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SOME COMPUTATIONS OF THE GENERALIZED HILBERT-KUNZ FUNCTION AND MULTIPLICITY

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ABSTRACT. Let R be a local ring of characteristic $p > 0$ which is F-finite and has perfect residue field. We compute the generalized Hilbert-Kunz invariant for certain modules over several classes of rings: hypersurfaces of finite representation type, toric rings, and weakly F-regular rings.

1. INTRODUCTION

Let R be a local ring of characteristic $p > 0$ which is F-finite and has perfect residue field. Let M be a finitely generated R-module. Let $F_R^n(M) = M \otimes_R {}^nR$ denote the n-fold iteration of the Frobenius functor given by base change along the Frobenius endomorphism. Let $\dim R = d$ and $q = p^n$. This paper constitutes a further study of

$$f_{gHK}^M(n) := \ell(\mathrm{H}_{\mathfrak{m}}^0(F^n(M)))$$

and

$$\mathrm{e}_{\mathrm{gHK}}(M) := \lim_{n\to\infty} \frac{f_{gHK}^M(n)}{p^{nd}},$$

which are called the *generalized Hilbert-Kunz function* and *generalized Hilbert-Kunz multiplicity* of M, respectively. These notions were first defined by Epstein-Yao in [8] and were studied in detail in [7]. For instance, it is now known ([7, Theorem 1.1]) that $\mathrm{e}_{\mathrm{gHK}}(M)$ exists for all modules over a Cohen-Macaulay isolated singularity.

It is a non-trivial and interesting problem to compute even the classical Hilbert-Kunz multiplicity. In this note we focus on computing f_{gHK}^M and the limit $\mathrm{e}_{\mathrm{gHK}}(M)$ for certain modules in a number of cases: when R is a normal domain of dimension 2 (section 2), a hypersurface of finite representation type (section 3) and when R is a toric ring (section 4). We also point out a connection between the generalized Hilbert-Kunz limits and tight closure theory in section 5. Namely, over weakly F-regular rings, these limits detect depths of the module M and all of its pull-backs along iterations of the Frobenius.

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2. Dimension two

In this section, we prove certain preliminary facts about behavior of f_{gHK}^M when R is normal and $M = R/I$ where I is reflexive. We then apply them to give a formula for $\mathrm{e_{gHK}}(R/I)$ when I represents a torsion element in the class group of R. For a reflexive ideal I, let $I^{(q)}$ denote the qth-symbolic power of I.

Lemma 2.1. *Let R be a local normal domain of dimension at least 2 and I a reflexive ideal that is locally free on the punctured spectrum. Then*

$$\ell(\mathrm{H}^0_{\mathfrak{m}}(R/I^{[q]})) = \ell(I^{(q)}/I^{[q]}) = \ell(I^{(q)}/I^q) + \ell(I^q/I^{[q]}) = \ell(\mathrm{H}^0_{\mathfrak{m}}(R/I^q)) + \ell(I^q/I^{[q]}).$$

Proof. Apply local cohomology functor to the sequence

$$0 \to \frac{I^{(q)}}{I^{[q]}} \to \frac{R}{I^{[q]}} \to \frac{R}{I^{(q)}} \to 0.$$

Note that $\mathrm{H}^0_{\mathfrak{m}}(R/I^{(q)}) = 0$ since $I^{(q)}$ is reflexive, and $\ell(I^{(q)}/I^q), \ell(I^q/I^{[q]}) < \infty$ as the ideals coincide on the punctured spectrum. ☐

Proposition 2.2. *Let R be a local normal domain of dimension 2 and I be a reflexive ideal. Then $e_{gHK}(R/I) = 0$ if and only if I is principal.*

Proof. Only one direction needs to be checked. Suppose $\mathrm{e_{gHK}}(R/I) = 0$. Let $\mu()$ denote the minimal number of generators of an R-module. We have that $\ell(\mathrm{H}^0_{\mathfrak{m}}(R/I^{[q]})) \geq \ell(\mathrm{H}^0_{\mathfrak{m}}(R/I^q))$ by Lemma 2.1. It follows that $\limsup \frac{\ell(\mathrm{H}^0_{\mathfrak{m}}(R/I^q))}{q^2} = 0$, so I has analytic spread one by [10, Theorem 4.7]; thus $[I]$ is principal. ☐

Remark 2.3. The number $\limsup \frac{\ell(\mathrm{H}^0_{\mathfrak{m}}(R/I^n))}{n^2}$ is known as the epsilon multiplicity of I, $\epsilon(I)$. It has now been proved to exist as a limit under mild conditions; see [4, Theorem 1.1]. Lemma 2.1 says that $\mathrm{e_{gHK}}(R/I) \geq \epsilon(I)$.

Lemma 2.4. *Let R be a local normal domain of dimension 2 and I a reflexive ideal. Assume that $[I]$ is torsion in $\mathrm{Cl}(R)$. Then $\ell(\mathrm{H}^0_{\mathfrak{m}}(R/I^n))$ has quasi-polynomial behavior for n large enough.*

Proof. Let r be some integer such that $r[I] = 0$ in $\mathrm{Cl}(R)$. Then $I^r = I_1 \cap I_2$, where I_1 is the determinant of I^r and thus principal, and I_2 is $\mathfrak{m}$-primary. Let $I_1 = (x)$; we then have $I^r = xJ$, where $J = I_2 : x$. Note that J is $\mathfrak{m}$-primary. For any integer n, let $n = ar + b$. We have that $I^n = I^{ar+b} = x^a J^a I^b$. Then

$$\mathrm{H}^0_{\mathfrak{m}}(R/I^n) \cong \mathrm{H}^1_{\mathfrak{m}}(I^n) \cong \mathrm{H}^1_{\mathfrak{m}}(J^a I^b) \cong \mathrm{H}^0_{\mathfrak{m}}(R/J^a I^b).$$

To calculate the last term we use

$$0 \to I^b/J^a I^b \to R/J^a I^b \to R/I^b \to 0.$$

The leftmost term has finite length, thus what we want is equal to $\ell(I^b/J^aI^b) + \ell(\mathrm{H}^0_{\mathfrak{m}}(R/I^b))$. Since b is periodic and a grows linearly with n, what we claimed follows. Note that the limit of $\ell(\mathrm{H}^0_{\mathfrak{m}}(R/I^n))/n^2$ is equal to $e(J)/2r^2$. ☐

3. The finite representation type case

We now describe how to compute $\mathrm{e_{gHK}}(M)$ when M is a module of positive depth over a Gorenstein local ring of finite Cohen-Macaulay type. We first need some definitions.

Definition 3.1. Let R be a Gorenstein complete local ring of finite Cohen-Macaulay type with perfect residue field (in particular, R must be a hypersurface singularity; see [15]). Let $X_1, \cdots, X_n$ be all the indecomposable non-free Cohen-Macaulay modules.

We define the *stable Cohen-Macaulay type* of M to be the vector $(u_1, \cdots, u_n)$ with $X = \bigoplus X_i^{u_i}$, here X is a Cohen-Macaulay approximation $0 \to M \to N \to X \to 0$ where $\mathrm{pd}_R N < \infty$. This is well defined since R is complete. As R is also a hypersurface, by taking syzygy one can see that X is stably equivalent to the e-syzygy of M where $e = 2\dim R$.

We also define $v_j = \lim_{n\to\infty} \frac{\#({}^nR, X_j)}{q^n}$, where $\#({}^nR, X_j)$ is the number of copies of X_j in the decomposition of ${}^nR$. This limit exists by [13, 14].

Proposition 3.2. *Using the setup of Definition* 3.1*, let M be an R-module of positive depth. One has*

$$e_{gHK}(M) = \sum_{1\le i,j\le n} u_i v_j \ell(\mathrm{Tor}_1^R(X_i, X_j)) = \sum_{1\le i,j\le n} u_i v_j \ell(\mathrm{Tor}_2^R(X_i, X_j)).$$

Proof. Taking an MCM approximation $0 \to M \to N \to X \to 0$ and tensor with ${}^nR$, we get

$$0 \to \mathrm{Tor}_1^R(X, {}^nR) \to M \otimes {}^nR \to N \otimes {}^nR.$$

Noting that $\mathrm{depth}(N \otimes {}^nR) = \mathrm{depth}\, N = \mathrm{depth}\, M > 0$ and $\mathrm{Tor}_1^R(X, {}^nR)$ has finite length as R must have isolated singularity, we get that $\ell(\mathrm{H}^0_{\mathfrak{m}}(M \otimes {}^nR)) = \ell(\mathrm{Tor}_1^R(X, {}^nR))$. The first equality is now obvious.

For the second equality we just need that $\ell(\mathrm{Tor}_1^R(X, {}^nR)) = \ell(\mathrm{Tor}_2^R(X, {}^nR))$ by [5]. □

Example 3.3. Let $R = k[[x, y, z]]/(xy - z^r)$, where k is a perfect field of characteristic $p > 0$. R has finite type with $X_i = (x, z^i)$, $1 \le i \le r-1$. It is not hard to check that $\ell(\mathrm{Tor}_1^R(X_i, X_j)) = \min\{i, j, r-i, r-j\}$. Also, it is known that $v_j = 1/r$. So for a module M with positive depth and stable CM type $(u_1, \ldots, u_n)$, one gets

$$\mathrm{e}_{\mathrm{gHK}}(M) = \frac{1}{r} \sum_{1\le i,j\le r-1} u_i \min\{i, j, r-i, r-j\}.$$

4. The toric case

In this section, we show how to compute the generalized Hilbert-Kunz multiplicity of R/I, where R is a normal toric ring and I is an ideal of R generated by monomials. We fix the following notation.

Notation 4.1. Let k be a field of chaaracteristic p and $M \cong \mathbb{Z}^d$ be a lattice and $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$. Let $\sigma \subset M_{\mathbb{R}}$ be a strongly convex rational polyhedral cone and $R = k[\sigma \cap M] = k[X^m | m \in \sigma \cap M]$ be a normal toric ring. Let $I = (X^{m_1}, \ldots, X^{m_s})$ be a monomial ideal of R. We let Γ_I be the convex hull of $\bigcup_{i=1}^s [m_i + \sigma]$ and $W_I = \bigcup_{i=1}^s [m_i + \sigma]$. We define a subset LC_I of $M_{\mathbb{R}}$ by

$$m \in LC_I \text{ iff } [m + \sigma] \setminus [m + \sigma] \cap W_I \text{ has finite volume.}$$

Proposition 4.2. *With the notation above:*

(1) *For $m \in M$, $x^m \in I : J^\infty$ (where J is the maximal ideal) if and only if $m \in LC_I$.*

(2) $LC_{I^{[q]}} = qLC_I$ *and* $W_{I^{[q]}} = qW_I$.
(3) $LC_I \setminus W_I$ *is a bounded region in* $M_{\mathbb{R}}$.

Proof. It is clear from the definition that $m \in LC_I$ iff $x^m J^t \subseteq I$ for $t \gg 0$. (2) is also clear. For (3), note that the region in question is defined by finitely many half planes. Thus, if it has infinite volume, there will be q large enough such that $LC_I \setminus W_I$ contains infinitely many points in $\frac{1}{q}\mathbb{Z}^d$. In other words, there are infinitely many integral points in $LC_{I^{[q]}} \setminus W_{I^{[q]}}$. But the integral points in that region simply correspond to the monomials in $\mathrm{H}^0_{\mathfrak{m}}(R/I^{[q]})$, a contradiction. $\square$

Remark 4.3. In Figure 1, $LC_I \setminus W_I$ can be seen as a combination of the red and green regions.

Theorem 4.4. *Let R and I be as above. Then $e_{gHK}(R/I) = \mathrm{vol}(LC_I \setminus W_I)$. In particular, $e_{gHK}(R/I) \in \mathbb{Q}$.*

Proof. The previous proposition tells us that $m \in LC_{I^{[q]}}$ iff $m/q \in LC_I$, from which the result follows. $\square$

We demonstrate the ideas of the last theorem with two concrete examples.

Proposition 4.5. *Let $R = k[[x^r, x^{r-1}y, \cdots, y^r]]$ be isomorphic to the r-Veronese of $k[[x,y]]$ and $I_m = (x^r, x^{r-1}y, \cdots, x^{r-m}y^m) \subset R$ be one of the reflexive ideals of R (note that I corresponds to the element $\overline{m} \in \mathbb{Z}/(r) \cong \mathrm{Cl}(R)$). Then*

$$e_{gHK}(R/I_m) = \frac{m(m+1)}{2r}.$$

Proof. Let $I = I_m$. Note that $I^r = (x^{r^2}, \cdots, x^{(r-m)r}y^{rm}) = x^{(r-m)r}\mathfrak{m}^m$. We use Lemmas 2.1 and 2.4 to calculate the relevant lengths. It follows that

$$\lim \ell(\mathrm{H}^0_{\mathfrak{m}}(R/I^q))/q^d = e(\mathfrak{m}^m)/2r^2 = m^2/2r.$$

The second part involves $\ell(I^q/I^{[q]})$. The monomials that are in I^q but not in $I^{[q]}$ are contained in the right triangles whose hypotenuses are the intervals $(iq, (r-i)q), ((i-1)q, (r-i+1)q)$ with $i = r, \cdots, r-m+1$. It is clear that the number of such monomials, which is the length we want, is of order $mq^2/2r$. So the second term contributes $m/2r$ to the limit. We conclude that

$$\mathrm{e_{gHK}}(R/I) = m^2/2r + m/2r = m(m+1)/2r.$$

$\square$

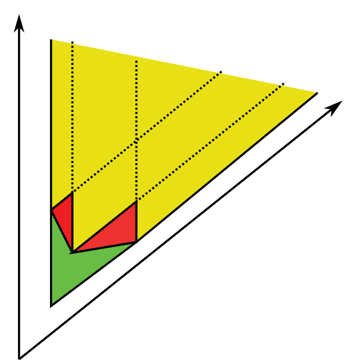

FIGURE 1

Proposition 4.6. *Let $R = k[[x, y, z]]/(xy - z^r)$ and $I_m = (x, z^m) \subset R$ $(m < r)$. Then*

$$e_{gHK}(R/I_m) = \frac{m(r-m)}{r}.$$

Proof. Let $I = I_m$. Note that $I^r = x^m(x^{r-m}, x^{r-m-1}z^m, \ldots, z^{r-m}y^{m-1}, y^m) = x^m J$. We again use Lemmas 2.1 and 2.4.

If we assign points $x \to (r, -1), y \to (0, 1), z \to (1, 0)$, the points corresponding to $(x^{r-m}, x^{r-m-1}z^r, \ldots, z^{r-m}y^{m-1}, y^m)$ are $(r(r-m), -(r-m)), \ldots, (r-m, m-1), (0, m)$, lying on a line of slope $1/(r-m)$. This line and the cone defined by $x \geq 0$ and $y \geq -x/r$ form a triangle of area $r(r-m)/2$; this means the multiplicity of the ideal J is $rm(r-m)$. Hence, $\lim \ell(\mathrm{H}^0_{\mathfrak{m}}(R/I^q))/q^d = m(r-m)/2r$.

On the other hand, $\ell(I^q/I^{[q]})$ corresponds to the triangle whose vertices are $(qm, 0), (qr, -q)$ and $(qr, -q(r-m)/r)$. The area is $m(r-m)q^2/r$. Summing up we have $\mathrm{e_{gHK}}(R/I) = \frac{m(r-m)}{r}$. $\square$

5. The weakly F-regular case

Lastly, we study a connection between generalized Hilbert-Kunz multiplicity and tight closure theory. We first recall the following criterion for tight closure due to Hochster-Huneke.

Lemma 5.1. *Let R be equidimensional and either complete or essentially of finite type over a field and let $N \subseteq L \subseteq G$ be finitely generated R-modules such that L/N has finite length. Then $e_{gHK}(G/N) \geq e_{gHK}(G/L)$, and equality occurs if and only if $L \subseteq N^*_G$.*

We now want to show:

Proposition 5.2. *Let R be weakly F-regular (i.e, all ideals are tightly closed) and M be a finitely generated R-module. The following are equivalent:*

(1) $e_{gHK}(M) = 0$.
(2) $\operatorname{depth} F^n(M) > 0$ *for all* $n \geq 0$.

Proof. We only need to show (1) implies (2). It is harmless to complete R and M (see Exercise 4.1 in [9]). Suppose there exists $n \geq 0$ such that $\operatorname{depth} F^n(M) = 0$; we need to prove that $\mathrm{e_{gHK}}(M) > 0$. Replacing M by $F^n(M)$ if neccessary, we may assume $\operatorname{depth} M = 0$. Now take a short exact sequence $0 \to N \to G \to M \to 0$ where G is free. Let $x \in G$ represent an element in the socle of M. We know that $L = (N, x) \nsubseteq N = N^*_G$, thus $\mathrm{e_{gHK}}(M) > \mathrm{e_{gHK}}(G/L) \geq 0$ by Lemma 5.1. $\square$

Remark 5.3. When R is strongly F-regular, one can prove the above proposition as follows. The assumption means that we have decompositions of R-modules ${}^nR = R^{a_q} \oplus M_q$ and $c = \lim_{n\to\infty} \frac{a_q}{q^d} > 0$. Then it is clear that $\mathrm{e_{gHK}}(M) \geq c\ell(\mathrm{H}^0_{\mathfrak{m}}(M))$, so the non-trivial direction (1) implies (2) is now easy to see.

Corollary 5.4. *Let R be weakly F-regular of dimension at least 2 and I be a reflexive ideal that is locally free on the punctured spectrum. Then $e_{gHK}(R/I) = 0$ if and only if I is principal.*

Proof. By Proposition 5.2 we only need to show that $\operatorname{depth} R/I^{[q]} = 0$ for some q. But suppose it is not the case. Then Lemma 2.1 implies that $I^q = I^{[q]}$ for all q; thus the analytic spread is one. $\square$

Before moving on we recall the following limits studied in [7]. Let $i \geq 0$ be an integer. Let

$$\mathrm{e}^i_{\mathrm{gHK}}(M) := \lim_{n\to\infty} \frac{\ell(\mathrm{H}^i_{\mathfrak{m}}(F^n(M)))}{p^{nd}}.$$

Let $\mathrm{IPD}(M)$ denote the set of prime ideals $\mathfrak{p}$ such that $\mathrm{pd}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = \infty$.

Lemma 5.5. *Let R be of depth d. Let N be an R-module such that $\mathrm{IPD}(N) \subseteq \{\mathfrak{m}\}$. Let M be a t-syzygy of N. Then $\mathrm{H}^{i+t}_{\mathfrak{m}}(F^n(M)) \cong \mathrm{H}^i_{\mathfrak{m}}(F^n(N))$ for $0 \leq i \leq d-t-1$.*

Proof. We begin by tensoring the exact sequence $0 \to \operatorname{syz} N \to F \to N \to 0$ with ${}^nR$ to get

$$0 \to \mathrm{Tor}^R_1(N, {}^nR) \to F^n(\operatorname{syz} N) \xrightarrow{f} F^n(F) \to F^n(N) \to 0,$$

which we break into

$$0 \to \mathrm{Tor}^R_1(N, {}^nR) \to F^n(\operatorname{syz} N) \to C \to 0$$

and

$$0 \to C \to F^n(F) \to F^n(N) \to 0.$$

Note that $\mathrm{Tor}^R_1(N, {}^nR)$ has finite length, so the long sequence of local cohomology for the first sequence gives $\mathrm{H}^i_{\mathfrak{m}}(F^n(\operatorname{syz} N)) \cong \mathrm{H}^i_{\mathfrak{m}}(C)$ for $i > 0$. For the second sequence, we have that $\mathrm{H}^i_{\mathfrak{m}}(F^n(N)) \cong \mathrm{H}^{i+1}_{\mathfrak{m}}(C)$ for $0 \leq i \leq d-2$. Thus

$$\mathrm{H}^i_{\mathfrak{m}}(F^n(N)) \cong \mathrm{H}^{i+1}_{\mathfrak{m}}(F^n(\operatorname{syz} N))$$

for $0 \leq i \leq d-2$. A simple induction finishes the proof. $\square$

Theorem 5.6. *Let R be weakly F-regular of dimension $d \geq 2$ and let $0 \leq a \leq b \leq d-1$ be integers. Let M be an R-module that is locally free on the punctured spectrum and $\operatorname{depth} M \geq a$. The following are equivalent:*

(1) *$e^i_{gHK}(M) = 0$ for $a \leq i \leq b$.*
(2) *$\mathrm{H}^i_{\mathfrak{m}}(F^n(M)) = 0$ for all $a \leq i \leq b$ and all $n \geq 0$.*

Proof. We use induction on $b-a$. It is enough to prove the case $b = a$, since the conclusion implies that $\operatorname{depth} M \geq a+1$, and we can replace a by $a+1$. As $\operatorname{depth} M \geq a$, we can push forward a times and write M as $\operatorname{syz}^a N$ for some module N. Proposition 5.2 and Lemma 5.5 finish the proof. $\square$

Corollary 5.7. *Let R be weakly F-regular and I be a reflexive ideal that is locally free on the punctured spectrum. If $[I]$ is torsion in the class group of R, then I is Cohen-Macaulay.*

Proof. We can assume R has dimension at least 3. We just note that the double dual of $F^n(I)$, $F^n(I)^{**}$, is isomorphic to $I^{(q)}$, which corresponds to the element $q[I]$ in $\mathrm{Cl}(R)$. The natural map $F^n(I) \to F^n(I)^{**}$ has kernel and cokernel of finite length. It follows that $\mathrm{H}^i_{\mathfrak{m}}(F^n(I)) \cong \mathrm{H}^i_{\mathfrak{m}}(F^n(I)^{**}) \cong \mathrm{H}^i_{\mathfrak{m}}(I^{(q)})$ for $i \geq 2$. But the isomorphism classes of $I^{(q)}$ will be periodic as $[I]$ is torsion. Thus $\mathrm{e}^i_{\mathrm{gHK}}(I) = 0$ for $2 \leq i \leq d-1$, and by Theorem 5.6, $\mathrm{H}^i_{\mathfrak{m}}(I) = 0$ for $2 \leq i \leq d-1$, which is all we need to prove. $\square$

Remark 5.8. If the order of $[I]$ is prime to the characteristic of R, the result was first proved, without condition that I is locally free on the punctured spectrum, for strongly F-regular rings in [16, Theorem 2.7 and Example 2.8]. The condition on the order of $[I]$ was removed in [12, Corollary 3.3]. All of these results will be extended in [6], with a more direct approach.

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