S]$ by Galois descent. Write $\mathcal{U} \subset S$ for the principal k-open subset defined by the non-vanishing of F; since $F \neq 0$, $\mathcal{U}$ is non-empty.

For $x \in S(k)$, we have $C_H^0(S) \subset C_H^0(x)$; thus we only must prove that the indicated centralizers have the same dimension for $x \in \mathcal{U}(k)$. In view of the smoothness, it is enough to observe that the infinitesimal centralizers $\mathfrak{c}_{\mathfrak{h}}(S) = \mathfrak{h}^S$ and $\mathfrak{c}_{\mathfrak{h}}(x) = \mathfrak{h}^x$ coincide for $x \in \mathcal{U}(k)$, which is clear by the choice of $\mathcal{U}$. □

Proposition 5.3. *Let ϕ be a cocharacter associated to X, and let $u = \varepsilon(X) \in G(k)$. Then $\phi(t)u$ is $G(k)$-conjugate to $\phi(t)$ for all but finitely many $t \in k^\times$.*

Proof. The nilpotent element X is a *distinguished* nilpotent element in $\mathrm{Lie}(L)$ where L is a Levi factor of a suitable k-parabolic subgroup and where the image of ϕ lies in L. It is enough to show the required conjugacy using an element of $L(k)$, thus we may and will suppose that X is *distinguished.* In that case, a maximal torus of the centralizer of X is central in G; in view of [JN04, Remark 2.10], we see that the connected k-subgroup N^0 is *nilpotent* (while [JN04] works in the setting in which k is algebraically closed, note that a linear algebraic group A over k is nilpotent if and only if $A_{k_{\mathrm{alg}}}$ is nilpotent, where k_{alg} is an algebraic closure of k).

Since N^0 is nilpotent, $N^0 = S \times U$ where $U = R_u(N) = R_u(C_G(X))$ is the unipotent radical of N (and of $C_G(X)$), and S is the unique maximal torus of N^0. After extending scalars to an algebraic closure, it follows from [JN04, Prop 5.10] that the image of ϕ has no fixed points on the unipotent radical of the centralizer of X other than the identity. Thus, working over the original field k we may apply Lemma 5.2 to learn that for all but possibly finitely many $t \in k^\times$, the element $\phi(t)$ has no fixed points on U. Suppose that t is such an element; thus $C_U(\phi(t)) = 1$. Now, the (geometric) U-orbit of $\phi(t)$ for the action by inner automorphisms is clearly contained in $\phi(t)U$; since U is a unipotent group, its orbits on the affine variety N are all closed [Ste74, Prop. 2.5]. Thus a dimension argument shows that geometrically, the U-orbit $\mathrm{Int}(U)\phi(t)$ is equal to $\phi(t)U$.

Now [McN04, Theorem 28] shows that U is a k-split unipotent group; it then follows from [McN04, Prop. 33] that $\mathrm{Int}(U(k))\phi(t) = \phi(t)U(k)$, and the proof is complete. □

Write $\mathfrak{g}(\phi; n)$ for the subspace on which the image of ϕ acts via the weight $n \in \mathbf{Z}$. Recall that X is said to be *even* provided that $\mathfrak{g}(\phi; 1) = 0$. If X is even, then $\mathfrak{g}(\phi; n) \neq 0$ implies that $n \in 2\mathbf{Z}$.

Proposition 5.4. *Let X be an* even *nilpotent element and choose a cocharacter ϕ associated to X. Then*

$$\dim_k C_G(X) = \dim_k C_G(\mathrm{im}(\phi)).$$

Proof. Since the image of ϕ is a torus, $\mathrm{Lie}(C_G(\mathrm{im}(\phi)))$ coincides with $\mathfrak{g}(\phi; 0)$. It follows from [JN04, Prop. 5.9] that the mapping

$$(\clubsuit) \qquad \mathrm{ad}(X) : \sum_{n \geq 0} \mathfrak{g}(\phi; n) \to \sum_{n \geq 2} \mathfrak{g}(\phi; n)$$

is surjective. Recall by Theorem 5.1 that the centralizer $C = C_G(X)$ of X is contained in $P(\phi)$. Since $\operatorname{Lie} P(\phi) = \sum_{n\geq 0} \mathfrak{g}(\phi; n)$, and since by Proposition 4.2 C is smooth, $\operatorname{Lie} C$ is equal to the kernel of the mapping $\operatorname{ad}(X)$ in (♣). Since X is even, $\mathfrak{g}(\phi; 1) = 0$, so that $\dim \operatorname{Lie} C = \dim \ker \operatorname{ad}(X) = \dim \mathfrak{g}(\phi; 0)$. Again since C is smooth, the proposition follows. □

6. The center of a nilpotent centralizer

Keep the assumptions of the previous section; thus G is a standard reductive group over the infinite field k, and $\mathfrak{g}$ is the Lie algebra of G.

Let $A \in \operatorname{Lie}(G)$, and recall from Proposition 4.2 that $C_G(A)$ is a smooth subgroup of G. Furthermore, we have the following result.

Proposition 6.1. *The center $Z(C_G(A))$ of $C_G(A)$ is a smooth group scheme over k. In particular, $A \in \operatorname{Lie}(Z(C_G(A)))$.*

Proof. The smoothness of $Z(C_G(A))$ is [MT09, Theorem A]. The smoothness implies that $\operatorname{Lie}(Z(C_G(A))) = \operatorname{Lie}(C_G(A))^{\operatorname{Ad} C_G(A))}$, and the remaining assertion is now clear. □

We recall the following result found in [LT11, Lemma 3.5]; since the proof is short, we repeat it here for completeness.

Lemma 6.2. *Let $A, B \in \operatorname{Lie}(G)$ with $[A, B] = 0$. Then $Z(C_G(A)) \subset C_G(B)$, and $[Z(C_G(A)), Z(C_G(B))] = 1$.*

Proof. The adjoint action of the group $Z(C_G(A))$ on $\operatorname{Lie}(C_G(A))$ is trivial; since $B \in \mathfrak{c}_{\mathfrak{g}}(A) = \operatorname{Lie}(C_G(A))$, where the equality holds by smoothness (Proposition 4.2), conclude that $Z(C_G(A))$ centralizes B, which verifies the result. □

Fix now a nilpotent element $X \in \mathfrak{g}$ together with a cocharacter ϕ associated with X. Consider the centralizer $C = C_G(X)$; of course, C is smooth by Proposition 4.2 and in particular $\operatorname{Lie}(C) = \mathfrak{c}_{\mathfrak{g}}(X)$. As in Theorem 5.1(c), denote by $M \subset C$ the Levi factor of C determined by the cocharacter ϕ.

The following result was proved in [LT11, Prop. 3.7] in case G is semisimple and the characteristic is *very good* for G; since we will use the result in a slightly more general situation, we give the proof.

Proposition 6.3. $C = \langle M, Z(C_G(Y)) \mid Y \in \operatorname{Lie}(R)\rangle$.

Proof. Write H for the group on the right-hand side of the stated equality. For any $Y \in \operatorname{Lie}(R) \subset \operatorname{Lie}(C)$, we have $[X, Y] = 0$ and so Lemma 6.2 shows that $Z(C_G(Y)) \subset C_G(X)$. Since also $M \subset C$, conclude that $H \subset C$.

On the other hand, for any $Y \in \operatorname{Lie}(R)$, recall that by Proposition 6.1, $Z(C_G(Y))$ is smooth; thus

$$\operatorname{Lie}(Z(C_G(Y))) = \operatorname{Lie}(C_G(Y))^{\operatorname{Ad} C_G(Y)}.$$

In particular, $Y \in \operatorname{Lie}(Z(C_G(Y)))$. This proves that $\operatorname{Lie}(R) \subset \operatorname{Lie}(H)$. Since also $\operatorname{Lie}(M) \subset \operatorname{Lie}(H)$, and since $\operatorname{Lie}(C) = \operatorname{Lie}(M) + \operatorname{Lie}(R)$, it follows that $\operatorname{Lie}(C) = \operatorname{Lie}(H)$. Since C is smooth, and since $H \subset C$, we conclude that $C^0 = H^0$.

Since R is connected and lies in C, it lies in $C^0 = H^0$; thus we have $M, R \subset H$. Since $C = M \cdot R$, the equality $C = H$ follows, as required. □

We require the following result, which extends [LT11, Theorem 3.9].

Theorem 6.4. $\mathrm{Lie}(Z(C)) = \mathfrak{z}(\mathrm{Lie}(C))^{\mathrm{Ad}(M)} = \mathfrak{z}(\mathfrak{c}_{\mathfrak{g}}(X))^{\mathrm{Ad}(M)}$.

Proof. Of course, the second equality just reflects the fact that the centralizer $C = C_G(X)$ is *smooth.*

For the first equality, note that

$$\mathrm{Lie}(Z(C)) \subset \mathfrak{z}(\mathrm{Lie}(C)) \quad \text{and} \quad \mathrm{Lie}(Z(C)) \subset \mathrm{Lie}(C)^M,$$

thus the inclusion "$\subset$" is clear.

We now prove the inclusion "$\supset$". Suppose that $A \in \mathfrak{z}(\mathrm{Lie}(C))^M$; we must argue that $A \in \mathrm{Lie}(Z(C))$. Now, for each $Y \in \mathrm{Lie}(R)$, $[A, Y] = 0$. Thus it follows from Lemma 6.2 that $Z(C_G(Y)) \subset C_G(A)$, so that $Z(C_G(Y))$ commutes with $Z(C_G(A))$. Since M centralizes A by assumption, we have $M \subset C_G(A)$ so that $Z(C_G(A))$ commutes with M. Thus applying Proposition 6.3, we see that $Z(C_G(A)) \subset Z(C)$; in particular, we have $\mathrm{Lie}(Z(C_G(A))) \subset \mathrm{Lie}(Z(C))$. According to Proposition 6.1, the group $Z(C_G(A))$ is smooth and in particular, $A \in \mathrm{Lie}(Z(C_G(A)))$. We now conclude that $A \in \mathrm{Lie}(Z(C))$. This completes the proof. $\square$

7. The main result

Before formulating the main result of this paper, we first observe the following:

Proposition 7.1. *Let G be a connected and reductive group over k and write ζ_G for its center (viewed as a group scheme over k). Then* $\mathrm{Lie}(\zeta_G)$ *coincides with the center* $\mathfrak{z}(\mathrm{Lie}(G))$ *of* $\mathrm{Lie}(G)$.

Proof. Write $\mathfrak{g} = \mathrm{Lie}(G)$. Since $\mathfrak{z}(\mathfrak{g})$ is equal to the kernel of the adjoint representation $\mathrm{Ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, the assertion follows from the fact – see [SGA3, Prop. 5.7.14] – that ζ_G is the (scheme theoretic) kernel of $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$. $\square$

We now adopt the assumptions of the previous section; in particular, X is an even nilpotent element in $\mathrm{Lie}(G)$.

Theorem 7.2. *We have*

$$\dim Z(C_G(X)) \geq \dim Z(C_G(\mathrm{im}(\phi))).$$

Proof. The group $Z(C_G(X))$ is smooth by Proposition 6.1. The centralizer $C_G(\mathrm{im}(\phi))$ is again a standard reductive group – see §5 – and, in particular, $Z(C_G(\mathrm{im}(\phi)))$ is smooth by Remark 4.4(iii). Thus, it suffices to prove that

$$\dim_k \mathrm{Lie}(Z(C_G(X))) \geq \dim_k \mathrm{Lie}(Z(C_G(\mathrm{im}(\phi)))).$$

Let $\Theta \subset k^\times$ be the set of those $t \neq 1$ for which $C_G(\phi(t))$ fails to be conjugate to $C_G(\phi(t)u)$ or for which $C_G(\phi(t))$ fails to coincide with $C_G(\mathrm{im}\,\phi)$; according to Proposition 5.3 and Lemma 5.2, the set Θ is finite.

According to Theorem 6.4, $\mathrm{Lie}(Z(C_G(X))) = \mathfrak{z}(\mathrm{Lie}(C))^{\mathrm{Ad}(M)}$. Moreover, according to Proposition 7.1, $\mathrm{Lie}(Z(C_G(\mathrm{im}(\phi))))$ coincides with $\mathfrak{z}(\mathfrak{c}_{\mathfrak{g}}(\phi(t)))$ for any $t \in k^\times - \Theta$. Thus to prove the theorem, it suffices to prove that

$$(\sharp) \qquad \dim \mathfrak{z}(\mathfrak{c}_{\mathfrak{g}}(X))^{\mathrm{Ad}(M)} = \dim \mathfrak{z}(\mathfrak{c}_{\mathfrak{g}}(X)) \cap \mathfrak{g}^{\mathrm{Ad}(M)} \geq \dim \mathfrak{z}(\mathfrak{c}_{\mathfrak{g}}(\phi(t)u))$$

for $t \in k^\times - \Theta$.

We are going to apply the results of §3; in particular, Proposition 3.2. Let $A = k[T, T^{-1}]$ so that $\mathcal{V}(k) = k^\times$. Let $J = G$ and $\sigma(T) = \phi(T)u$, viewed as a morphism $\sigma : \mathcal{V} \to G$. Finally, let

$$\mathfrak{b} = \mathfrak{g} \qquad \text{and} \qquad \mathfrak{h} = \mathfrak{g}^M.$$

It only remains to verify conditions (A1), (A2) and (A3).

Condition (A1) is a consequence of Proposition 5.4 and the choice of Θ. Now (A2) follows since for $t \in k^\times - (\Theta \cup \{1\})$ the element $\phi(t)u$ is a semisimple element conjugate to $\phi(t)$. Thus $C_G(\phi(t)u)$ is a standard reductive subgroup which is conjugate to $C_G(\operatorname{im}\phi)$. In particular, the dimension of the center of $C_G(\phi(t)u)$ is independent of $t \in k^\times - (\Theta \cup \{1\})$.

Finally, since M centralizes both u and the image of ϕ, it is contained in $C_G(\phi(t)u)$ for all t. Thus the center of $C_G(\phi(t)u)$ is centralized by M for all $t \in k^\times$, hence (A3) holds.

Now ($\sharp$) – and thus the conclusion of the theorem – follows at once from Proposition 3.2. □

Remark 7.3. Suppose that k has characteristic zero, and write $H = d\phi(1)$, so that X and H may be prolonged to an $\mathfrak{sl}_2$-triple whose span we write as $\mathfrak{s}$. We can apply the results of §3 to this setting to give a second proof of Theorem 7.2 valid under these hypotheses; in this way we recover arguments of Simion-Testerman [ST 14].

It follows from Theorem 6.4 that $\mathfrak{z}(\mathfrak{c}_\mathfrak{g}(X))^M = \operatorname{Lie}(Z(C_G(X)))$. It remains to argue that

$$(\flat\flat) \qquad \dim \mathfrak{z}(\mathfrak{c}_\mathfrak{g}(X))^M = \dim \mathfrak{z}(\mathfrak{c}_\mathfrak{g}(X)) \cap \mathfrak{g}^M \geq \dim \mathfrak{z}(\mathfrak{c}_\mathfrak{g}(X + tH))$$

for $t \neq 0$; for this, we are going to use Proposition 3.2.

Take $A = k[T]$, so that $\mathcal{V}(k) = k$. Write $\mathfrak{b} = \mathfrak{g}$ and $\mathfrak{h} = \mathfrak{g}^M$, and consider the regular map $\sigma : \mathcal{V} \to \mathfrak{b}$ given by $t \mapsto X + tH$.

We now verify conditions (A1), (A2) and (A3). The equality

$$\dim C_G(X) = \dim C_G(X + tH) \qquad \text{for} \quad t \in k$$

can be verified as in [ST 14, Lemma 2.7]; it is essentially a consequence of the representation theory of $\mathfrak{s}$. This verifies (A1).

Moreover, for $t \neq 0$, $X + tH$ is a semisimple element of $\mathfrak{s}$ which is conjugate to H. Thus the dimension of the center of $C_G(X + tH)$ is constant for $t \neq 0$ and (A2) is verified. Finally B centralizes both X and H, so (A3) is immediate. Now ($\flat\flat$) follows at once from Proposition 3.2.

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