A DYNAMIC PROGRAMMING APPROACH
TO THE PARISI FUNCTIONAL

AUKOSH JAGANNATH AND IAN TOBASCO

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ABSTRACT. G. Parisi predicted an important variational formula for the thermodynamic limit of the intensive free energy for a class of mean field spin glasses. In this paper, we present an elementary approach to the study of the Parisi functional using stochastic dynamic programing and semi-linear PDE. We give a derivation of important properties of the Parisi PDE avoiding the use of Ruelle Probability Cascades and Cole-Hopf transformations. As an application, we give a simple proof of the strict convexity of the Parisi functional, which was recently proved by Auffinger and Chen.

1. INTRODUCTION

Consider the mixed p-spin glass model on the hypercube $\Sigma_N = \{-1, 1\}^N$, which is given by the Hamiltonian

$$H_N(\sigma) = H'_N(\sigma) + h \sum_i \sigma_i,$$

where H'_N is the centered gaussian process on Σ_N with covariance

$$\mathbb{E} H'_N(\sigma^1) H'_N(\sigma^2) = N\xi((\sigma^1, \sigma^2)/N).$$

The parameter ξ satisfies $\xi(t) = \sum_{p \geq 2} \beta_p^2 t^p$, where we assume there is a positive ϵ such that $\xi(1 + \epsilon) < \infty$, and h is a non-negative real number. It was predicted by Parisi [13], and later proved rigorously by Talagrand [19], and Panchenko [16], that the thermodynamic limit of the intensive free energy is given by

$$\lim_{N \to \infty} \frac{1}{N} \log \sum_{\sigma \in \Sigma_N} e^{H_N(\sigma)} = \inf_{\mu \in \mathrm{Pr}[0,1]} \mathcal{P}(\mu; \xi, h) \quad \text{a.s.}$$

Here $\mathrm{Pr}([0, 1])$ is the space of probability measures on $[0, 1]$, and the Parisi functional, $\mathcal{P}$, is given by

$$\mathcal{P}(\mu; \xi, h) = u_\mu(0, h) - \frac{1}{2} \int_0^1 \xi''(t)\mu[0, t]t \, dt,$$

where u_μ solves the Parisi PDE:

$$\begin{cases} \partial_t u_\mu(t,x) + \frac{\xi''(t)}{2}\left(\partial_{xx}u_\mu(t,x) + \mu\,[0,t]\,(\partial_x u_\mu(t,x))^2\right) = 0, & (t,x) \in (0,1) \times \mathbb{R}, \\ u_\mu(1,x) = \log\cosh(x). \end{cases}$$

In the case that μ has finitely many atoms, the existence of a solution of the Parisi PDE and its regularity properties are commonly proved using the Cole-Hopf transformation and Ruelle Probability Cascades. A continuity argument is then used to extend the definition of u_μ to general μ and to prove corresponding regularity properties. Such approaches do not address the question of uniqueness of solutions. See [1, 2, 15, 21] for a summary of these results.

In this paper, we present a different approach. In Section 2, we prove the existence, uniqueness, and regularity of the Parisi PDE using standard arguments from semi-linear parabolic PDEs.

Theorem 1. *The Parisi PDE admits a unique weak solution which is continuous, differentiable in time at continuity points of μ, and smooth in space.*

See Section 2 for the precise statement of this result, and in particular for the definition of weak solution. Due to the non-linearity of the Parisi PDE, low regularity of the coefficients, loss of uniform ellipticity at $t = 0$, and unboundedness of the initial data, the proof of Theorem 1 requires the careful application of many different (though relatively standard) arguments in tandem.

The presentation of a PDE driven approach to the study of this functional is not only of interest to experts in the field of spin glasses, but may also be of interest to practitioners of the calculus of variations, PDEs, and stochastic optimal control. There are many important, purely analytical questions surrounding this functional that must be addressed before further progress on questions in spin glasses can be made. See [18, 20, 21] for a discussion. Some of these questions are thought to be intractable to the methods currently used in the spin glass literature but appear to be well suited to the techniques of the aforementioned fields. As such it is important to present the study of this functional in a language that is both basic and palatable to their practitioners.

Besides its intrinsic interest, the preceding theorem has useful applications to the study of the Parisi functional. After proving the existence of a sufficiently regular solution to the above PDE, we can use elementary arguments from stochastic analysis to prove many important and basic properties of this functional, such as fine estimates on the solution of the Parisi PDE and the strict convexity of the Parisi functional itself.

As a first application of this type, we further develop the well-posedness theory of the Parisi PDE by quantitatively proving the continuity of the solution in the measure μ. We also prove sharp bounds on some of the derivatives of the solution. Such bounds are important to the proofs of many important results regarding the Parisi functional; see for example Talagrand's proof of the Parisi formula in [21] and also [1, 2, 18]. They were previously proved using manipulations of the Cole-Hopf transformation and Ruelle Probability Cascades [21]. This is presented in Section 2.4.

As a further demonstration how Theorem 1 can be combined with methods from stochastic optimal control, we present a simple proof of the strict convexity of the Parisi functional. As background, recall the prediction by Parisi [13] that

the minimizer of the Parisi functional should be unique and should serve the role of the order parameter in these systems. The question of the strict convexity of $\mathcal{P}$ was first posed by Panchenko in [14] as a way to prove this uniqueness. It was studied by Panchenko [14], Talagrand [18, 19], Bovier and Klimovsky [4], and Chen [5], and finally resolved by Auffinger and Chen in their fundamental work [2]. The work of Auffinger and Chen rested on a variational representation of the log-moment generating functional of Brownian motion [3, 7], which they combine with approximation arguments to give a variational representation for the solution of the Parisi PDE. We note here that an early version of this variational representation appeared in [4], where it is shown, using the theory of viscosity solutions, to hold when the coefficient $\mu[0, t]$ is piecewise continuous with finitely many jumps.

Since the Parisi PDE is a Hamilton-Jacobi-Bellman equation, it is natural to obtain the desired variational representation for its solution as an application of the dynamic programming principle from stochastic optimal control theory. The required arguments are elementary and are commonly used in studying non-linear parabolic PDEs of the type seen above. We prove the variational representation in Section 3, and then deduce from it the strict convexity of the Parisi functional in Section 4.

Theorem 2. *The functional $\mathcal{P}(\mu; \xi, h)$ is strictly convex for all choices of ξ and h.*

The variational representation which was discussed above is given in Lemma 18. From this it follows immediately that one has the following representation for the Parisi Formula.

Proposition 3. *The Parisi Formula has the representation*

$$\lim_{N \to \infty} \frac{1}{N} \log \sum_{\sigma \in \Sigma_N} e^{H_N(\sigma)}$$

$$= \inf_{\mu \in \Pr([0,1])} \sup_{\alpha \in \mathcal{A}_0} \mathbb{E} \left[\log \cosh \left(\int_0^1 \xi''(s)\mu[0, s]\alpha_s ds + \int_0^t \sqrt{\xi''(s)} dW_s + h \right) \right.$$

$$\left. - \frac{1}{2} \int_0^1 \xi''(s)\mu[0, s] \left(\alpha_s^2 + s \right) ds \right],$$

where $\mathcal{A}_0$ consists of all bounded processes on $[0, 1]$ that are progressively measurable with respect to the filtration of Brownian motion.

2. Well-posedness of the Parisi PDE

Let $u : [0, 1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function with essentially bounded weak derivative $\partial_x u$. We call u a *weak solution* of the Parisi PDE if it satisfies

$$0 = \int_0^1 \int_{\mathbb{R}} -u\partial_t \phi + \frac{\xi''(t)}{2} \left(u\partial_{xx}\phi + \mu[0, t] (\partial_x u)^2 \phi \right) dxdt + \int_{\mathbb{R}} \phi(1, x) \log \cosh x \, dx$$

for every $\phi \in C_c^\infty ((0, 1] \times \mathbb{R})$. We now state the precise version of Theorem 1 from the introduction.

Theorem 4. *There exists a unique weak solution u to the Parisi PDE. The solution u has higher regularity:*
- *$\partial_x^j u \in C_b ([0, 1] \times \mathbb{R})$ for $j \geq 1$,*
- *$\partial_t \partial_x^j u \in L^\infty ([0, 1] \times \mathbb{R})$ for $j \geq 0$.*

For all $j \geq 1$, the derivative $\partial_x^j u$ is a weak solution to

$$\begin{cases} \partial_t \partial_x^j u + \frac{\xi''(t)}{2} \left(\partial_{xx} \partial_x^j u + \mu[0,t] \partial_x^j \left(\partial_x u \right)^2 \right) = 0, & (t,x) \in (0,1) \times \mathbb{R}, \\ \partial_x^j u(1,x) = \frac{d^j}{dx^j} \log \cosh x, & x \in \mathbb{R}. \end{cases}$$

Remark 5. The solution described in [1] can be shown to be a weak solution of the Parisi PDE, using the approximation methods developed there. It was also shown in [1] that this solution has the higher regularity described above.

Remark 6. The reader may notice that the essential boundedness of $\partial_x u$ is not strictly necessary to make sense of the definition of weak solutions. It is used in the proof of uniqueness in an essential way, however we do not claim that this proof is optimal by any means.

Continous dependence is proved in Section 2.4.

We begin the proof of Theorem 4. After performing the time change $t \to s(t) = \frac{1}{2}(\xi'(1) - \xi'(t))$ and extending the time-changed CDF $\mu[0, s^{-1}(t)]$ by zero, we are led to consider the semi-linear parabolic PDE

$$(1) \qquad \begin{cases} \partial_t u - \Delta u = m(t) u_x^2, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ u(0,x) = g(x), & x \in \mathbb{R}, \end{cases}$$

where $g(x) = \log \cosh x$ and $m(t) = \mu[0, s^{-1}(t)] 1_{t \leq (\xi'(1)/2)}$. We carry over the definition of a weak solution from before: a continuous function $u : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ with essentially bounded weak derivative $\partial_x u$ is a *weak solution* to (1) if it satisfies

$$0 = \int_0^\infty \int_{\mathbb{R}} u \partial_t \phi + u \partial_{xx} \phi + m(t) \left(\partial_x u \right)^2 \phi \, dx dt + \int_{\mathbb{R}} \phi(0,x) g(x) \, dx$$

for every $\phi \in C_c^\infty([0,\infty) \times \mathbb{R})$. Evidently, the existence, uniqueness, and regularity theory of weak solutions to the Parisi PDE is captured by that of (1).

Our proof of the well-posedness of (1) boils down to the study of a certain fixed point equation, which we introduce now. Let $e^{t\Delta}$ be the heat semi-group on $\mathbb{R}$, i.e.

$$\left(e^{t\Delta} h \right)(x) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{|x-y|^2}{4t}} h(y) \, dy.$$

Then, u weakly solves (1) if and only if u satisfies

$$(2) \qquad u(t) = e^{t\Delta} g + \int_0^t e^{(t-s)\Delta} m(s) u_x^2(s) \, ds.$$

This is an application of Duhamel's principle (see e.g. [6, Ch. 2]). For completeness, we present this in Proposition 24.

In Sections 2.1-2.3 below, we prove the existence, uniqueness, and regularity of fixed points of (2) on a certain complete metric space. The properties of g and m we will be using are that

- $g' \in L^\infty$ and $\frac{d^j}{dx^j} g \in L^2 \cap L^\infty$ for $j \geq 2$,
- m is a monotonic function of time alone and $||m||_\infty \leq 1$.

These properties will influence our choice of space on which to study (2). The exact bound on m does not matter, but we include it for convenience.

Once Theorem 4 is established, one can give a quick proof of the final component of wellposedness, namely the continuity of the map from μ to the corresponding solution of the Parisi PDE, using standard SDE techniques. This is in Section 2.4.

The notation $\lesssim_c$ denotes an inequality that is true up to a universal constant that depends only on c. Throughout the proofs below, we will use two elementary estimates for the heat kernel which we record here:

$$(3) \qquad ||e^{t\Delta}||_{L^p \to L^p} \leq 1 \quad \text{and} \quad ||\partial_x e^{t\Delta}||_{L^p \to L^p} \lesssim \frac{1}{\sqrt{t}}.$$

2.1. **Existence of a fixed point.** We prove the existence of a fixed point to (2). First we show that there exists a solution for short-times $t < T_*$; then by using an a priori estimate we prove that a solution exists for all time.

Short-time existence comes via a contraction mapping argument. Define the Banach space

$$\mathcal{X} = \{\psi \in L^\infty(\mathbb{R})\} \cap \{\psi_x \in L^\infty(\mathbb{R})\} \cap \{\psi_{xx} \in L^2(\mathbb{R})\}$$

with the norm

$$||\psi||_{\mathcal{X}} = ||\psi||_\infty \vee ||\psi_x||_\infty \vee ||\psi_{xx}||_2,$$

and for each $T > 0$ define the complete metric space

$$X_T^h = \{e^{t\Delta}h + \phi \: : \: \phi \in L^\infty([0,T];\mathcal{X})\}$$
$$\cap \{||\phi_x||_{L^\infty([0,T]\times\mathbb{R})} \leq ||h'||_\infty, ||\phi_{xx}||_{L^\infty([0,T];L^2(\mathbb{R}))} \leq ||h''||_2\}$$

with the distance

$$d_{X_T^h}(u,v) = ||u-v||_{L^\infty([0,T];\mathcal{X})}.$$

The symbol h in the definition of the space refers to the initial data, which is assumed to satisfy $h' \in L^\infty$ and $h'' \in L^2$.

Given $u \in X_T^h$ define the map

$$(4) \qquad A[u] = e^{t\Delta}h + \int_0^t e^{(t-s)\Delta}m(s)u_x^2(s)\,ds.$$

Lemma 7 (Short-time existence). *Let*

$$(5) \qquad T_*(h) = \min\left\{1, [C \cdot (||h'||_\infty + ||h''||_2)]^{-2}\right\},$$

where $C \in \mathbb{R}_+$ is a universal constant. Then for all $T \in (0, T_*)$,
- *(self-map) $A : X_T^h \to X_T^h$.*
- *(strict contraction) There exists $\alpha < 1$ such that*

$$d_{X_T^h}(A[u], A[v]) \leq \alpha \cdot d_{X_T^h}(u,v), \quad u,v \in X_T^h.$$

Therefore for every $T < T_*(h)$ there exists $u \in X_T^h$ satisfying $u = A[u]$.

Proof. First we prove A is a self-map. Let $u \in X_T^h$ and say

$$\psi = A[u] - e^{t\Delta}h = \int_0^t e^{(t-s)\Delta}m(s)u_x^2(s)\,ds.$$

Note that

$$\psi_x = \int_0^t \partial_x e^{(t-s)\Delta}m u_x^2(s)\,ds,$$

$$\psi_{xx} = \int_0^t \partial_x e^{(t-s)\Delta}2m u_x u_{xx}(s)\,ds.$$

The estimates in (3) and the definition of X_T^h imply the bounds

$$||\psi||_{L^\infty([0,T]\times\mathbb{R})} \lesssim T||h'||_\infty^2,$$

$$||\psi_x||_{L^\infty([0,T]\times\mathbb{R})} \lesssim T^{1/2}||h'||_\infty^2,$$

$$||\psi_{xx}||_{L^\infty([0,T];L^2(\mathbb{R}))} \lesssim T^{1/2}||h'||_\infty||h''||_2.$$

Therefore there is a universal constant $C \in \mathbb{R}_+$ such that $A : X_T^h \to X_T^h$ whenever

$$T \le T_0(h) = (C||h'||_\infty)^{-2}.$$

Now we prove that A is a strict contraction. Let $u, v \in X_T^h$ and say

$$D = A[u] - A[v] = \int_0^t e^{(t-s)\Delta}m(s)\left(u_x^2(s) - v_x^2(s)\right)\,ds.$$

The estimates in (3) and the definition of X_T^h give

$$d_{X_T^h}(A[u], A[v]) \le C \cdot \max\left\{T||h'||_\infty, T^{1/2}||h'||_\infty, T^{1/2}\left(||h''||_2 + ||h'||_\infty\right)\right\} d_{X_T^h}(u, v),$$

where $C \in \mathbb{R}_+$ is a universal constant. Therefore, if

$$T_1(h) = \min\left\{1, [C \cdot (||h'||_\infty + ||h''||_2)]^{-2}\right\},$$

then A is a strict contraction on X_T^h for all $T < T_0 \wedge T_1$. Since $T_1 \le T_0$ we may take $T_* = T_1$. $\qquad\square$

To prove the existence of a global-in-time solution to (2) we will work in the space

$$X_T = \left\{e^{t\Delta}g + \phi : \phi \in L^\infty([0,T];\mathcal{X})\right\}$$

defined for each $T \in \mathbb{R}_+$. Note that $X_T^g \subset X_T$, so that by Lemma 7, if we take $T < T_*(g)$, then there exists $u \in X_T$ satisfying the fixed point equation (2). To extend u to all of time we require the following a priori estimates.

Lemma 8 (A priori estimates). *Let* $T \in \mathbb{R}_+$ *and assume* $u \in X_T$ *satisfies* (2). *Then*

$$||u_x||_{L^\infty([0,T]\times\mathbb{R})} \le ||g'||_\infty,$$

$$||u_{xx}||_{L^\infty([0,T];L^2(\mathbb{R}))} \le ||g''||_2 \exp\left(||g'||_\infty^2 T\right).$$

Proof. The estimate on u_x is derived by the maximum principle. By Corollary 11 (see below) we have

$$\partial_t^\pm u_x(t,x) - \Delta u_x(t,x) = 2m(t\pm)u_x\partial_x u_x(t,x), \quad \forall\,(t,x) \in (0,T)\times\mathbb{R},$$

and by assumption u_x is bounded. Now the usual proof of the maximum principle for linear parabolic PDE in unbounded domains goes through [10].

For the estimate on u_{xx} observe that

$$u_x = e^{t\Delta}g' + \int_0^t e^{(t-s)\Delta}2mu_x u_{xx}(s)\,ds,$$

so by a standard energy estimate (see Lemma 12 below) we have for almost every $t \le T$,

$$||u_{xx}||_{L^2(\mathbb{R})}^2(t) \le 2||u_x||_{L^\infty([0,T]\times\mathbb{R})}^2 \int_0^t ||u_{xx}||_{L^2(\mathbb{R})}^2(s)ds + ||g''||_2^2.$$

The desired result follows from Gronwall's inequality [6] and the a priori bound on u_x. □

Corollary 9 (Global existence). *For each $T \in \mathbb{R}_+$, there exists $u_T \in X_T$ satisfying* (2). *The solutions $\{u_T\}_{T \in \mathbb{R}_+}$ so produced agree on their common domains.*

Proof. Define the maximal time of existence T_M to be the supremum over $T \in \mathbb{R}_+$ such that there exists $u_T \in X_T$ satisfying (2). If $T_M < \infty$, then by Lemma 7 we must have

$$\limsup_{T \uparrow T_M} ||(u_T)_x||_{L^\infty([0,T] \times \mathbb{R})} + ||(u_T)_{xx}||_{L^\infty([0,T];L^2(\mathbb{R}))} = \infty,$$

otherwise we could construct a solution extending for times beyond T_M. Therefore by Lemma 8 we must have $T_M = \infty$.

A quick application of Lemma 13 shows that $u_T = u_{T'}$ for $t \leq T \wedge T'$. □

2.2. Regularity of fixed points.
One proves the higher regularity of the fixed point u by a parabolic bootstrapping procedure.

Lemma 10 (Higher regularity). *Assume $u \in X_T$ satisfies* (2). *Then u satisfies*
- $\partial_x^j u \in L^\infty\left([0,T]; L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})\right)$ *for $j \geq 2$,*
- $\partial_t u \in L^\infty([0,T] \times \mathbb{R})$ *and* $\partial_t \partial_x^j u \in L^\infty\left([0,T]; L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})\right)$ *for $j \geq 1$.*

Proof. Let us describe the first step of the argument. Since $u \in X_T$ we have $u_x \in L^\infty_{tx}$ and $u_{xx} \in L^\infty_t L^2_x$. Our goal will be to deduce $u_{xx} \in L^\infty_{tx}$ and $u_{xxx} \in L^\infty_t L^2_x$. It will be important to note that we are working on the finite-time domain $[0,T] \times \mathbb{R}$, so that in particular $L^\infty_t L^2_x \subset L^2_{tx}$.

Start by writing

$$u_x = e^{t\Delta} g' + \int_0^t e^{(t-s)\Delta} 2m u_x u_{xx}(s) \, ds;$$

then by Lemma 12 we get $u_{xxx} \in L^2_{tx}$. Since $m u_x u_{xx} \in L^\infty_t L^2_x$, $g'' \in L^\infty$ and

$$u_{xx} = e^{t\Delta} g'' + \int_0^t \partial_x e^{(t-s)\Delta} 2m u_x u_{xx}(s) \, ds,$$

we conclude that $u_{xx} \in L^\infty_{tx}$. Here we have used that $\int_0^t \partial_x e^{(t-s)\Delta} \, ds \, : \, L^\infty_t L^2_x \to L^\infty_{tx}$, which follows from (3).

Now

$$\partial_x(m u_x u_{xx}) = m\left(u_{xx}^2 + u_x u_{xxx}\right) \in L^2_{tx},$$

$$\text{so that } u_{xx} = e^{t\Delta} g'' + \int_0^t e^{(t-s)\Delta} 2m \left(u_{xx}^2 + u_x u_{xxx}\right) \, ds,$$

and finally we conclude $u_{xxx} \in L^\infty_t L^2_x$ using Lemma 12 again.

The rest of the estimates on $\partial_x^j u$ are proved in the same way; the $\partial_t \partial_x^j u$ estimates follow easily. □

There is a sense in which the weak solution u is a classical solution.

Corollary 11. *Let $u \in X_T$ satisfy* (2). *Then for all $j \geq 0$ we have*
- $\partial_x^j u$ *exists pointwise and is continuous,*
- *the left/right derivatives $\partial_t^{\pm} \partial_x^j u$ exist pointwise, and $\partial_t \partial_x^j u$ exists at continuity points of m.*

Moreover, we have that

$$\partial_t^\pm \partial_x^j u\,(t,x) - \Delta \partial_x^j u\,(t,x) = m\,(t\pm)\,\partial_x^j\,\left[u_x^2\right](t,x)\,,\quad \forall\,(t,x) \in (0,T) \times \mathbb{R}.$$

For completeness, we record the energy estimate which was used above. The proof is standard (see [6]) and is omitted.

Lemma 12. *Let h be weakly differentiable with $h' \in L^2$ and let $f \in L^2([0,T] \times \mathbb{R})$. Then*

$$\psi(t) = e^{t\Delta}h + \int_0^t e^{(t-s)\Delta}f(s)\,ds$$

satisfies

$$||\psi_x||_{L^\infty([0,T];L^2(\mathbb{R}))}^2 + ||\psi_{xx}||_{L^2([0,T]\times\mathbb{R})}^2 \le ||f||_{L^2([0,T]\times\mathbb{R})}^2 + ||h'||_{L^2(\mathbb{R})}^2.$$

2.3. **Uniqueness of fixed points.** Since we used a contraction mapping argument to construct fixed points for (4) in the spaces X_T^h, we have implicitly demonstrated a uniqueness theorem there. The following result achieves uniqueness without mention of the second derivative u_{xx}.

Lemma 13. *Assume $u,v : [0,T] \times \mathbb{R} \to \mathbb{R}$ are weakly differentiable and that u_x, v_x are essentially bounded. Then if u,v satisfy the fixed point equation (2), it follows that $u = v$.*

Proof. In the following, C denotes a universal constant which may change from line to line. Let $d = u - v$; then by assumption we have

$$d\,(t) = \int_0^t e^{(t-s)\Delta}m\,(s)\,(u_x + v_x)\,d_x\,(s)\,\,ds,\quad t \le T.$$

Therefore

$$d_x(t) = \int_0^t \partial_x e^{(t-s)\Delta}m(s)(u_x + v_x)d_x(s)\,ds,\quad t \le T.$$

Using the second heat kernel estimate in (3), we conclude the contractive estimate

$$||d_x||_{L^\infty([0,t]\times\mathbb{R})} \le C||u_x + v_x||_{L^\infty([0,T]\times\mathbb{R})} \int_0^t \frac{1}{\sqrt{t-s}}||d_x\,(s)||_{L^\infty(dx)}\,ds$$

for all $t \le T$. It now follows from an iterative argument that $d_x = 0$ and hence that $d = 0$. To see this note that if $d_x = 0$ on $[0,t_1] \times \mathbb{R}$, then by the contractive estimate above,

$$||d_x||_{L^\infty([t_1,t]\times\mathbb{R})} \le C||u_x + v_x||_{L^\infty([0,T]\times\mathbb{R})}\sqrt{t - t_1}||d_x||_{L^\infty([t_1,t]\times\mathbb{R})}$$

for all $t \in [t_1,T]$. Therefore $d_x = 0$ on $[0,t_1 + \epsilon]$, where ϵ depends only on the L^∞ bounds on u_x, v_x. This completes the proof. □

2.4. **Continuous dependence of solutions.** For convenience we metrize the weak topology on the space of probability measures on the interval $\Pr[0,1]$ with the metric

$$d\,(\mu,\nu) = \int_0^1 |\mu\,[0,s] - \nu\,[0,s]|\,ds.$$

Lemma 14. *Let $\mu,\nu \in \Pr[0,1]$ and u,v be the corresponding solutions to the Parisi PDE. Then*

$$||u - v||_\infty \le \xi''\,(1)\,d(\mu,\nu),$$
$$||u_x - v_x||_\infty \le \exp\,(\xi'\,(1))\,\xi''\,(1)\,d(\mu,\nu).$$

Remark 15. The first inequality is originally due to Guerra [11].

Proof. Let u, v solve the Parisi PDE weakly; then $w = u - v$ solves

$$\begin{cases} w_t + \frac{\xi''}{2} \left(w_{xx} + \mu[0,t] \left(u_x + v_x \right) w_x + \left(\mu[0,t] - \nu[0,t] \right) v_x^2 \right) = 0, & (t,x) \in (0,1) \times \mathbb{R}, \\ w(1,x) = 0, & x \in \mathbb{R}, \end{cases}$$

weakly. Since u_x, v_x are Lipschitz in space uniformly in time and bounded in time, we can solve the SDE

$$dX_t = \xi''(t)\, \mu[0,t]\, \frac{u_x + v_x}{2}\, (t, X_t)\, dt + \sqrt{\xi''(t)} dW_t.$$

Furthermore, as w weakly solves the above PDE and has the same regularity as u and v, we can write

$$w(t,x) = \mathbb{E}_{X_t = x} \left(\int_t^1 \frac{1}{2} \xi''(s) \left(\mu[0,s] - \nu[0,s] \right) v_x^2(s, X_s)\, ds \right)$$

by Proposition 22. Therefore

$$||w||_\infty \leq \xi''(1)\, d(\mu, \nu)$$

since ξ'' is non-decreasing and $||v_x||_\infty^2 \leq 1$ by Lemma 16.

By differentiating the PDE for w in x, one finds by similar arguments in Proposition 22 that w_x has the representation

$$w_x(t,x) = \mathbb{E}_{X_t = x} \left(\int_t^1 E(t,s)\, \xi''(s) \left(\mu[0,s] - \nu[0,s] \right) v_x v_{xx}(s, X_s)\, ds \right),$$

where

$$E(t,s) = \exp \left(\int_t^s \xi''(\tau)\, \mu[0,\tau]\, \frac{v_{xx} + u_{xx}}{2}\, (\tau, X_\tau)\, d\tau \right).$$

Using $||v_x||_\infty \leq 1$ and $||u_{xx}||_\infty \vee ||v_{xx}||_\infty \leq 1$ from Lemma 16, and since ξ'' is non-decreasing,

$$||w_x||_\infty \leq e^{\xi'(1)} \xi''(1)\, d(\nu, \mu).$$

$\square$

Lemma 16. *The solution u to the Parisi PDE satisfies $|u_x| < 1$ and $0 < u_{xx} \leq 1$.*

Remark 17. The Auffinger-Chen SDE and the corresponding Itô formulas for u_x and u_{xx} used in the proof below were first proved in [2] using approximation arguments.

Proof. Using the PDEs for u_x, u_{xx} given in Theorem 4, along with Proposition 22, we can write

$$u_x(t,x) = \mathbb{E}_{X_t = x} \left(\tanh X_1 \right),$$

$$u_{xx}(t,x) = \mathbb{E}_{X_t = x} \left(\text{sech}^2 X_1 + \int_t^1 \xi''(s)\, \mu[0,s]\, u_{xx}^2(s, X_s)\, ds \right),$$

where X_t solves the Auffinger-Chen SDE

$$dX_t = \xi''(t)\, \mu[0,t]\, u_x(t, X_t)\, dt + \sqrt{\xi''(t)} dW_t.$$

The first equality immediately implies the bound on u_x, and the second equality implies $u_{xx} > 0$. Then by a rearrangement one finds

$$u_{xx}(t,x) = 1 - \mu[0,t] u_x^2(t,x) - \mathbb{E}_{X_t = x} \left(\int_t^1 u_x^2(s, X_s)\, d\mu(s) \right),$$

and $u_{xx} \leq 1$ follows.

$\square$

3. A variational formulation for the Parisi PDE

In this section we use the methods of dynamic programming (see e.g. [8]) to give a new proof of the variational formula for the solution of the Parisi PDE.

Lemma 18. *Let u_μ solve the Parisi PDE as above and define the class $\mathcal{A}_t$ of processes α_s on $[t, 1]$ that are bounded and progressively measurable with respect to Brownian motion. Then*

$$
(6) \qquad u_\mu(t, x) = \sup_{\alpha \in \mathcal{A}_t} \mathbb{E}_{X_t^\alpha = x} \left[-\frac{1}{2} \int_t^1 \xi''(s)\mu[0, s]\alpha_s^2 ds + \log \cosh(X_1^\alpha) \right],
$$

where X_s^α solves the SDE

$$
(7) \qquad dX_s^\alpha = \xi''(s)\mu[0, s]\alpha_s ds + \sqrt{\xi''(s)}dW_s
$$

with initial data $X_t^\alpha = x$. Furthermore, the optimal control satisfies

$$
\mu[0, s]\alpha_s^* = \mu[0, s]u_x(s, X_s) \quad a.s.,
$$

where X_s solves the Auffinger-Chen SDE with the same initial data:

$$
dX_s = \xi''(s)\mu[0, s]\partial_x u(s, X_s)ds + \sqrt{\xi''(s)}dW_s.
$$

Remark 19. This formula was first proved by Auffinger and Chen in [2]. Taking advantage of the Cole-Hopf representation in the case of atomic μ, they proved the lower bound for every α using Girsanov's lemma and Jensen's inequality. They then verified that their optimal control achieves the supremum, by an application of Itô's lemma. The uniqueness follows from a convexity argument. In contrast, we recognize the Parisi PDE as a specific Hamilton-Jacobi-Bellman equation. It is well known that the solution of such an equation can be seen as the value function of a stochastic optimal control problem. As such, this representation can be obtained by a textbook application of "the verification argument". This argument simultaneously gives the variational representation and a characterization of the optimizer. We also note that the argument presented here is more flexible, as is evidenced by replacing the nonlinearity u_x^2 with $F(u_x)$ in the Parisi PDE, where F is smooth, strictly convex, and has super linear growth. In particular, observe that one cannot use the Cole-Hopf transformation on the resulting PDE, but the arguments of this paper follow through *mutatis mutandis*.

Proof. Let u solve the Parisi PDE. Notice that the nonlinearity is convex, so if we let

$$
(8) \qquad
\begin{aligned}
L(t, \lambda) &= -\xi''(t)\mu[0, t]\frac{\lambda^2}{2}, \\
f(t, \lambda) &= \xi''(t)\mu[0, t]\lambda,
\end{aligned}
$$

then by the Legendre transform we have

$$
\begin{aligned}
\xi''(t)\mu[0, t]\frac{(\partial_x u)^2}{2} &= \xi''(t)\mu[0, t]\sup_{\lambda \in \mathbb{R}}\left\{-\lambda^2/2 + \lambda\partial_x u\right\} \\
&= \sup_{\lambda \in \mathbb{R}}\left\{L(t, \lambda) + f(t, \lambda)\partial_x u\right\}.
\end{aligned}
$$

Therefore, we can write the Parisi PDE as a Hamilton-Jacobi-Bellman equation:

$$
0 = \partial_t u + \frac{\xi''(t)}{2}\partial_{xx}u + \sup_{\lambda \in \mathbb{R}}\left\{L(t, \lambda) + f(t, \lambda)\partial_x u\right\}.
$$

Since α_s in $\mathcal{A}_t$ is bounded and progressively measurable, we can consider the process, X^α, which solves the SDE

$$dX^\alpha = f(s, \alpha_s)ds + \sqrt{\xi''(t)}dW$$

with initial data $X_t^\alpha = x$. This process has the corresponding infinitesimal generator

$$\mathcal{L}(t, \alpha) = \frac{1}{2}\xi''(t)\partial_{xx} + f(t, \alpha)\partial_x.$$

Notice that u is a (weak) sub-solution to

$$\partial_t u + \mathcal{L}(t, \alpha)u + L(t, \alpha) \leq 0$$

with the regularity obtained in Theorem 4. It follows from Itô's lemma (Proposition 22) that

$$u(t, x) \geq \sup_{\alpha \in \mathcal{A}_t} \mathbb{E}_x \left[\int_t^1 L(s, \alpha_s) \, ds + \log \cosh (X_1^\alpha) \right].$$

The result now follows upon observing that the control $u_x(s, X_s)$ achieves equality in the above since it achieves equality in the Legendre transform. That this control is in the class $\mathcal{A}_t$ can be seen by an application of the parabolic maximum principle (Lemma 16). Uniqueness follows from the fact that λ achieves equality in the Legendre transform if and only if

$$\xi''(t)\mu[0, t]\lambda = \xi''(t)\mu[0, t]u_x.$$

$\square$

Applying this representation to the Parisi formula gives Proposition 3.

4. STRICT CONVEXITY

As an application of the above ideas, we give a simple proof of strict convexity of $\mathcal{P}$.

Theorem 20. *The Parisi functional is strictly convex.*

Proof. We will prove that $\mu \to u_\mu(0, h)$ is strictly convex. Then

$$\mathcal{P}(\mu) = u_\mu(0, h) - \frac{1}{2} \int_0^1 \xi''(t)\mu[0, t]s \, ds$$

will be the sum of a strictly convex and a linear functional, so $\mathcal{P}$ will be strictly convex.

Recall

$$u_\mu(0, h) = \sup_{\alpha \in \mathcal{A}_0} \mathbb{E}_h \left[\int_0^1 -\xi''(s)\mu[0, s]\frac{\alpha_s^2}{2} \, ds + \log \cosh (X_1^\alpha) \right].$$

Fix distinct $\mu, \nu \in \Pr[0, 1]$ and let $\mu_\theta = \theta\mu + (1 - \theta)\nu$, $\theta \in (0, 1)$. Let α^θ be the optimal control for the Parisi PDE associated to μ_θ so that

$$u_{\mu_\theta}(0, h) = \mathbb{E}_h \left[\int_0^1 -\xi''(s)\mu_\theta[0, s]\frac{(\alpha_s^\theta)^2}{2} \, ds + \log \cosh \left(X_1^{\alpha^\theta} \right) \right].$$

Consider the auxiliary processes $Y_t^{\alpha^\theta}$ and $Z_t^{\alpha^\theta}$ given by solving

$$dY_t = \xi''(t)\mu[0, t]\alpha_t^\theta dt + \sqrt{\xi''(t)}dW_t \quad \text{and} \quad dZ_t = \xi''(t)\nu[0, t]\alpha_t^\theta dt + \sqrt{\xi''(t)}dW_t$$

with initial data $Y_0 = Z_0 = h$, and note that

$$X_t^{\alpha^\theta} = \theta Y_t^{\alpha^\theta} + (1-\theta) Z_t^{\alpha^\theta}.$$

By the lemma below, $P(Y_1 \neq Z_1) > 0$. Therefore by the strict convexity of $\log\cosh$ and the variational representation (6),

$$u_{\mu_\theta}(0, h) = \mathbb{E}_h \left[\int_0^1 -\xi''(s) \mu_\theta[0, s] \frac{(\alpha_s^\theta)^2}{2} ds + \log\cosh\left(X_1^{\alpha^\theta}\right) \right]$$

$$< \theta \left(\mathbb{E}_h \left[\int_0^1 -\xi''(s) \mu[0, s] \frac{(\alpha_s^\theta)^2}{2} ds + \log\cosh\left(Y_1^{\alpha^\theta}\right) \right] \right)$$

$$+ (1-\theta) \left(\mathbb{E}_h \left[\int_0^1 -\xi''(s) \nu[0, s] \frac{(\alpha_s^\theta)^2}{2} ds + \log\cosh\left(Z_1^{\alpha^\theta}\right) \right] \right)$$

$$\leq \theta u_\mu(0, h) + (1-\theta) u_\nu(0, h),$$

as desired. $\qquad\square$

Lemma 21. *Let Y_t and Z_t be as above. Then $P(Y_1 \neq Z_1) > 0$.*

Proof. It suffices to show that

$$Var(Y_1 - Z_1) > 0.$$

By definition we have

$$Y_1 - Z_1 = \int_0^1 \xi''(s)(\mu[0, s] - \nu[0, s]) \alpha_s^\theta \, ds.$$

Observe that by the PDE for u_x in Theorem 4 and Itô's lemma (see Proposition 22), the optimal control $\alpha_t^\theta = (u_{\mu_\theta})_x$ is a martingale,

$$\alpha_t^\theta - \alpha_0^\theta = \int_0^t \sqrt{\xi''(s)} u_{xx}(s, X_s) dW_s.$$

Therefore if we call $\Delta_s = \xi''(s)(\mu[0, s] - \nu[0, s])$, then

$$Var(Y_1 - Z_1) = \mathbb{E}_h \left(\int_0^1 \Delta_s(\alpha_s^\theta - \alpha_0^\theta) ds \right)^2 = \int_{[0,1]^2} \Delta_s \Delta_t K(s, t) \, ds dt,$$

where

$$K(s, t) = \mathbb{E}_h \left[\left(\alpha_s^\theta - \alpha_0^\theta \right) \cdot \left(\alpha_t^\theta - \alpha_0^\theta \right) \right].$$

Now since $\Delta_s \in L_2[0, 1]$, it suffices to show that $K(s, t)$ is positive definite. We have

$$K(s, t) = \mathbb{E}_h \left[\int_0^s \sqrt{\xi''(s)} u_{xx}\left(s, X_s^{\alpha^\theta} \right) dW_s \cdot \int_0^t \sqrt{\xi''(t)} u_{xx}\left(t, X_t^{\alpha^\theta} \right) dW_t \right]$$

$$= \int_0^{t \wedge s} \xi''(t') \mathbb{E}_h u_{xx}^2(t', X_{t'}^{\alpha^\theta}) dt' = p(t \wedge s) = p(t) \wedge p(s),$$

where

$$p(s) = \int_0^s \xi''(t') \mathbb{E}_h u_{xx}^2(t', X_{t'}^{\alpha^\theta}) dt'.$$

By the maximum principle (Lemma 16), $u_{xx} > 0$, so that $p(t)$ is strictly increasing. Since this kernel corresponds to a monotonic time change of a Brownian motion, it is positive definite. $\qquad\square$

5. Appendix

We will say that a function $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ with *at most linear growth* if it satisfies an inequality of the form

$$|f(t, x)| \lesssim_T 1 + |x| \quad \forall T \in \mathbb{R}_+, \ (t, x) \in [0, T] \times \mathbb{R}.$$

We will say the same in the case that $f : \mathbb{R} \to \mathbb{R}$ with the obvious modifications. In the following we fix a probability space $(\Omega, \mathcal{F}, P)$ and let W_t be a standard brownian motion with respect to P. Let $\mathcal{F}_t$ be the filtration corresponding to W_t.

To make this paper self-contained, we present a version of Itô's lemma in a lower regularity setting. The argument is a modification of [17, Cor. 4.2.2].

Proposition 22. *Let $a, b : [0, T] \times (\Omega, \mathcal{F}, P) \to \mathbb{R}$ be be bounded and progressively measurable with respect to $\mathcal{F}_t$ and let $a \geq 0$. Let X_t solve*

$$dX_t = \sqrt{a(t)}dW_t + b(t)dt$$

with initial data $X_0 = x$. Let $L = \frac{1}{2}a(t, \omega)\Delta + b(t, \omega)\partial_x$. Finally assume that we have u satisfying:

(1) $u \in C([0, T] \times \mathbb{R})$ *with at most linear growth.*
(2) $u_x, u_{xx} \in C_b([0, T] \times \mathbb{R})$.
(3) u *is weakly differentiable in t with essentially bounded weak derivative u_t, which has a representative that is Lipschitz in x uniformly in t.*

Then u satisfies Ito's lemma:

$$u(t, X_t) - u(s, X_s) = \int_s^t (\partial_t + L) u(s', X_{s'})ds' + \int_s^t u_x(s', X_{s'})\sqrt{a(s')}dW_{s'}.$$

Remark 23. This result is applied throughout the paper to the solution u from Theorem 1 and its spatial derivatives. We note here that, given the regularity in Theorem 4, the weak derivatives $\partial_t \partial_x^j u$, $j \geq 0$, have representatives satisfying the above Lipschitz property.

Proof. To prove this, we will smooth u by a standard mollification-in-time procedure and apply Itô's lemma. Without loss of generality, assume $T = 1$ and $s = 0$. Extend u to all of space-time by

$$u(t, x) = \begin{cases} u(0, x), & t < 0, \\ u(1, x), & t > 1. \end{cases}$$

Abusing notation, we call the extension u and note that it satisfies each of the assumptions above. Let $\phi(y) \in C_c^\infty(-1, 1)$ with $0 \leq \phi \leq 1$ and $\int \phi = 1$, and define $\phi_\epsilon(s) = \phi(s/\epsilon)/\epsilon$. Define the time-mollified version of u as

$$u^\epsilon(t, x) = \int_\mathbb{R} \phi_\epsilon(s)u(t - s, x)ds.$$

Since $u^\epsilon \in C^{1,2}$ has bounded derivatives and grows at most linearly, Ito's lemma implies that

$$u^\epsilon(t, X_t) - u^\epsilon(0, x) = \int_0^t (\partial_t + L) u^\epsilon(s, X_s)ds + \int_0^t u_x^\epsilon(s, X_s)\sqrt{a}dW_s$$

$$= \int_0^t u_t^\epsilon(s, X_s)ds + \int_0^t Lu^\epsilon(s, X_s)ds \int_0^t u_x^\epsilon(s, X_s)\sqrt{a}dW_s$$

$$= A_\epsilon + B_\epsilon + C_\epsilon$$

for all $\epsilon > 0$. Since these quantities are well defined at $\epsilon = 0$, it suffices to show their convergence.

First we show that the left-hand side converges. Note u is Lipschitz with constant $||\nabla u||_\infty$. Therefore,

$$\sup_{x \in \mathbb{R}} \sup_t |u^\epsilon(t, x) - u(t, x)| = \sup_{x \in \mathbb{R}} \sup_t |\int \phi(y)\, (u(t - \epsilon y, x) - u(t, x))\, dy| \leq ||\nabla u||_\infty \epsilon.$$

Thus $u^\epsilon(t, X_t) \to u(t, X_t)$ uniformly P-a.s.

Now we consider the right-hand side. For A_ϵ, note that since u_t is Lipschitz in x uniformly in t, by an application of Lebesgue's differentiation theorem, we have that $u_t^\epsilon \to u_t$ for all x, Lebesgue-a.s. in t. Thus by the bounded convergence theorem, we have that

$$\sup_{t \in [0,1]} |\int_0^t u_t^\epsilon(s, X_s)ds - \int_0^t u_t(s, X_s)ds| \leq \int_0^1 |u_t^\epsilon(s, X_s) - u_t(s, X_s)|ds \to 0.$$

Thus, $A_\epsilon \to A$ uniformly P-a.s.

The convergence for B_ϵ follows from a similar argument. Since $u_x, u_{xx} \in C_b$, commuting derivatives with mollification shows that u_x^ϵ and u_{xx}^ϵ converge to u_x and u_{xx} pointwise. Then, the bounded convergence theorem implies that $B_\epsilon \to B$ uniformly P-a.s. just as before.

Now we prove uniform a.s. convergence of C_ϵ to C. Combining the above arguments proves that C_ϵ is uniformly a.s. convergent, so it suffices to check its convergence to C in probability. By Doob's inequality and Ito's isometry,

$$P\left(\sup_{t \in [0,1]} |\int_0^t u_x^\epsilon \sqrt{a}dW_s - \int_0^t u_x \sqrt{a}dW_s| \geq \eta \right) \lesssim_a \frac{1}{\eta^2} \int_0^1 \mathbb{E}|u_x^\epsilon - u_x|^2 \to 0,$$

where the last convergence is again by the bounded convergence theorem. $\square$

We finish with a discussion of Duhamel's principle, which justifies the introduction of the fixed point equation (2) in the proof of Theorem 4. Note that since our weak solutions satisfy $\partial_x u \in L^\infty$ by definition, they have at most linear growth.

Proposition 24. *Suppose that $u, f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ have at most linear growth. Assume that f is Borel measurable and that u and g are continuous. Then*
(9)
$$0 = \int_0^\infty \int_\mathbb{R} u \partial_t \phi + u \partial_{xx} \phi + f\phi\, dxdt + \int_\mathbb{R} \phi(0, x)\, g(x)\, dx \quad \forall \phi \in C_c^\infty([0, \infty) \times \mathbb{R})$$

if and only if

$$(10) \qquad u(t) = e^{t\Delta} g + \int_0^t e^{(t-s)\Delta} f(s)\, ds \quad \forall t \in [0, \infty).$$

Remark 25. Although the assumption of linear growth is not optimal, it will be sufficient for our application. Implicit here is a uniqueness theorem for weak solutions of the heat equation with at most linear growth. Recall that even classical solutions fail to be unique without certain growth conditions at $|x| = \infty$ (see e.g. [12, Ch. 7]).

Proof. That u satisfies (9) if it satisfies (10) is clear in the case that f, g are smooth and compactly supported. Then, a cutoff and mollification argument upgrades the result to the given class.

In the other direction, suppose that u satisfies (9). Define the function

$$\Theta(t, x) = u(t, x) - \left[e^{t\Delta} g(\cdot) \right](x) - \int_0^t \left[e^{(t-s)\Delta} f(s, \cdot) \right](x) \, ds,$$

which is continuous and satisfies $\Theta(0, \cdot) = 0$. By a similar argument as above, Θ satisfies the heat equation in the sense of distributions on $\mathbb{R}_+ \times \mathbb{R}$. Since the heat operator is hypoelliptic, it follows that Θ is a classical solution [9]. By its definition, Θ grows at most linearly since the same is true for u, f, and g. By the maximum principle for the heat equation in unbounded domains [12], we conclude that $\Theta = 0$. □

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Courant Institute of Mathematical Sciences, 251 Mercer Street, New York, New York 10012

E-mail address: `aukosh@cims.nyu.edu`

Courant Institute of Mathematical Sciences, 251 Mercer Street, New York, New York 10012

E-mail address: `tobasco@cims.nyu.edu`