

NON-LEFT-ORDERABLE SURGERIES ON TWISTED TORUS KNOTS

KATHERINE CHRISTIANSON, JUSTIN GOLUBOFF, LINUS HAMANN,
AND SRIKAR VARADARAJ

(Communicated by Martin Scharlemann)

ABSTRACT. Boyer, Gordon, and Watson have conjectured that an irreducible rational homology 3-sphere is an L-space if and only if its fundamental group is not left-orderable. Since large classes of L-spaces can be produced from Dehn surgery on knots in S^3 , it is natural to ask what conditions on the knot group are sufficient to imply that the quotient associated to Dehn surgery is not left-orderable. Clay and Watson developed a criterion for determining the left-orderability of this quotient group and used it to verify the conjecture for surgeries on certain L-space twisted torus knots. We generalize a recent theorem of Ichihara and Temma to provide another such criterion. We then use this new criterion to generalize the results of Clay and Watson and to verify the conjecture for a much broader class of L-spaces obtained by surgery on twisted torus knots.

1. INTRODUCTION

For a closed, connected, orientable 3-manifold Y , let $\widehat{\text{HF}}(Y)$ denote the Heegaard Floer homology of Y , as defined in [OS04b]. We begin with a definition.

Definition 1. A closed, connected, orientable 3-manifold Y is an *L-space* if it is a rational homology sphere satisfying $\text{rk } \widehat{\text{HF}}(Y) = |H_1(Y; \mathbb{Z})|$.

A result due to Ozsváth and Szabó [OS04a, Proposition 5.1] gives $\text{rk } \widehat{\text{HF}}(Y) \geq |H_1(Y; \mathbb{Z})|$ for any rational homology 3-sphere Y . Thus, we can understand L-spaces as spaces with minimal Heegaard Floer homology. L-spaces derive their name from lens spaces, which were the first class of spaces observed to have minimal Heegaard Floer homology; however, many other spaces, such as those which admit an elliptic geometry [OS05, Proposition 2.3], are also L-spaces.

It is interesting to consider whether L-spaces may be characterized using properties unrelated to their Heegaard Floer homologies. We recall the following definition.

Definition 2. A nontrivial group G is *left-orderable* if there exists a strict total ordering $>$ of the elements of G that is left-invariant: whenever $g > h$ then $fg > fh$, for all $g, h, f \in G$.

Received by the editors October 18, 2014 and, in revised form, May 15, 2015 and June 12, 2015.

2010 *Mathematics Subject Classification.* Primary 57M25; Secondary 20F60, 57M50.

Key words and phrases. Left-orderability, twisted torus knots, Dehn surgery.

The authors were partially funded by NSF grant DMS-0739392.

Boyer, Gordon, and Watson established that a closed, connected, Seifert fibred 3-manifold is an L-space if and only if its fundamental group cannot be left-ordered [BGW13]. After providing further examples to support this correspondence, they proposed the following conjecture.

Conjecture 3 ([BGW13, Conjecture 3]). *An irreducible rational homology 3-sphere is an L-space if and only if its fundamental group is not left-orderable.*

In order to investigate this conjecture, it is useful to consider Dehn surgery on knots in S^3 , since this process provides large classes of 3-manifolds. Boyer, Rolfsen, and Wiest [BRW05, Theorem 1.1] demonstrated that the fundamental group of a P^2 -irreducible, connected, compact 3-manifold is left-orderable if and only if it has a nontrivial homomorphic image which is left-orderable. Since the abelianization of any knot group is \mathbb{Z} , we have that any knot group is left-orderable. However, the fundamental group of a manifold produced by Dehn surgery is a quotient of the knot group, which may or may not be left-orderable. In light of these observations and Conjecture 3, it is natural to ask the following question (cf. [CW13, Question 1.4]).

Question 4. Given a knot K in S^3 and a rational number r , what conditions on the knot group of K are sufficient to imply that r -surgery on K yields a manifold with non-left-orderable fundamental group?

In [CW13], Clay and Watson answer Question 4 with the following sufficient condition. We denote by $S_K^3(r)$ the manifold produced by r -surgery on a knot K .

Theorem 5 ([CW13, Theorem 1.5]). *Let \mathbb{Q}^+ be the set of positive rational numbers. Let K be a nontrivial knot in S^3 , let μ and λ be a meridian and 0-framed longitude, respectively, of K , and let $\frac{p_0}{q_0}, \frac{p_1}{q_1} \in \mathbb{Q}^+$ with $p_i, q_i > 0$. If $\mu^{p_0} \lambda^{q_0} > 1$ implies $\mu^{p_1} \lambda^{q_1} > 1$ for every left ordering $>$ of the knot group of K , then $\pi_1(S_K^3(p/q))$ is not left-orderable for any $\frac{p}{q} \in \mathbb{Q}^+$ such that $p, q > 0$ and $\frac{p}{q} \in \left(\frac{p_0}{q_0}, \frac{p_1}{q_1}\right)$.*

In order to consider other sufficient conditions that answer Question 4, we require the following well-known equivalent condition for left-orderability (see, for instance, [Ghy01, Theorem 6.8]).

Theorem 6. *Let G be a countable group. Then the following are equivalent:*

- G acts effectively on the real line by order-preserving homeomorphisms.
- G is left-orderable.

Let us denote by $\text{Homeo}^+(\mathbb{R})$ the group of order-preserving homeomorphisms of \mathbb{R} . Then, the first condition in Theorem 6 is equivalent to the existence of an injective homomorphism $\Phi : G \rightarrow \text{Homeo}^+(\mathbb{R})$. For such homomorphisms, we will sometimes abuse notation and write $g(t)$ for $\Phi(g)t$ for elements $g \in G$ and $t \in \mathbb{R}$.

We are interested in studying global fixed points of such a homomorphism, i.e. points $t \in \mathbb{R}$ such that $\Phi(g)t = t$ for all $g \in G$. The following lemma due to Boyer, Rolfsen, and Wiest demonstrates the importance of these points.

Lemma 7 ([BRW05, Lemma 5.1]). *If there is a homomorphism $\Phi : G \rightarrow \text{Homeo}^+(\mathbb{R})$ with nontrivial image, then there is another such homomorphism with no global fixed points.*

With this lemma and Theorem 6, the following criterion for non-left-orderability is straightforward.

Proposition 8. *If G is a countable group and every homomorphism $\Phi : G \rightarrow \text{Homeo}^+(\mathbb{R})$ has a global fixed point, then G is not left-orderable.*

Proof. By contradiction, assume G is left-orderable. Then, by Theorem 6, there exists $\Phi : G \rightarrow \text{Homeo}^+(\mathbb{R})$ injective. We can then apply Lemma 7 to conclude that there exists $\Phi' : G \rightarrow \text{Homeo}^+(\mathbb{R})$ with no global fixed points. But this contradicts our hypothesis. \square

Ichihara and Temma use exactly the reasoning of Proposition 8 in [IT14] to demonstrate the following criterion for non-left-orderability of the fundamental groups of surgery manifolds. Their work was motivated by that of Nakae in [Nak13].

Theorem 9 ([IT14]). *Let K be a knot in S^3 . Suppose that the knot group $\pi_1(S^3 - K)$ is of the form*

$$\langle x, y \mid (w_1 x^c w_1^{-1}) y^{-r} (w_2^{-1} x^d w_2) y^{r-\ell}, \mu \lambda \mu^{-1} \lambda^{-1} \rangle$$

where $c, d \geq 0$, $r \in \mathbb{Z}$, $\ell \geq 0$, $\mu = x$, $\lambda = x^{-m} w x^{-n}$, w is a word which excludes x^{-1} and y^{-1} , $m, n \geq 0$, and $p/q \geq m + n$. Then, Dehn surgery along the slope p/q yields a closed 3-manifold with non-left-orderable fundamental group.

The criteria used in the proof of Theorem 9 actually apply to a more general class of group presentations. The main result of our paper is an extraction of these criteria which reframes the theorem in a more widely applicable manner. The proof of Theorem 10 closely follows the proof of Theorem 9.

Theorem 10. *Let K be a nontrivial knot in S^3 . Let G denote the knot group of K , and let $G(p/q)$ be the quotient of G resulting from p/q -surgery. Let μ be a meridian of K and s be a v -framed longitude with $v > 0$. Suppose that G has two generators, x and y , such that $x = \mu$ and s is a word which excludes x^{-1} and y^{-1} and contains at least one x . Suppose further that every homomorphism $\Phi : G(p/q) \rightarrow \text{Homeo}^+(\mathbb{R})$ satisfies $\Phi(x)t > t$ for all $t \Rightarrow \Phi(y)t \geq t$ for all t . If $p, q > 0$, then, for $p/q \geq v$, $G(p/q)$ is not left-orderable.*

Remark. When applying Theorem 10, it is sufficient to demonstrate that every homomorphism $\Phi : G \rightarrow \text{Homeo}^+(\mathbb{R})$ satisfies $\Phi(x)t > t$ for all $t \Rightarrow \Phi(y)t \geq t$ for all t , since this implies the final hypothesis in the statement of the theorem. As a result, we can understand Theorem 10, like Theorems 5 and 9, to be a set of conditions on the knot group.

Proof of Theorem 10. By Proposition 8, it suffices to show that every homomorphism $\Phi : G(p/q) \rightarrow \text{Homeo}^+(\mathbb{R})$ has a global fixed point. First, note that since $G(p/q)$ has the relation $x^{p-qv} s^q = 1$, we have

$$(1) \quad x^{qv-p} = s^q.$$

Now, assume that $x(t) = t$ (equivalently, $x^{-1}(t) = t$) for some $t \in \mathbb{R}$. Assume $y(t) \neq t$; then, we can pick an order such that $y(t) > t$, or equivalently, $y^{-1}(t) < t$. By hypothesis, s^{-1} contains only x^{-1} and y^{-1} , which are order-preserving homeomorphisms of \mathbb{R} . Thus, by applying $x^{-1}(t) = t$ and $y^{-1}(t) < t$ repeatedly, we can conclude that $s^{-q}(t) < t$. (Note that s must contain at least one y : otherwise, s would be a power of the meridian x , so s and x could not generate the peripheral subgroup.) But then

$$t > s^{-q}(t) = x^{p-qv}(t) = t$$

which is a contradiction. Thus, $y(t) = t$, and we have a global fixed point.

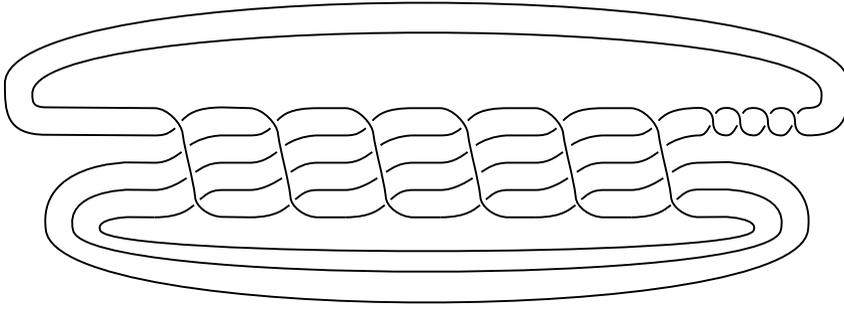


FIGURE 1. The $(5, 6, 2, 2)$ -twisted torus knot, denoted $T_{5,6}^{2,2}$.

Now, we are left with the case $x(t) \neq t$ for all t . We prove this is impossible. Since x has no fixed points, we can pick an order such that $x(t) > t$ for all t . By assumption, then, $y(t) \geq t$ for all t . Now, s contains only x and y , and we have $y(x(t)) > y(t) \geq t$. So, $s(t) > t$ for all t , since s contains at least one x by assumption. By equation (1), $s^q(t) = x^{qv-p}(t)$, so we must have $qv - p > 0$, or $v > p/q$. But this contradicts the assumption $p/q \geq v$. \square

We note that this theorem can be restated in a purely group-theoretic sense. Consider a group G that has a $\mathbb{Z} \oplus \mathbb{Z}$ -subgroup with distinguished generators μ and s . If we assume the hypotheses of Theorem 10, then $G/\langle\langle \mu^{p-qv} s^q \rangle\rangle$ is not left-orderable.

Returning to the motivation for Question 4, we can consider how these various criteria for non-left-orderability of the fundamental groups of surgery manifolds can help verify Conjecture 3 for these manifolds. We say that a knot K in S^3 which admits a positive L-space surgery is an *L-space knot*. It is known [OS11, Corollary 1.4] that if K is an L-space knot, then r -surgery on K produces an L-space exactly when $r \geq 2g(K) - 1$, where $g(K)$ denotes the genus of K . We will focus on demonstrating that surgeries larger than this bound on known L-space knots produce manifolds with non-left-orderable fundamental groups.

The specific knots we will consider are families of L-space twisted torus knots. We denote by $T_{p,q}^{\ell,m}$ the twisted torus knot obtained from the (p, q) -torus knot by twisting ℓ strands m times. We will call this twisted torus knot the (p, q, ℓ, m) -twisted torus knot. Figure 1, for instance, shows $T_{5,6}^{2,2}$.

We denote the 3-manifold produced by r -surgery on $T_{p,q}^{\ell,m}$ as $M_{p,q}^{\ell,m}(r)$. Throughout, we will assume $p, q, \ell, m, r > 0$, $r \in \mathbb{Q}$.

The following theorem due to Vafaee [Vaf13] provides a class of twisted torus knots which are known to be L-space knots.

Theorem 11 ([Vaf13]). $T_{p,pk\pm 1}^{\ell,m}$ is an L-space knot if and only if

- (1) $\ell = p - 1$,
- (2) $\ell = p - 2$ and $m = 1$, or
- (3) $\ell = 2$ and $m = 1$.

In [CW13], Clay and Watson apply Theorem 5 to certain subfamilies of the L-space knots specified in Theorem 11. Their progress is summarized by the following theorem.

Theorem 12 ([CW13, Theorems 4.5 and 4.7]).

- (a) For sufficiently large r , $M_{3,5}^{2,m}(r)$ has a non-left-orderable fundamental group.
- (b) Suppose that q is a positive integer congruent to 2 modulo 3. For sufficiently large r , $M_{3,q}^{2,1}(r)$ has a non-left-orderable fundamental group.

We note that the knots considered by Clay and Watson are all of the form $T_{p,p(k+1)-1}^{p-1,m}$. Thus, both theorems verify Conjecture 3 for surgeries on subfamilies of the first case of twisted torus knots specified by Theorem 11.

In [IT14], Ichihara and Temma similarly apply Theorem 9 to prove the following theorem, generalizing the work of Clay and Watson.

Theorem 13 ([IT14, Corollary 1.2]). *Suppose that q is a positive integer congruent to 2 modulo 3. For sufficiently large r , then $M_{3,3k-1}^{2,m}(r)$ has non-left-orderable fundamental group.*

In this paper, we apply Theorem 10 to prove the following result.

Theorem 14. *For sufficiently large r , $M_{p,pk\pm 1}^{p-1,m}(r)$ and $M_{p,pk\pm 1}^{p-2,1}(r)$ have non-left-orderable fundamental group.*

Theorem 14 answers Question 4 for cases (1) and (2) of the twisted torus knots specified in Theorem 11. Case (1) is a generalization of Theorems 12 and 13; case (2) is an entirely new family of twisted torus knots. In light of Theorem 11, these results support Conjecture 3.

The paper is organized as follows. In Section 2, we compute the knot groups of $T_{p,pk\pm 1}^{\ell,m}$ and their corresponding peripheral subgroups. In Section 3, we prove Theorem 14.

2. COMPUTING KNOT GROUPS AND PERIPHERAL SUBGROUPS

First, we fix some notation. For the knot groups of twisted torus knots, we will generalize the notation of [CW13] by defining

$$G_{p,q}^{\ell,m} = \pi_1(S^3 - T_{p,q}^{\ell,m}).$$

We will denote by $G_{p,q}^{\ell,m}(r)$ the fundamental group of $M_{p,q}^{\ell,m}(r)$.

It is a well-known fact (see, for instance, [Lic97]) that, for a nontrivial knot, the fundamental group of the boundary of the knot complement injects into the knot group. Its image (up to conjugation) is the peripheral subgroup, which means that the peripheral subgroup is abelian.

We now derive the knot groups of two general cases, $T_{p,pk+1}^{\ell,m}$ and $T_{p,pk-1}^{\ell,m}$. From these, we can then get the knot groups of the L-space twisted torus knots specified in Theorem 11 by plugging in specific values of ℓ and m . Our approach to both cases is the same as that of Clay and Watson [CW13]: we will use the Seifert-van Kampen Theorem applied to a genus-two Heegaard splitting, with the knot appearing on the Heegaard surface (see Figures 2 and 7, which depict $T_{5,5k-1}^{2,m}$ and $T_{5,5k+1}^{2,m}$ as examples).

Proposition 15. *For the $(p, pk - 1, \ell, m)$ -twisted torus knot, the following hold.*

- (a) *The knot group is*

$$\begin{aligned} G_{p,pk-1}^{\ell,m} &= \langle a, b \mid a^{p-\ell} (a(b^{1-k(p-\ell)} a^{p-\ell})^m)^{\ell-1} a \\ &= b^{k(p-\ell)-1} (b^k (b^{1-k(p-\ell)} a^{p-\ell})^m)^{\ell-1} b^k \rangle. \end{aligned}$$

(b) The peripheral subgroup is generated by the meridian

$$\mu = a^{-1}b^k$$

and the surface framing

$$s = \mu^{p(pk-1)+\ell^2 m} \lambda = a^{p-\ell-1} (a(b^{1-k(p-\ell)} a^{p-\ell})^m)^\ell a.$$

Proof. Let $S^3 = U \cup_\Sigma V$ be the genus-two Heegaard splitting of S^3 specified by Figure 2. Then $\pi_1(U)$ is the free group on the generators a and c , and $\pi_1(V)$ is the free group on the generators b and d (see Figure 3). Using the Seifert-Van Kampen Theorem, we can then express $G_{p,pk-1}^{\ell,m}$ as a free product with amalgamation of $\pi_1(U)$ and $\pi_1(V)$. To do this, we need the images of the generators of $\pi_1(\Sigma \setminus \nu(T_{p,pk-1}^{\ell,m}))$ under inclusion into $\pi_1(U)$ and $\pi_1(V)$.

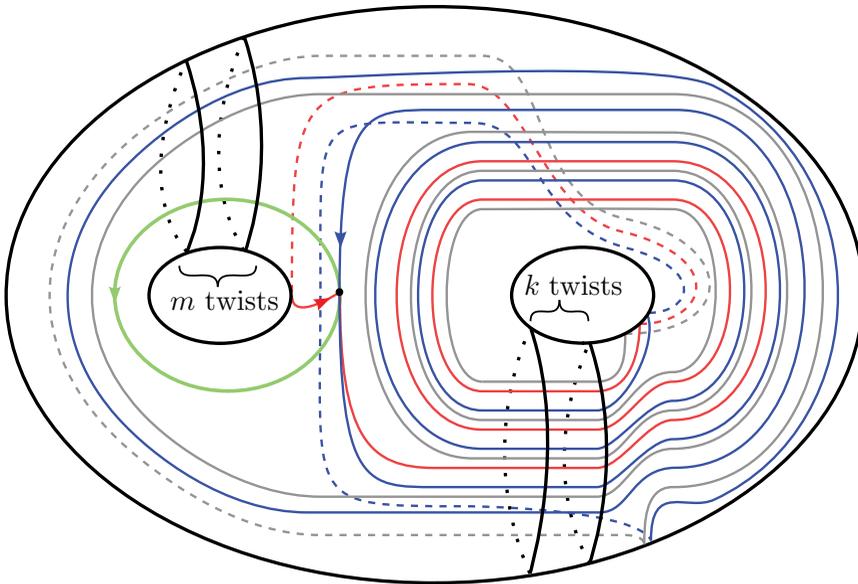


FIGURE 2. Generators for the fundamental group of $\Sigma \setminus \nu(T_{p,pk-1}^{\ell,m})$. Here, we show $T_{5,5k-1}^{2,m}$ as an example. The knot is shown in gray.

Denote by R , G , and B the red, green, and blue oriented curves, respectively, in Figure 3. Now, $\Sigma \setminus \nu(T_{p,pk-1}^{\ell,m})$ deformation retracts onto the wedge product of R , G , and BR^{-1} (understood as an operation in $\pi_1(\Sigma \setminus \nu(T_{p,pk-1}^{\ell,m}))$), so R , G , and B generate the fundamental group of $\Sigma \setminus \nu(T_{p,pk-1}^{\ell,m})$. The green loop has image c in $\pi_1(U)$ and image d^m in $\pi_1(V)$, so we get the relation

$$c = d^m.$$

Likewise, the red loop gives

$$a^{p-\ell} = b^{(p-\ell)k-1} d$$

and the blue loop gives

$$a^{p-\ell} (ac)^{\ell-1} a = b^{(p-\ell)k-1} (b^k d^m)^{\ell-1} b^k.$$

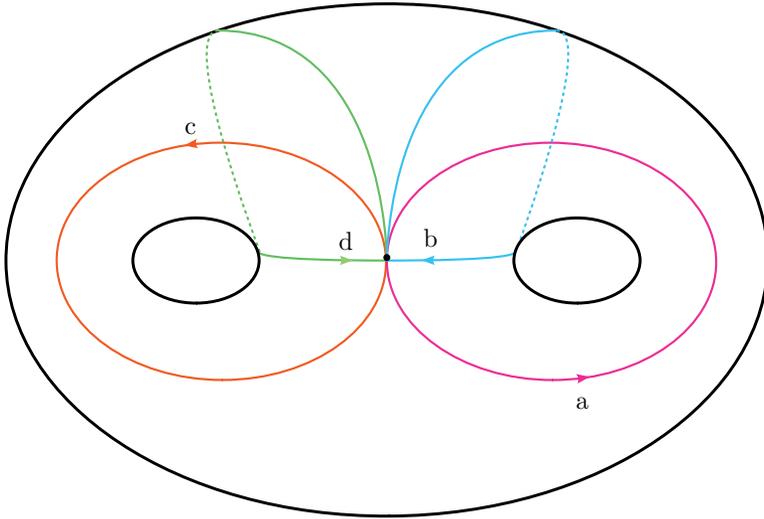


FIGURE 3. Generators for the fundamental group of the Heegaard surface Σ .

Using the first two relations to solve for c and d , we are left with only one relation:

$$a^{p-\ell}(a(b^{1-k(p-\ell)}a^{p-\ell})^m)^{\ell-1}a = b^{k(p-\ell)-1}(b^k(b^{1-k(p-\ell)}a^{p-\ell})^m)^{\ell-1}b^k.$$

Thus, we have

$$\begin{aligned} G_{p,pk-1}^{\ell,m} &= \langle a, b \mid a^{p-\ell}(a(b^{1-k(p-\ell)}a^{p-\ell})^m)^{\ell-1}a \\ &= b^{k(p-\ell)-1}(b^k(b^{1-k(p-\ell)}a^{p-\ell})^m)^{\ell-1}b^k \rangle. \end{aligned}$$

For the peripheral subgroup, we will compute the meridian μ and the surface framing s as specified in Figure 4. From Figure 4, it is immediately clear that

$$s = a^{p-\ell}c(ac)^{\ell-1}a = a^{p-\ell-1}(a(b^{1-k(p-\ell)}a^{p-\ell})^m)^{\ell}a.$$

In order to compute μ , we focus on the right half of the handlebody in Figure 4. This part of the knot is shown along with a and b in Figure 5.

As a base case, we consider $k = 1$. Figure 6 demonstrates that the word $a^{-1}b$ is homotopic to μ in this case. For larger k , we note that b is homotopic to the core of one full twist, as seen in Figure 5. So, for each of the $k - 1$ twists added to the $k = 1$ base case, we must append one extra copy of b to the end of the word $a^{-1}b$ in order to create a word which is homotopic to μ . Thus,

$$\mu = a^{-1}b^k.$$

Finally, we note that the linking number between $T_{p,q}^{\ell,m}$ and a push-off along Σ , by the construction of the twisted torus knot, is $pq + \ell^2m$, which gives us

$$s = \mu^{p(pk-1)+\ell^2m}\lambda.$$

□

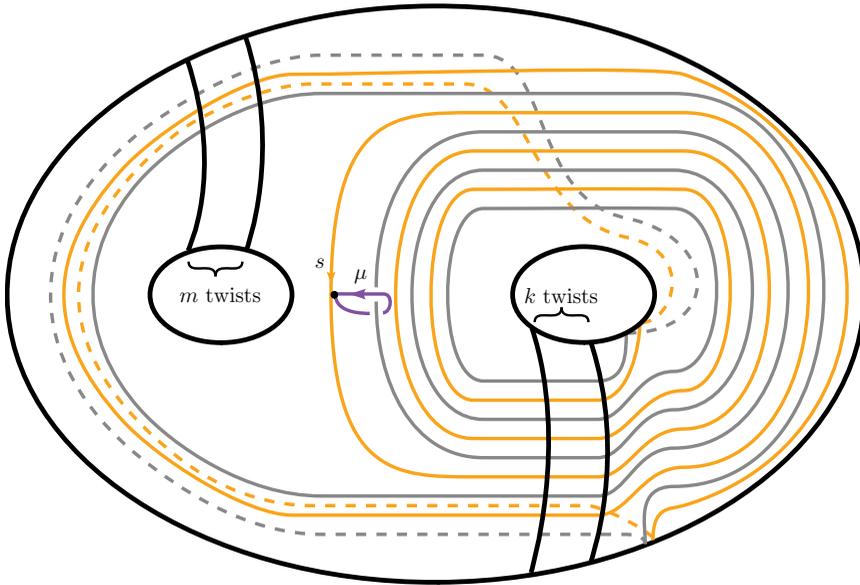


FIGURE 4. Generators for the peripheral subgroup of $T_{p,pk-1}^{\ell,m}$. Here, we show $T_{5,5k-1}^{2,m}$ as an example. The knot is shown in gray.

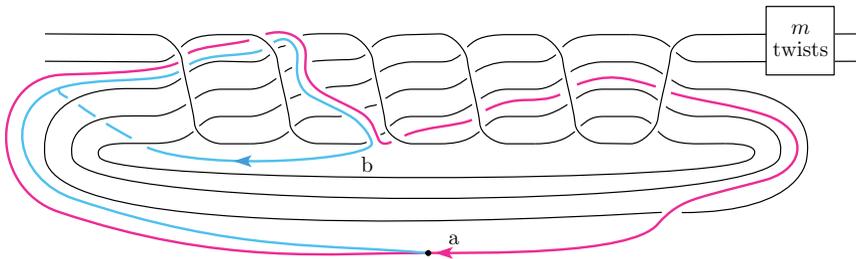


FIGURE 5. Generators a and b for the knot group of $T_{5,5k-1}^{2,m}$. We show $k = 1$ as a base case. Note that this braid is a view from the back of the above handlebody diagrams.

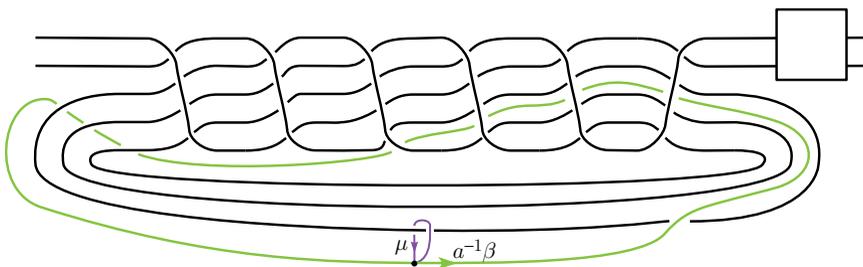


FIGURE 6. Homotopy between μ and $a^{-1}b$ for the base case $k = 1$.

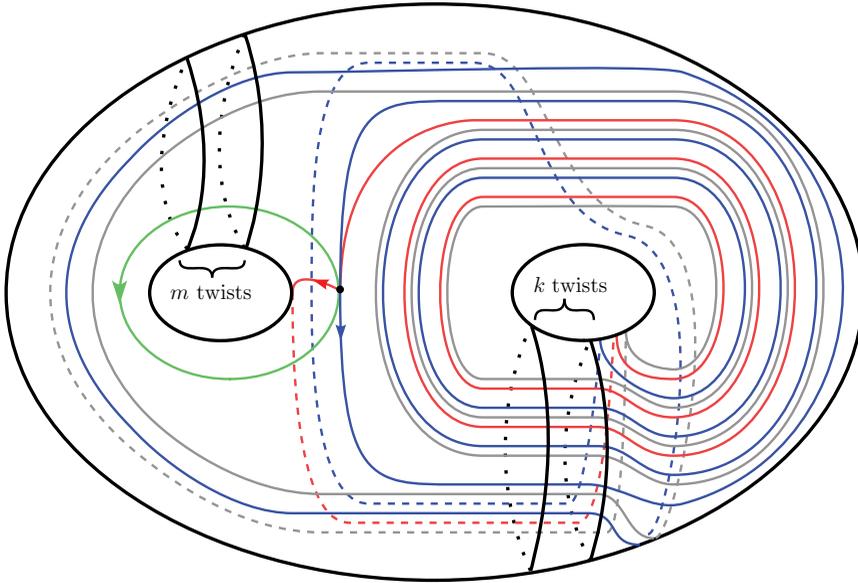


FIGURE 7. Generators for the fundamental group of $\Sigma \setminus \nu(T_{p,pk+1}^{\ell,m})$. Here, we show $T_{5,5k+1}^{2,m}$ as an example. The knot is shown in gray.

Proposition 16. *For the $(p, pk + 1, \ell, m)$ -twisted torus knot, the following hold.*

(a) *The knot group is*

$$G_{p,pk+1}^{\ell,m} = \langle a, b \mid a((b^{k(p-\ell)+1}a^{\ell-p})^m a)^{\ell-1} a^{p-\ell} = b^k((b^{k(p-\ell)+1}a^{\ell-p})^m b^k)^{\ell-1} b^{k(p-\ell)+1} \rangle.$$

(b) *The peripheral subgroup is generated by the meridian*

$$\mu = b^{-k} a$$

and the surface framing

$$s = \mu^{p(pk+1)+\ell^2 m} \lambda = ((b^{k(p-\ell)+1}a^{\ell-p})^m a)^{\ell} a^{p-\ell}.$$

Proof. Let $S^3 = U \cup_{\Sigma} V$ be the genus-two Heegaard splitting of S^3 specified by Figure 7. We use the same reasoning as in Proposition 15 to write $G_{p,pk+1}^{\ell,m}$ as a free product with amalgamation of $\langle a, c \rangle$ and $\langle b, d \rangle$ (see Figure 3).

The generators for the fundamental group of $\Sigma \setminus \nu(T_{5,5k+1}^{2,m})$ are shown in Figure 7. From the green loop, we get

$$c = d^m,$$

from the red loop, we get

$$a^{p-\ell} = d^{-1} b^{k(p-\ell)+1},$$

and from the blue loop, we get

$$(ac)^{\ell-1} a^{p-\ell+1} = (b^k d^m)^{\ell-1} b^{k(p-\ell)+1}.$$

Using the first two relations to solve for c and d , we are left with only one relation:

$$(a(b^{k(p-\ell)+1}a^{\ell-p})^m)^{\ell-1} a^{p-\ell+1} = (b^k(b^{k(p-\ell)+1}a^{\ell-p})^m)^{\ell-1} b^{k(p-\ell)+1}.$$

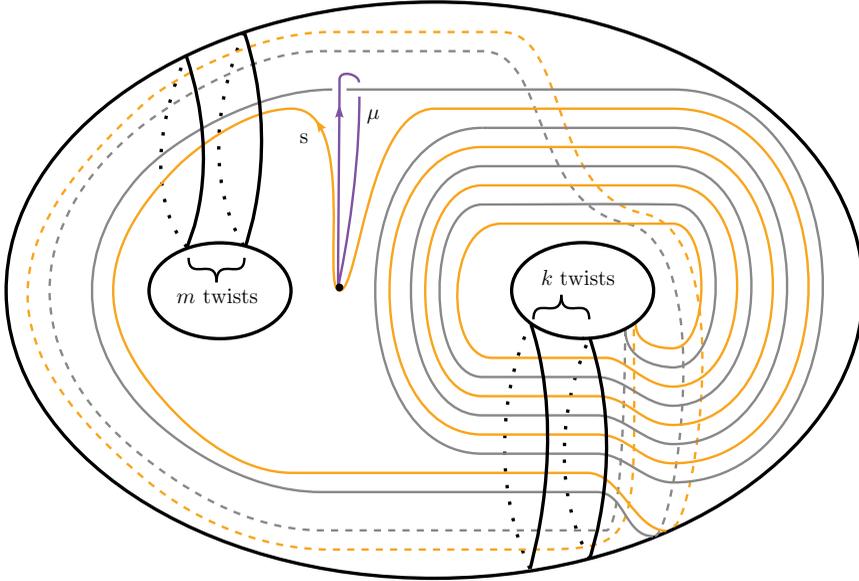


FIGURE 8. Generators for the peripheral subgroup of $T_{p,pk+1}^{\ell,m}$. Here, we show $T_{5,5k+1}^{2,m}$ as an example. The knot is shown in gray.

Rewriting this slightly to create the same form as the group relation in Proposition 15, we get

$$\begin{aligned} G_{p,pk+1}^{\ell,m} &= \langle a, b \mid a((b^{k(p-\ell)+1}a^{\ell-p})^m a)^{\ell-1} a^{p-\ell} \\ &= b^k((b^{k(p-\ell)+1}a^{\ell-p})^m b^k)^{\ell-1} b^{k(p-\ell)+1} \rangle. \end{aligned}$$

The reasoning for the peripheral subgroup also follows that of Proposition 15 very closely. It is immediate from Figure 8 that

$$s = (ca)^\ell a^{p-\ell} = ((b^{k(p-\ell)+1}a^{\ell-p})^m a)^\ell a^{p-\ell}.$$

To compute μ , we focus on the right half of the handlebodies in Figure 8. This part of the knot is shown along with a and b in Figure 9. We consider $k = 1$ as a base case: here, Figure 10 demonstrates that the word $b^{-1}a$ is homotopic to μ . For larger k , we note that b is homotopic to the core of one full twist, as depicted in Figure 9. So, for each of the $k - 1$ twists added to the $k = 1$ base case, we must append one extra copy of b^{-1} to the start of the word $b^{-1}a$ in order to create a word which is homotopic to μ . Thus,

$$\mu = b^{-k}a.$$

Finally, the linking number between $T_{p,pk+1}^{\ell,m}$ and a push-off along Σ is $p(pk + 1) + \ell^2 m$, so that

$$s = \mu^{p(pk+1)+\ell^2 m} \lambda.$$

□

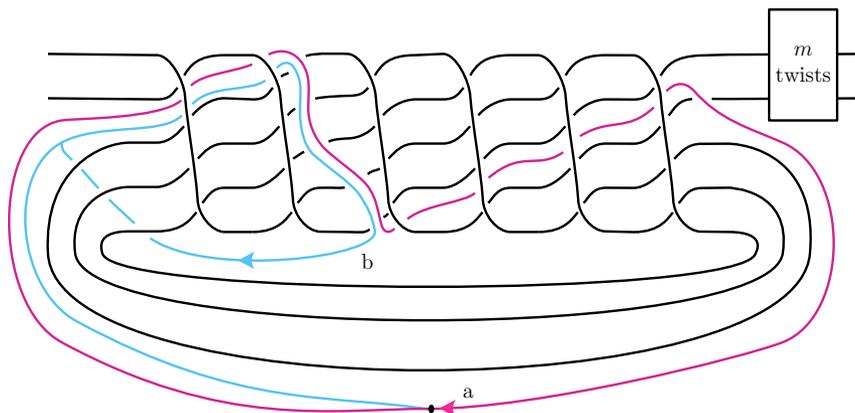


FIGURE 9. Generators a and b for the knot group of $T_{5,5k+1}^{2,m}$. Here, we show $k = 1$ as a base case. Note that this braid is a view from the back of the above handlebody diagrams.

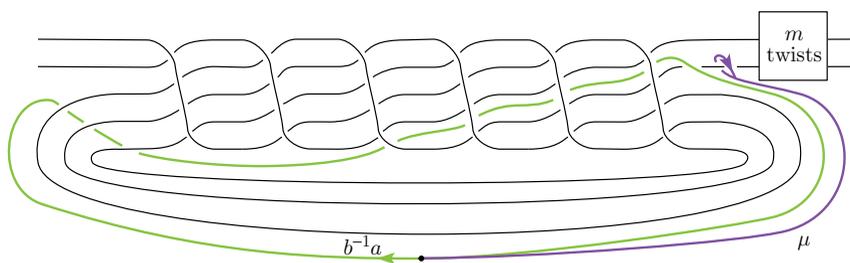


FIGURE 10. Homotopy between μ and $b^{-1}a$ for the base case $k = 1$.

3. NON-LEFT-ORDERABILITY OF L-SPACE TWISTED TORUS KNOTS

In this section, we prove Theorem 14. The proof relies on applications of Theorem 10 to the twisted torus knots specified in cases (1) and (2) of Theorem 11.

First, we note that for all the $T_{p,q}^{\ell,m}$ which are L-space knots, we have knot groups with 2 generators, and we have computed expressions for the meridian μ and the $pq + \ell^2 m$ -framed longitude s (see Propositions 15 and 16). To apply Theorem 10, then, we have to find generators x and y such that

- (1) for any homomorphism $\Phi : G_{p,q}^{\ell,m}(r) \rightarrow \text{Homeo}^+(\mathbb{R})$, $x(t) > t$ for all $t \in \mathbb{R}$ implies $y(t) \geq t$ for all $t \in \mathbb{R}$; and
- (2) x is (a conjugate of) μ , and (the corresponding conjugate of) s can be written with only positive powers of x and y and at least one x .

Then, we can apply Theorem 10 to conclude that r -surgery on $T_{p,q}^{\ell,m}$ yields a 3-manifold with non-left-orderable fundamental group for all $r \geq pq + \ell^2 m$.

We begin with two lemmas that verify condition (1) of the above list for the first two cases in Theorem 11.

Lemma 17. Consider $G_{p,pk-1}^{\ell,m}$ as in Proposition 15. Let $x = a^{-1}b^k$ and $y = b^{1-k}a$. Then, x is a meridian of $T_{p,pk-1}^{\ell,m}$, x and y generate $G_{p,pk-1}^{\ell,m}$, and for any homomorphism $\Phi : G_{p,pk-1}^{\ell,m}(r) \rightarrow \text{Homeo}^+(\mathbb{R})$, $x(t) > t$ for all real t implies that $y(t) \geq t$ for all real t .

Proof. By Proposition 15, x is a meridian of $T_{p,pk-1}^{\ell,m}$. Moreover, $b = yx$ and $a = (yx)^{k-1}y$, so x and y generate $G_{p,pk-1}^{\ell,m}$. Next, we examine the group relation from Proposition 15 in terms of x and y :

$$((yx)^{k-1}y)^{p-\ell}((yx)^{k-1}yC^m)^{\ell-1}(yx)^{k-1}y = (yx)^{k(p-\ell)-1}((yx)^kC^m)^{\ell-1}(yx)^k$$

where $C = b^{1-k(p-\ell)}a^{p-\ell}$. If we assume $x(t) > t$ for all t , we can add x anywhere we want on one side of the relation to get a strict inequality on any t . In symbols, if we have $w_1(t) = w_2(t)$ and $w_3w_4 = w_1$, then we know that $xw_4(t) > w_4(t)$, so $w_3xw_4(t) > w_3w_4(t) = w_1(t) = w_2(t)$. Adding x multiple times to the left side of the relation, we get

$$[(yx)^{k(p-\ell)-1}y((yx)^kC^m)^{\ell-1}(yx)^k](t) > [(yx)^{k(p-\ell)-1}((yx)^kC^m)^{\ell-1}(yx)^k](t).$$

Since every word corresponds to a homeomorphism on \mathbb{R} , we know that for all $t' \in \mathbb{R}$ there exists t such that $((yx)^kC^m)^{\ell-1}(yx)^k(t) = t'$. Thus, we have

$$[(yx)^{k(p-\ell)-1}y](t') > [(yx)^{k(p-\ell)-1}](t')$$

which implies that

$$y(t') > t'$$

for all $t' \in \mathbb{R}$. □

Lemma 18. Consider $G_{p,pk+1}^{\ell,m}$ as in Proposition 16. Let $x = b^{-k}a$ and $y = a^{-1}b^{k+1}$. Then, x is a meridian of $T_{p,pk+1}^{\ell,m}$, x and y generate $G_{p,pk+1}^{\ell,m}$, and for any homomorphism $\Phi : G_{p,pk+1}^{\ell,m}(r) \rightarrow \text{Homeo}^+(\mathbb{R})$, $x(t) > t$ for all real t implies $y(t) \geq t$ for all real t .

Proof. By Proposition 16, x is a meridian of $T_{p,pk+1}^{\ell,m}$. Moreover, $b = xy$ and $a = (xy)^kx$, so x and y generate $G_{p,pk+1}^{\ell,m}$. Using the group relation from Proposition 16 and rewriting in terms of x and y , we get

$$(xy)^kx(C^m(xy)^kx)^{\ell-1}((xy)^kx)^{p-\ell} = (xy)^k(C^m(xy)^k)^{\ell-1}(xy)^{k(p-\ell)+1}$$

where $C = b^{k(p-\ell)+1}a^{\ell-p}$. Assume $xt > t$ for all $t \in \mathbb{R}$. Then, adding x 's to the right-hand side of the above equation gives us

$$[(xy)^kx(C^m(xy)^kx)^{\ell-1}((xy)^kx)^{p-\ell}](t) < [(xy)^kx(C^m(xy)^kx)^{\ell-1}((xy)^kx)^{p-\ell}]y(t).$$

Now, the word $(xy)^kx(C^mxy)^kx)^{\ell-1}((xy)^kx)^{p-\ell}$ corresponds to an order-preserving homeomorphism on \mathbb{R} . By order preservation, the above inequality can only be true if

$$t < y(t)$$

for all $t \in \mathbb{R}$. □

Theorem 14 follows from the next four propositions, which verify condition (2) of the list outlined above.

Proposition 19. $M_{p,pk-1}^{p-1,m}(r)$ has a non-left-orderable fundamental group for $r \geq p(pk-1) + (p-1)^2m$.

Proof. To satisfy the hypotheses of Theorem 10, by Lemma 17, it is sufficient to check that s can be written with only positive powers of x and y , with x and y as defined in the lemma. In terms of x and y , the expression for s in Proposition 15 with $\ell = p - 1$ becomes

$$s = (a(b^{1-k}a)^m)^{p-1}a = ((yx)^{k-1}y^{m+1})^{p-1}(yx)^{k-1}y$$

which contains only positive powers of x and y and at least one x . We can then apply Theorem 10, noting that $s = \mu^{p(pk-1)+(p-1)^2m}\lambda$. \square

Proposition 20. $M_{p,pk-1}^{p-2,1}(r)$ has a non-left-orderable fundamental group for $r \geq p(pk - 1) + (p - 2)^2$

Proof. It is again sufficient to check that s only contains positive powers of x and y , with x and y as in Lemma 17. Now, we consider the expression for s from Proposition 15 with $\ell = p - 2$ and $m = 1$ and rewrite it using the group relation:

$$s = a^{-1}a^2(ab^{-2k+1}a^2)^{p-2}a = a^{-1}b^{2k-1}(b^{1-k}a^2)^{p-2}a.$$

In terms of x and y , this becomes

$$s = x(yx)^{k-1}(y(yx)^{k-1}y)^{p-2}(yx)^{k-1}y$$

which contains only positive powers of x and y and at least one x . We can then apply Theorem 10, noting that $s = \mu^{p(pk-1)+(p-2)^2}\lambda$. \square

Proposition 21. $M_{p,pk+1}^{p-1,m}(r)$ has a non-left-orderable fundamental group for $r \geq p(pk + 1) + (p - 1)^2m$.

Proof. To satisfy the hypotheses of Theorem 10, by Lemma 18, it is sufficient to check that s can be written with only positive powers of x and y , with x and y as defined in the lemma. Using the expression for s in Proposition 16 with $\ell = p - 1$, we get

$$s = ((b^{k+1}a^{-1})^ma)^{p-1}a = (b^{k+1}(a^{-1}b^{k+1})^{m-1})^{p-1}a = ((xy)^{k+1}y^{m-1})^{p-1}(xy)^kx$$

which contains only positive powers of x and y , with at least one x . We can then apply Theorem 10, noting that $s = \mu^{p(pk+1)+(p-1)^2m}\lambda$. \square

Proposition 22. $M_{p,pk+1}^{p-2,1}(r)$ has a non-left-orderable fundamental group for $r \geq p(pk + 1) + (p - 2)^2$.

Proof. It is again sufficient to check that s can be written with only positive powers of x and y , with x and y as in Lemma 18. Using the expression for s in Proposition 16 with $\ell = p - 2$ and $m = 1$, we get

$$s = (b^{2k+1}a^{-1})^{p-2}a^2 = b^{2k+1}(a^{-1}b^{2k+1})^{p-3}a = (xy)^{2k+1}(y(xy)^k)^{p-3}(xy)^kx$$

which contains only positive powers of x and y and at least one x . We can then apply Theorem 10, noting that $s = \mu^{p(pk+1)+(p-2)^2}\lambda$. \square

ACKNOWLEDGEMENTS

The authors thank Jennifer Hom and Mike Wong for providing the resources and background material necessary for their work and for their guidance throughout the research process. The authors also thank Liam Watson for his support and for useful discussions and they thank the Columbia University math REU program for giving them the opportunity to pursue this research.

REFERENCES

- [BGW13] Steven Boyer, Cameron McA. Gordon, and Liam Watson, *On L -spaces and left-orderable fundamental groups*, Math. Ann. **356** (2013), no. 4, 1213–1245, DOI 10.1007/s00208-012-0852-7. MR3072799
- [BRW05] Steven Boyer, Dale Rolfsen, and Bert Wiest, *Orderable 3-manifold groups* (English, with English and French summaries), Ann. Inst. Fourier (Grenoble) **55** (2005), no. 1, 243–288. MR2141698 (2006a:57001)
- [CW13] Adam Clay and Liam Watson, *Left-orderable fundamental groups and Dehn surgery*, Int. Math. Res. Not. IMRN **12** (2013), 2862–2890. MR3071667
- [Ghy01] Étienne Ghys, *Groups acting on the circle*, Enseign. Math. (2) **47** (2001), no. 3-4, 329–407. MR1876932 (2003a:37032)
- [IT14] Kazuhiro Ichihara and Yuki Temma, *Non left-orderable surgeries and generalized Baumslag–Solitar relators*, Preprint (2014), <http://arxiv.org/abs/1406.4700>.
- [Lic97] W. B. Raymond Lickorish, *An introduction to knot theory*, Graduate Texts in Mathematics, vol. 175, Springer-Verlag, New York, 1997. MR1472978 (98f:57015)
- [Nak13] Yasuharu Nakae, *A good presentation of $(-2, 3, 2s + 1)$ -type pretzel knot group and \mathbb{R} -covered foliation*, J. Knot Theory Ramifications **22** (2013), no. 1, 1250143, 23, DOI 10.1142/S021821651250143X. MR3024026
- [OS04a] Peter Ozsváth and Zoltán Szabó, *Holomorphic disks and three-manifold invariants: properties and applications*, Ann. of Math. (2) **159** (2004), no. 3, 1159–1245, DOI 10.4007/annals.2004.159.1159. MR2113020 (2006b:57017)
- [OS04b] Peter Ozsváth and Zoltán Szabó, *Holomorphic disks and topological invariants for closed three-manifolds*, Ann. of Math. (2) **159** (2004), no. 3, 1027–1158, DOI 10.4007/annals.2004.159.1027. MR2113019 (2006b:57016)
- [OS05] Peter Ozsváth and Zoltán Szabó, *On knot Floer homology and lens space surgeries*, Topology **44** (2005), no. 6, 1281–1300, DOI 10.1016/j.top.2005.05.001. MR2168576 (2006f:57034)
- [OS11] Peter S. Ozsváth and Zoltán Szabó, *Knot Floer homology and rational surgeries*, Algebr. Geom. Topol. **11** (2011), no. 1, 1–68, DOI 10.2140/agt.2011.11.1. MR2764036 (2012h:57056)
- [Vaf13] Faramarz Vafaei, *On the knot Floer homology of twisted torus knots*, Preprint (2013), <http://arxiv.org/abs/1311.3711>.

DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY, NEW YORK, NEW YORK 10027
E-mail address: mac2370@columbia.edu

DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY, NEW YORK, NEW YORK 10027
E-mail address: goluboff@bc.edu

DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY, NEW YORK, NEW YORK 10027
E-mail address: dlh2145@columbia.edu

DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY, NEW YORK, NEW YORK 10027
E-mail address: sv2423@columbia.edu