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NOTE ABOUT SQUARE FUNCTION ESTIMATES AND UNIFORMLY RECTIFIABLE MEASURES

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ABSTRACT. We generalise and offer a different proof of a recent L^2 square function estimate on uniformly rectifiable sets by Hofmann, Mitrea, Mitrea and Morris. The proof is a short argument using the α-numbers of Tolsa.

1. INTRODUCTION

We will deal with certain square function estimates in $\mathbb{R}^d$ involving the following class of kernels:

1.1. **Definition.** Let $\gamma_1, \gamma_2 > 0$. We say that $S \in K_{\gamma_1,\gamma_2}(\mathbb{R}^d)$ if $S\colon \mathbb{R}^d \times \mathbb{R}^d \setminus \{(x,x)\colon x \in \mathbb{R}^d\} \to \mathbb{R}$ satisfies for some $C < \infty$ that

$$|S(x,y)| \le \frac{C}{|x-y|^{\gamma_1}}$$

and

$$|S(x,y) - S(x,y')| \le C\frac{|y-y'|^{\gamma_2}}{|x-y|^{\gamma_1+\gamma_2}}$$

whenever $|y-y'| \le |x-y|/2$. For $\gamma > 0$ we set $K_\gamma(\mathbb{R}^d) = K_{\gamma,1}(\mathbb{R}^d)$.

In this paper we denote $A \lesssim B$ if $A \le CB$ for some absolute constant C, and $A \sim B$ if $B \lesssim A \lesssim B$.

Suppose $0 < n < d$. We say that a Radon measure μ in $\mathbb{R}^d$ is n-Ahlfors-David Regular (or n-ADR) if $\mu(B(x,r)) \sim r^n$ for all $x \in \operatorname{spt}\mu$ and $0 < r \le \operatorname{diam}(\operatorname{spt}\mu)$. Consider a kernel $S \in K_{n+\beta,\gamma}(\mathbb{R}^d)$ for some $\beta, \gamma > 0$. Given $f \in L^2(\mu) \cup L^\infty(\mu)$ and $x \in \mathbb{R}^d \setminus \operatorname{spt}\mu$ we define the absolutely convergent integral

$$T_{S,\mu}f(x) = \int S(x,y)f(y)\,d\mu(y).$$

In what follows we denote $E = \operatorname{spt}\mu$. For minor convenience, we shall always assume $\operatorname{diam}(E) = \infty$. We are interested in the square function estimate

$$\int_{\mathbb{R}^d\setminus E} |T_{S,\mu}f(x)|^2 \operatorname{dist}(x,E)^{2\beta-(d-n)}\,dx \lesssim \int_E |f(y)|^2\,d\mu(y), \qquad f \in L^2(\mu). \tag{1.2}$$

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David and Semmes [5], and more recently Hofmann, Mitrea, Mitrea and Morris [8] have studied the square function estimate (1.2) in the context of uniformly rectifiable sets in co-dimension one (i.e. $d = n + 1$).

1.3. **Definition.** Let n be an integer. A measure μ in $\mathbb{R}^d$ is n-UR (uniformly rectifiable) if it is n-ADR and it satisfies the big pieces of Lipschitz images (BPLI) property. This means that there exist $\theta, M > 0$ such that for all $x \in E = \operatorname{spt} \mu$ and $r > 0$ we have a Lipschitz mapping g from the ball $B_n(0, r) \subset \mathbb{R}^n$ to $\mathbb{R}^d$ with $\operatorname{Lip}(g) \leq M$ and

$$\mu(B(x, r) \cap g(B_n(0, r))) \geq \theta r^n.$$

A closed set $E \subset \mathbb{R}^d$ is n-ADR if $\mu = \mathcal{H}^n|_E$ is n-ADR, and it is called uniformly n-rectifiable if, further, $\mathcal{H}^n|_E$ is uniformly n-rectifiable. Here $\mathcal{H}^n$ denotes the n-dimensional Hausdorff measure in $\mathbb{R}^d$.

Hofmann, Mitrea, Mitrea and Morris [8, Corollary 5.7] proved that the square function estimate (1.2) holds, if $\mu = \mathcal{H}^n|_E$ for a given n-UR set $E \subset \mathbb{R}^{n+1}$ and $S(x, y) = (\partial_j K)(x-y)$ for some fixed $j \in \{1, \ldots, n+1\}$, where $K \in C^2(\mathbb{R}^{n+1} \setminus \{0\})$, K is odd and $K(\lambda x) = \lambda^{-n} K(x)$ for $\lambda > 0$, $x \in \mathbb{R}^{n+1} \setminus \{0\}$. Such kernels S are kernels of *convolution* form in $K_{n+1}(\mathbb{R}^{n+1})$ and satisfy

$$T_{S,L}1(x) := \int_L S(x, y)\, d\mathcal{H}^n(y) = 0 \tag{1.4}$$

for every n-plane L and $x \notin L$. Much earlier, David and Semmes [5] had proved the case where $K(x) = x_i/|x|^{n+1}$ for some $i \in \{1, \ldots, n + 1\}$. The proof in [8] is relatively complicated having the following steps:

(1) Using a general local Tb theorem one proves that having big pieces of square function estimates (BPSFE) is enough for (1.2). See Definition 4.1 of [8] for the definition of BPSFE.
(2) One proves (1.2) in the case that $S(x, y) = (\partial_j K)(x - y)$ like above and E is a Lipschitz graph. This uses, among other things, Fourier analysis and borrows some techniques from the earlier papers [2], [6], [7] and [9].
(3) One then considers a set E which has big pieces of Lipschitz graphs (BPLG), or rather $(\mathrm{BP})^k\mathrm{LG}$ for some k (big pieces of big pieces of...). Then (1.2) follows (for S like above) from the Lipschitz graph case using the theorem about the sufficiency of BPSFE. See Definition 5.1 of [8] for the definitions of BPLG and $(\mathrm{BP})^k\mathrm{LG}$.
(4) Finally, one uses a deep geometric fact by Azzam and Schul [1] which says that a UR set E has $(\mathrm{BP})^2\mathrm{LG}$.

Our aim is to provide a new and much simpler proof of the above result, and also to prove a more general result.

We now discuss the $T1$ theorem in our setting. It is an important tool in our proof, since it is extremely useful for verifying the square function estimate (1.2). It works assuming only that the measure μ is n-ADR. Before stating it, we need to introduce some dyadic notation. Let $\mathcal{D}(E)$ be a dyadic structure in $E = \operatorname{spt} \mu$ (that is, a collection of David or Christ cubes). We refer to [3] for the construction of dyadic systems in metric spaces. This means that $\mathcal{D}(E) = \bigcup_j \mathcal{D}_j(E)$, and each cube (this is just terminology) $Q \in \mathcal{D}_j(E)$ satisfies $Q \subset E$, $c^{-1}2^{-j} \leq \operatorname{diam}(Q) \leq 2^{-j}$ and $\mu(Q) \sim 2^{-jn}$. For $Q \in \mathcal{D}_j$ we set $\ell(Q) = 2^{-j}$. These sets enjoy the usual structural properties of dyadic cubes i.e. for two cubes $Q, R \in \mathcal{D}(E)$ either $Q \cap R = \emptyset$ or one of them is contained in the other. For a dyadic cube $R \in \mathcal{D}(E)$, $R^{(k)}$ denotes the unique cube $S \in \mathcal{D}(E)$ such that $R \subset S$ and $\ell(S) = 2^k \ell(R)$.

For a true cube $W \subset \mathbb{R}^d$ we denote its side length also by $\ell(W)$. Let $\mathcal{W}$ denote the collection of maximal cubes W from the standard dyadic grid of $\mathbb{R}^d$ for which there holds that $3W \subset \mathbb{R}^d \setminus E$. Then we have that $\operatorname{dist}(x, E) \sim \ell(W)$ for every $x \in 2W$. To each $W \in \mathcal{W}$ we associate precisely one $Q(W) \in \mathcal{D}(E)$ for which $\operatorname{dist}(Q(W), W) \sim \ell(W) \sim \ell(Q(W))$. For every $Q \in \mathcal{D}(E)$ we then define the Whitney region associated to Q by setting

$$W_Q = \bigcup \{W \in \mathcal{W} \colon\, Q(W) = Q\}.$$

The Carleson box $\widehat{R}$ is defined by

$$\widehat{R} = \bigcup_{\substack{Q \in \mathcal{D}(E) \\ Q \subset R}} W_Q.$$

The above is a way to produce Whitney regions which works in this generality. The exact way of producing them is not of great importance. Rather, it is the properties that they enjoy which we shall now list. We have that the sets W_Q, $Q \in \mathcal{D}(E)$, are disjoint, $\mathbb{R}^d \setminus E = \bigcup_{W \in \mathcal{W}} W = \bigcup_{Q \in \mathcal{D}(E)} W_Q$, $\operatorname{dist}(x, E) \sim \ell(Q)$ if $x \in W_Q$, and $|W_Q| \sim \ell(Q)^d$ (for $W_Q \neq \emptyset$).

We are ready to state the $T1$ theorem from [8, Theorem 3.2]. It says that the square function estimate (1.2) is equivalent to

$$\sup_{R \in \mathcal{D}(E)} \frac{1}{\mu(R)} \int_{\widehat{R}} |T_{S,\mu} 1(x)|^2 \operatorname{dist}(x, E)^{2\beta - (d-n)}\, dx < \infty. \tag{1.5}$$

We now formulate our theorem about square function estimates for UR measures.

1.6. **Theorem.** *Let n, d be integers and $0 < n < d$. Suppose S is a kernel which satisfies $S \in K_{n+\beta}(\mathbb{R}^d)$ for some $\beta > 0$. Let μ be an n-UR measure in $\mathbb{R}^d$ with $E = \operatorname{spt} \mu$. If the Carleson condition*

$$\sup_{R \in \mathcal{D}(E)} \frac{1}{\ell(R)^n} \int_{\widehat{R}} \sup_{L\colon\, \operatorname{dist}(x,L) \sim \operatorname{dist}(x,E)} |T_{S,L} 1(x)|^2 \operatorname{dist}(x, E)^{2\beta - (d-n)}\, dx < \infty \tag{1.7}$$

holds, then the square function estimate (1.2) *holds. Here, for a given x, the supremum is taken over all the n-planes $L \subset \mathbb{R}^d$ for which* $\operatorname{dist}(x, L) \sim \operatorname{dist}(x, E)$. *The notation $T_{S,L}$ is defined in* (1.4).

The $T1$ theorem is extremely useful for verifying the square function estimate (1.2). However, the object

$$T_{S,\mu} 1(x) = \int_E S(x, y)\, d\mu(y)$$

might not be so easy to get a hold of if E is not something "geometrically simple". In the case that μ is not only n-ADR but also n-UR and $S \in K_{n+\beta}(\mathbb{R}^d)$ $(= K_{n+\beta,1}(\mathbb{R}^d))$, the Carleson type condition (1.7), where one only needs to integrate over n-planes, turns out to be sufficient for (1.2). Such a condition can be preferable when S is seen to have some special cancellation on n-planes L (for example, when $T_{S,L} 1$ vanishes).

Theorem 1.6 is stated in all co-dimensions, but it is at least interesting in the case of co-dimension one (meaning that $d = n + 1$). Indeed, it generalises Corollary 5.7

of [8], which we discussed above. Recall that $T_{S,L}1$ vanishes for their kernels. The proof of Theorem 1.6 is completely different from the proof of the above-referenced result in [8]. We work directly with the given UR measure μ making no reductions. Inspired by [4] we use the technology of α-numbers without resorting to Fourier analysis. In particular, we don't have to restrict to convolution form kernels. We note that [8] does also include some results about variable coefficient kernels for which they need their convolution form theorem combined with some spherical harmonics extensions.

We will now introduce the α-numbers, which are needed for our proof. The big pieces of Lipschitz images property stated in Definition 1.3 seems to be the preferred definition of uniform rectifiability. However, it is equivalent to a huge plethora of different conditions. For us the crucial one is the one using the so called α-numbers of Tolsa [10]. For two Borel measures σ and ν in $\mathbb{R}^d$ and a closed ball $B \subset \mathbb{R}^d$ we set

$$\operatorname{dist}_B(\sigma, \nu) = \sup \Big\{ \Big| \int f \, d\sigma - \int f \, d\nu \Big| : \operatorname{Lip}(f) \le 1, \operatorname{spt} f \subset B \Big\}.$$

We fix $M \sim 1$ so that given $Q \in \mathcal{D}(E)$ we have $2W \subset B(c_Q, M\ell(Q))$ if $Q(W) = Q$. Here c_Q denotes the centre of Q – it is a point in Q such that $d(c_Q, E \setminus Q) \gtrsim \ell(Q)$. To each cube $Q \in \mathcal{D}(E)$ we also associate the ball $B_Q = B(c_Q, 2M\ell(Q))$. For $Q \in \mathcal{D}(E)$, we define

$$\alpha(Q) = \frac{1}{\ell(Q)^{n+1}} \inf_{c \ge 0, L} \operatorname{dist}_{B_Q}(\mu, c\mathcal{H}^n|_L),$$

where the infimum is taken over all the constants $c \ge 0$ and all the n-planes L for which $L \cap \frac{1}{2}B_Q \neq \emptyset$. The constant $\alpha(Q)$ measures in a scale invariant way how close μ is to a flat n-dimensional measure in the ball B_Q. We also choose c_Q and L_Q which minimise $\alpha(Q)$. We always have that $c_Q \lesssim 1$, and that if $\alpha(Q)$ is small enough, then also $c_Q \gtrsim 1$ (see [10]). The key result of [10] for us is that if μ is n-UR, then for all $R \in \mathcal{D}(E)$ we have that

$$\sum_{\substack{Q \in \mathcal{D}(E) \\ Q \subset R}} \alpha(Q)^2 \mu(Q) \lesssim \mu(R). \tag{1.8}$$

1.9. *Remark.* Notice that if $S \in K_{n+\beta}$, $\beta > 0$, has the cancellation property $T_{S,L}1 \equiv 0$ for every n-plane, then (1.2) holds for every n-UR measure μ. Our Carleson condition (1.7) is a formal relaxation of this. The absolutely most naive relaxation would be to assume that

$$\int_{\mathbb{R}^d \setminus L} |T_{S,L} f(x)|^2 \operatorname{dist}(x, L)^{2\beta - (d-n)} \, dx \le C \int_L |f(y)|^2 \, d\mathcal{H}^n(y), \qquad f \in L^2(L),$$

for every n-plane L and some constant $C < \infty$ independent of L. But such a property does not imply much: the square function estimate can then fail on a sphere even for a positive kernel S. For example, define

$$S(x, y) := \frac{1_{B_1}(x)}{H(x) + |x - y|^2}, \qquad x, y \in \mathbb{R}^2,\, x \neq y.$$

Here $B_1 := B(0, 1)$ and $H \colon \mathbb{R}^2 \to [0, \infty]$, $H(x) := h(\operatorname{dist}(x, O))$, where $O = \partial B_1$,

$$h(t) := t^2 \left(1 + \log^+ \frac{1}{t} \right),$$

and $\log^+ t = \max\{\log t, 0\}$ for $t > 0$. This is a minor side note, which is still somewhat tedious to check. T. Orponen provided us with this example and the calculations verifying it. We give only minor hints of the details.

To check the failure of the square function estimate on the sphere notice that

$$T_{S,O}1(x) \gtrsim H(x)^{-1/2}$$

for $99/100 < |x| < 1$. Therefore, we have that

$$\begin{aligned}\int_{\mathbb{R}^2\setminus O} |T_{S,O}1(x)|^2 \operatorname{dist}(x,O)\,dx &\gtrsim \int_{99/100<|x|<1} \frac{dx}{\operatorname{dist}(x,O)(1+\log(1/\operatorname{dist}(x,O)))} \\ &\sim \int_0^{1/100} \frac{dr}{r(1+\log(1/r))} = \infty\end{aligned}$$

proving that the square function estimate does not hold for the UR set O. But we can show that it does hold uniformly on every line L.

The verification of this is more involved. Fix a line L, and an arbitrary dyadic interval $I_0 \subset L$. By the $T1$ theorem it is enough to verify the estimate

$$\int_{\widehat{I_0}} |T_{S,L}1(x)|^2 \operatorname{dist}(x,L)\,dx \lesssim \ell(I_0).$$

We rewrite and estimate the left hand side as follows:

$$\begin{aligned}\int_{\widehat{I_0}} |T_{S,L}1(x)|^2 \operatorname{dist}(x,L)\,dx &\sim \sum_{I\subset I_0} \ell(I) \int_{W_I} |T_{S,L}1(x)|^2\,dx \\ &\lesssim \sum_{I\subset I_0} \ell(I) \int_{W_I\cap B_1} \min\left(\frac{1}{H(x)}, \frac{1}{\ell(I)^2}\right)dx.\end{aligned}$$

Further estimation of this is somewhat involved (we need to handle separately long and small intervals I). We omit the details.

2. Proof of the square function estimate

In this section we give a proof of the square function estimate (1.2) under the UR hypothesis and (1.7). The proof is short so we are quite generous with the details.

Proof of Theorem 1.6. We will verify the $T1$ condition (1.5). To this end, fix $R \in \mathcal{D}(E)$. Recalling the definition of $\widehat{R}$ we need to prove that

$$\operatorname{Car}(R) := \sum_{\substack{Q\in\mathcal{D}(E)\\ Q\subset R}} \ell(Q)^{2\beta-(d-n)} \int_{W_Q} |T_{S,\mu}1(x)|^2\,dx \lesssim \mu(R).$$

For every $Q \in \mathcal{D}(E)$ and $x \in W_Q$ we want to prove that

$$\begin{aligned}(2.1)\quad |T_{S,\mu}1(x)| &\lesssim \frac{1}{\ell(Q)^\beta} \sum_{\substack{P\in\mathcal{D}(E)\\ P\supset Q}} \left(\frac{\ell(Q)}{\ell(P)}\right)^\beta \alpha(P) + \sup_{L:\ \operatorname{dist}(x,L)\sim\operatorname{dist}(x,E)} |T_{S,L}1(x)| \\ &=: U_1(Q) + U_2(x).\end{aligned}$$

Indeed, notice that

$$\sum_{\substack{Q\in\mathcal{D}(E)\\ Q\subset R}} \ell(Q)^{2\beta-(d-n)} \int_{W_Q} U_1(Q)^2\,dx \lesssim \sum_{Q\colon Q\subset R} \mu(Q) \sum_{P\colon Q\subset P\subset R} \Big(\frac{\ell(Q)}{\ell(P)}\Big)^{\beta} \alpha(P)^2$$

$$+ \sum_{Q\colon Q\subset R} \mu(Q) \sum_{P\colon R\subset P} \Big(\frac{\ell(Q)}{\ell(P)}\Big)^{\beta} \alpha(P)^2 = I_1 + I_2.$$

Using the Carleson property of the α-numbers (1.8) we see that

$$I_1 = \sum_{P\colon P\subset R} \alpha(P)^2 \sum_{Q\colon Q\subset P} \Big(\frac{\ell(Q)}{\ell(P)}\Big)^{\beta} \mu(Q) \lesssim \sum_{P\colon P\subset R} \alpha(P)^2\mu(P) \lesssim \mu(R).$$

The term I_2 is completely elementary (we just estimate $\alpha(P) \lesssim 1$ and do not need the Carleson property of the α-numbers):

$$I_2 \lesssim \sum_{Q\colon Q\subset R} \mu(Q)\Big(\frac{\ell(Q)}{\ell(R)}\Big)^{\beta} \sum_{P\colon R\subset P} \Big(\frac{\ell(R)}{\ell(P)}\Big)^{\beta} \lesssim \sum_{Q\colon Q\subset R} \mu(Q)\Big(\frac{\ell(Q)}{\ell(R)}\Big)^{\beta} \lesssim \mu(R).$$

Next, notice that using (1.7) we have that

$$\sum_{\substack{Q\in\mathcal{D}(E)\\ Q\subset R}} \ell(Q)^{2\beta-(d-n)} \int_{W_Q} U_2(x)^2\,dx$$

$$\sim \int_{\widehat{R}} \sup_{L\colon \operatorname{dist}(x,L)\sim\operatorname{dist}(x,E)} |T_{S,L}1(x)|^2 \operatorname{dist}(x,E)^{2\beta-(d-n)}\,dx \lesssim \ell(R)^n \sim \mu(R).$$

Combining the estimates, we have that (2.1) implies that $\operatorname{Car}(R) \lesssim \mu(R)$. So it only remains to prove (2.1).

Fix $Q \in \mathcal{D}(E)$ and $x \in W_Q$. If $\alpha(Q) \ge c_0$ we have the trivial estimate

$$|T_{S,\mu}1(x)| \lesssim \operatorname{dist}(x,E)^{-\beta} \sim \ell(Q)^{-\beta} \lesssim \ell(Q)^{-\beta}\alpha(Q) \le \frac{1}{\ell(Q)^{\beta}} \sum_{P\colon P\supseteq Q} \Big(\frac{\ell(Q)}{\ell(P)}\Big)^{\beta} \alpha(P).$$

Therefore, we may assume that $\alpha(Q) < c_0$ for a small parameter c_0 to be chosen. We will use this to show that $B(x, c_1\ell(Q)) \cap L_Q = \emptyset$ (here $c_1 > 0$ is a small enough dimensional constant). To this end, let us first show that

$$\sup_{y\in L_Q\cap B(c_Q, M\ell(Q))} \frac{\operatorname{dist}(y,E)}{M\ell(Q)} \le Cc_0^{1/(n+1)} = \epsilon_0. \tag{2.2}$$

Suppose $y \in B(c_Q, M\ell(Q)) \cap L_Q$ and set $\tau = \tau_y = \operatorname{dist}(y,E)/M\ell(Q)$. Notice here that $\tau \in [0,1]$ and $B(y, \tau M\ell(Q)) \subset \mathbb{R}^d \setminus E$. Let ϕ satisfy $1_{B(y,\tau M\ell(Q)/2)} \le \phi \le 1_{B(y,\tau M\ell(Q))}$ and $\operatorname{Lip}(\phi) \sim (\tau\ell(Q))^{-1}$. Note that $\operatorname{spt}\phi \subset B(c_Q, 2M\ell(Q)) = B_Q$ and $\operatorname{spt}\phi \subset \mathbb{R}^d \setminus E$. Therefore, we have that

$$\tau^{-1}c_0\ell(Q)^n \gtrsim \operatorname{Lip}(\phi)\alpha(Q)\ell(Q)^{n+1} \ge c_Q \int \phi\, d\mathcal{H}^n|_{L_Q} \gtrsim_M \tau^n\ell(Q)^n.$$

Here we used that $c_Q \gtrsim 1$ since $\alpha(Q)$ is small. This establishes (2.2).

Suppose then that $B(x, c_1\ell(Q)) \cap L_Q \ne \emptyset$. Then there exists $W \in \mathcal{W}$ so that $Q(W) = Q$ and there exists $y \in 2W \cap L_Q \subset B(c_Q, M\ell(Q)) \cap L_Q$ (the constant c_1 is so small that $B(x, c_1\ell(Q)) \subset 2W$ if $x \in W$). But, in view of (2.2), this means that

$$\ell(W) \sim \operatorname{dist}(y,E) \le \epsilon_0 M\ell(Q) \sim \epsilon_0\ell(W),$$

which is a contradiction for a small enough ϵ_0 i.e. for a small enough c_0. We thus conclude that $B(x, c_1\ell(Q)) \cap L_Q = \emptyset$.

Recalling that $c_Q \lesssim 1$ we estimate

$$|T_{S,\mu}1(x)| \lesssim \Big| \int S(x,y)\, d(\mu - c_Q\mathcal{H}^n|_{L_Q})(y)\Big| + \Big| \int S(x,y)\, d\mathcal{H}^n|_{L_Q}(y)\Big| = P_1 + P_2.$$

We first deal with P_1. To this end, notice that

$$P_1 \le \sum_{k\ge 1} \Big| \int \gamma_k(y) S(x,y)\, d(\mu - c_Q\mathcal{H}^n|_{L_Q})(y)\Big|,$$

where $\sum_{k\ge 0} \gamma_k = 1$, γ_k is smooth, supported on those y for which $|x-y| \sim 2^k\ell(Q)$ and satisfies $\|\nabla \gamma_k\|_\infty \lesssim (2^k\ell(Q))^{-1}$. The key thing is that the corresponding function γ_0 is not needed, since it is supported on $B(x, c_1\ell(Q))$ and this does not intersect the support of the measure i.e. $E \cup L_Q$. We further estimate

$$\begin{aligned} P_1 \le \sum_{k\ge 1} &\Big| \int \gamma_k(y) S(x,y)\, d(\mu - c_{Q^{(k+s_0)}}\mathcal{H}^n|_{L_{Q^{(k+s_0)}}})(y)\Big| \\ &+ \sum_{k\ge 1} \Big| \int \gamma_k(y) S(x,y)\, d(c_{Q^{(k+s_0)}}\mathcal{H}^n|_{L_{Q^{(k+s_0)}}} - c_Q\mathcal{H}^n|_{L_Q})(y)\Big| = J_1 + J_2 \end{aligned}$$

for some $s_0 \sim 1$ such that $\operatorname{spt} \gamma_k \subset B_{Q^{(k+s_0)}}$.

The function $y \mapsto \gamma_k(y)S(x,y)$ is Lipschitz with

$$\operatorname{Lip}(\gamma_k(\cdot)S(x,\cdot)) \lesssim (2^k\ell(Q))^{-n-\beta-1}.$$

This follows easily by using the size, Lipschitz and support properties of the involved functions. Therefore, we have that

$$J_1 \lesssim \sum_{k\ge 1} \alpha(Q^{(k+s_0)})\ell(Q^{(k+s_0)})^{n+1}(2^k\ell(Q))^{-n-\beta-1} \lesssim \frac{1}{\ell(Q)^\beta} \sum_{P\colon P\supset Q} \Big(\frac{\ell(Q)}{\ell(P)}\Big)^\beta \alpha(P).$$

Let us then estimate J_2. Let $f_k(y) := (2^k\ell(Q))^{n+1+\beta}\gamma_k(y)S(x,y)$ so that $\operatorname{Lip}(f_k) \lesssim 1$ and

$$J_2 \le \sum_{k\ge 1} (2^k\ell(Q))^{-n-1-\beta} \sum_{j=1}^{k+s_0} \Big| \int f_k(y)\, d(c_{Q^{(j)}}\mathcal{H}^n|_{L_{Q^{(j)}}} - c_{Q^{(j-1)}}\mathcal{H}^n|_{L_{Q^{(j-1)}}})(y)\Big|.$$

For a fixed k and j we estimate

$$\begin{aligned} \Big| \int f_k(y)\, &d(c_{Q^{(j)}}\mathcal{H}^n|_{L_{Q^{(j)}}} - c_{Q^{(j-1)}}\mathcal{H}^n|_{L_{Q^{(j-1)}}})(y)\Big| \\ &\le c_{Q^{(j)}} \Big| \int f_k(y)\, d(\mathcal{H}^n|_{L_{Q^{(j)}}} - \mathcal{H}^n|_{L_{Q^{(j-1)}}})(y)\Big| \\ &+ |c_{Q^{(j)}} - c_{Q^{(j-1)}}| \Big| \int f_k(y)\, d\mathcal{H}^n|_{L_{Q^{(j-1)}}}(y)\Big| = K_1 + K_2. \end{aligned}$$

To estimate K_1 we use that $c_{Q^{(j)}} \lesssim 1$ and that

$$d_H(L_{Q^{(j-1)}} \cap B_{Q^{(k+s_0)}}, L_{Q^{(j)}} \cap B_{Q^{(k+s_0)}}) \lesssim \alpha(Q^{(j)})\ell(Q^{(k+s_0)}).$$

Here d_H stands for the Hausdorff distance and we have used Lemma 3.4 of [10]. Using this one sees that

$$K_1 \lesssim \alpha(Q^{(j)})(2^k\ell(Q))^{n+1}.$$

Lemma 3.4 of [10] also gives that $|c_{Q^{(j)}} - c_{Q^{(j-1)}}| \lesssim \alpha(Q^{(j)})$. After this it is clear that also

$$K_2 \lesssim \alpha(Q^{(j)})(2^k \ell(Q))^{n+1}.$$

We conclude that

$$\begin{aligned} J_2 &\lesssim \sum_{k \geq 1} (2^k \ell(Q))^{-\beta} \sum_{j=1}^{k+s_0} \alpha(Q^{(j)}) \\ &\lesssim \sum_{R\colon R \supset Q} \ell(R)^{-\beta} \sum_{P\colon Q \subset P \subset R} \alpha(P) \\ &= \sum_{P\colon P \supset Q} \alpha(P) \ell(P)^{-\beta} \sum_{R\colon R \supset P} \Big(\frac{\ell(P)}{\ell(R)}\Big)^\beta \\ &\lesssim \sum_{P\colon P \supset Q} \alpha(P) \ell(P)^{-\beta} = \frac{1}{\ell(Q)^\beta} \sum_{P\colon P \supset Q} \Big(\frac{\ell(Q)}{\ell(P)}\Big)^\beta \alpha(P). \end{aligned}$$

Combining the estimates for J_1 and J_2 we have that

$$P_1 \lesssim \frac{1}{\ell(Q)^\beta} \sum_{P\colon P \supset Q} \Big(\frac{\ell(Q)}{\ell(P)}\Big)^\beta \alpha(P).$$

For P_2 notice that we have that $\operatorname{dist}(x, L_Q) \sim \ell(Q) \sim \operatorname{dist}(x, E)$. Therefore, we have that

$$P_2 \lesssim \sup_{L\colon\ \operatorname{dist}(x,L) \sim \operatorname{dist}(x,E)} |T_{S,L} 1(x)|.$$

Combining the estimates for P_1 and P_2 we have that (2.1) holds. Therefore, we are done. □

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