PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 144, Number 9, September 2016, Pages 3887–3902
http://dx.doi.org/10.1090/proc/13030
Article electronically published on March 17, 2016

(VOLUME) DENSITY PROPERTY OF A FAMILY OF COMPLEX MANIFOLDS INCLUDING THE KORAS-RUSSELL CUBIC THREEFOLD

MATTHIAS LEUENBERGER

(Communicated by Franc Forstneric)

Abstract. We present modified versions of existing criteria for the density property and the volume density property of complex manifolds. We apply these methods to show the (volume) density property for a family of manifolds given by $x^2y = a(\bar{z}) + xb(\bar{z})$ with $\bar{z} = (z_0, \ldots, z_n) \in \mathbb{C}^{n+1}$ and holomorphic volume form $\mathrm{d}x/x^2 \wedge \mathrm{d}z_0 \wedge \ldots \wedge \mathrm{d}z_n$. The key step is to show that in certain cases transitivity of the action of (volume preserving) holomorphic automorphisms implies the (volume) density property, and then to give sufficient conditions for the transitivity of this action. In particular, we show that the Koras-Russell cubic threefold $\{x^2y + x + z_0^2 + z_1^3 = 0\}$ has the density property and the volume density property.

1. Introduction

The density property and the volume density property are properties of Stein manifolds with a huge amount of applications in complex geometry in several variables. They were introduced by Varolin in [17]. The fact that $\mathbb{C}^n$ has the density property for $n \geq 2$ was already used by Andersén and Lempert in [1] where they showed that holomorphic automorphisms can be approximated by some special family of automorphisms. Varolin realized that the main observation of Andersén and Lempert may be formalized and can be applied to more general complex manifolds and different problems. Especially, Rosay and Forstnerič contributed a lot to this progress in [4]. This area of complex analysis in several variables is nowadays called Andersén-Lempert theory. The numerous applications of the density property are due to the Main Theorem of Andersén-Lempert theory which states that on manifolds with density property any local phase flow on a Runge domain can be approximated uniformly on compacts by global automorphisms. The analogous statement holds in the volume preserving case. For a deeper view into this topic we refer to the comprehensive texts [3, 8, 11].

Definition 1.1. Let X be a Stein manifold. If the Lie algebra $\mathrm{Lie}(\mathrm{CVF}_{\mathrm{hol}}(X))$ generated by complete (= globally integrable) holomorphic vector fields $\mathrm{CVF}_{\mathrm{hol}}(X)$

Received by the editors July 14, 2015 and, in revised form, November 1, 2015 and November 5, 2015.

2010 *Mathematics Subject Classification.* Primary 32M05, 32M25, 14R10.

Key words and phrases. Density property, holomorphic automorphisms, Koras-Russell cubic threefold.

The author was partially supported by Schweizerischer Nationalfond Grant 200021-140235/1.

on X is dense (in compact-open topology) in the Lie algebra of all holomorphic vector fields $\mathrm{VF}_{\mathrm{hol}}(X)$ on X, then X has the density property.

Let X be a Stein manifold equipped with a holomorphic volume form[1] ω. If the Lie algebra $\mathrm{Lie}(\mathrm{CVF}^{\omega}_{\mathrm{hol}}(X))$ generated by complete volume preserving (= vanishing ω-divergence) holomorphic vector fields $\mathrm{CVF}^{\omega}_{\mathrm{hol}}(X)$ on X is dense in the Lie algebra of all volume preserving holomorphic vector fields $\mathrm{VF}^{\omega}_{\mathrm{hol}}(X)$ on X, then X has the volume density property.

Recall that a vector field ν is called volume preserving if the Lie derivative $L_\nu\omega$ vanishes where the Lie derivative is given by the formula $L_\nu = \mathrm{d}i_\nu + i_\nu \mathrm{d}$ and i_ν is the interior product of a form with ν.

In Section 2 we present a criterion that implies the (volume) density property. For the definition of (semi-)compatible pairs and (ω-)generating sets see Definitions 2.9, 2.11 and 2.4. For a vector field ν and a point $p \in X$, we denote by $\nu[p] \in \mathrm{T}_pX$ the tangent vector of ν at p.

Theorem 1.2. (1) *Let X be a Stein manifold such that the holomorphic automorphisms $\mathrm{Aut}_{\mathrm{hol}}(X)$ act transitively on X. If there are compatible pairs $(\nu_i, \mu_i), i = 1, \ldots, N$, such that there is a point $p \in X$ where the vectors $\{\mu_i[p] : i = 1, \ldots, N\}$ form a generating set of T_pX, then X has the density property.*

(2) *Let X be a Stein manifold with a holomorphic volume form ω such that the volume preserving holomorphic automorphisms $\mathrm{Aut}^{\omega}_{\mathrm{hol}}(X)$ act transitively on X and $\mathrm{H}^{n-1}(X, \mathbb{C}) = 0$ (where $n = \dim X$). If there are semi-compatible pairs $(\nu_i, \mu_i), i = 1, \ldots, N$, of volume preserving vector fields such that there is a point $p \in X$ where the vectors $\{\nu_i[p] \wedge \mu_i[p] : i = 1, \ldots, N\}$ form a generating set of $\mathrm{T}_pX \wedge \mathrm{T}_pX$, then X has the volume density property.*

Note that this criterion and its proof is very much inspired by the criteria in [6,9] for the algebraic (volume) density property (see Definition 5.3). Actually, e.g. for (1), the only difference is that instead of requiring the algebraic automorphisms to act transitively on X we require the holomorphic automorphisms to act transitively.

In Section 3 we investigate the transitivity of the action by (volume preserving) holomorphic automorphisms $\mathrm{Aut}_{\mathrm{hol}}(X)$ (resp. $\mathrm{Aut}^{\omega}_{\mathrm{hol}}(X)$) where X is given by $x^2y = a(\bar{z}) + xb(\bar{z})$ with $\bar{z} = (z_0, \ldots, z_n) \in \mathbb{C}^{n+1}$ for some $n \geq 0$, $\deg_{z_0}(a) \leq 2$ and $\deg_{z_0}(b) \leq 1$. We show that (after possibly reordering the z_i) the condition

(A) there is some $k \geq 0$ such that $\deg_{z_i}(a) \leq 2$ and $\deg_{z_i}(b) \leq 1$ for all $i \leq k$ and for all common zeroes $\bar{q} = (q_0, \ldots, q_n)$ of $a, \frac{\partial a}{\partial z_0}, \ldots, \frac{\partial a}{\partial z_k}$, we have $b(\bar{q}) \neq 0$, and there is some $j \leq k$ such that $\frac{\partial a}{\partial z_j}$ does not vanish along the curve $\{z_i = q_i \text{ for all } i \neq j\} \subset \mathbb{C}^{n+1}$

is a sufficient condition for $\mathrm{Aut}_{\mathrm{hol}}(X)$ to act transitively on X. If additionally

(B) there is some $k \geq 0$ such that $\deg_{z_i}(a) \leq 2$ and $\deg_{z_i}(b) \leq 1$ for all $i \leq k$ and there is no $c \in \mathbb{C}^*$ for which the polynomials $\frac{\partial a}{\partial z_i} + c\frac{\partial b}{\partial z_i}$ for $i \leq k$ are all constant to zero

holds, then also $\mathrm{Aut}^{\omega}_{\mathrm{hol}}(X)$ acts transitively (Proposition 3.4).

In Section 4 we apply Theorem 1.2 to these kinds of surfaces for $n > 0$. This leads to our Main Theorem.

[1]A holomorphic volume form on a manifold of dimension n is a holomorphic n-form which is nowhere vanishing.

Main Theorem. *Let $n \geq 0$ and $a, b \in \mathbb{C}[z_0, \ldots, z_n]$ such that $\deg_{z_0}(a) \leq 2$, $\deg_{z_0}(b) \leq 1$ and not both of $\deg_{z_0}(a)$ and $\deg_{z_0}(b)$ are equal to zero. Let $\bar{z} = (z_0, \ldots, z_n)$. Then the hypersurface $X = \{x^2 y = a(\bar{z}) + x b(\bar{z})\}$ has the density property provided that the holomorphic automorphisms $\mathrm{Aut}_{\mathrm{hol}}(X)$ act transitively on X. In particular X has the density property if* (A) *holds or if $n = 0$.*

Moreover, if $\mathrm{H}^{n+1}(X, \mathbb{C}) = 0$ and the volume preserving holomorphic automorphisms $\mathrm{Aut}^{\omega}_{\mathrm{hol}}(X)$ act transitively on X, then X has the volume density property for the volume form $\omega = \mathrm{d}x/x^2 \wedge \mathrm{d}z_0 \wedge \ldots \wedge \mathrm{d}z_n$. In particular the transitivity condition holds if (A) *and* (B) *hold or if $n = 0$.*

The proof of the Main Theorem is finished in Section 5 where the case $n = 0$ is done by explicit calculations, not using the methods described in Section 2.

It is worth pointing out that the Main Theorem together with Corollary 3.5 implies that the Koras-Russell cubic threefold $C = \{x^2 y + x + z_0^2 + z_1^3 = 0\}$ has the (volume) density property. The threefold C is a famous example of an affine variety which is diffeomorphic to $\mathbb{R}^6$ but not algebraically isomorphic to $\mathbb{C}^3$; e.g. see [14]. As an affine algebraic variety, C (in particular the algebraic automorphism group of C) is well understood; e.g. see [2]. For example, it is known that the algebraic automorphisms do not act transitively on C. The density property implies that the situation in the holomorphic context is completely different. However, it is still unclear if C is biholomorphic to $\mathbb{C}^3$. Related to this question is a conjecture of Tóth and Varolin. The conjecture [16] states that a manifold which has the density property and which is diffeomorphic to $\mathbb{C}^n$ is automatically biholomorphic to $\mathbb{C}^n$. If the conjecture holds, then the Main Theorem would imply that C is isomorphic to $\mathbb{C}^3$.

2. Proof of Theorem 1.2

Let X be a Stein manifold of dimension n, and let $\mathcal{O}_X$ be the sheaf of holomorphic functions on X.

2.1. **Preliminaries.** Let $\mathfrak{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules, and let $s_1, \ldots, s_N \in \mathfrak{F}(X)$ be global sections. The following lemmas are standard applications of sheaf theory.

Lemma 2.1. *Let $p \in X$, and let $\mathfrak{m}_p \subset \mathcal{O}_X(X)$ be the corresponding ideal. If the elements $s_i + \mathfrak{m}_p \mathfrak{F}(X)$ span the vector space $\mathfrak{F}(X)/\mathfrak{m}_p \mathfrak{F}(X)$, then the localizations $(s_i)_p$ generate the stalk $\mathfrak{F}_p$.*

Proof. Let $\mathfrak{G}_p$ be the $(\mathcal{O}_X)_p$-submodule of $\mathfrak{F}_p$ generated by $(s_1)_p, \ldots, (s_N)_p$. The assumption implies that we have $(\mathfrak{G}_p + \mathfrak{m}_p \mathfrak{F}_p)/\mathfrak{m}_p \mathfrak{F}_p = \mathfrak{F}_p/\mathfrak{m}_p \mathfrak{F}_p$. Let $\mathfrak{L}_p = \mathfrak{F}_p/\mathfrak{G}_p$. We get

$$\begin{aligned}\frac{\mathfrak{L}_p}{\mathfrak{m}_p \mathfrak{L}_p} &= \frac{\mathfrak{F}_p/\mathfrak{G}_p}{\mathfrak{m}_p(\mathfrak{F}_p/\mathfrak{G}_p)} = \frac{\mathfrak{F}_p/\mathfrak{G}_p}{(\mathfrak{m}_p \mathfrak{F}_p + \mathfrak{G}_p)/\mathfrak{G}_p} \\ &= \frac{\mathfrak{F}_p}{\mathfrak{m}_p \mathfrak{F}_p + \mathfrak{G}_p} = \frac{\mathfrak{F}_p/\mathfrak{m}_p \mathfrak{F}_p}{(\mathfrak{m}_p \mathfrak{F}_p + \mathfrak{G}_p)/\mathfrak{m}_p \mathfrak{F}_p} = 0.\end{aligned}$$

The ring $(\mathcal{O}_X)_p$ is local, so by the Nakayama Lemma we may lift a basis of $\mathfrak{L}_p/\mathfrak{m}_p \mathfrak{L}_p$ to a generating set of $\mathfrak{L}_p$, and thus $\mathfrak{L}_p = 0$. This yields $\mathfrak{G}_p = \mathfrak{F}_p$, which shows the claim. $\square$

Lemma 2.2. *If the elements $(s_i)_p$ generate the stalks $\mathfrak{F}_p$ for all points $p \in X$, then every global section $\nu \in \mathfrak{F}(X)$ is of the form $\sum f_i s_i$ for some global holomorphic functions $f_i \in \mathcal{O}_X(X)$.*

Proof. Consider the morphisms of sheaves $\varphi : \mathcal{O}_X^N \to \mathfrak{F}$ given by $(f_i) \mapsto \sum f_i s_i$. By assumption φ is surjective on the level of stalks. Therefore we get the following short exact sequence of coherent sheaves:

$$0 \to \ker \varphi \to \mathcal{O}_X^N \to \mathfrak{F} \to 0.$$

Indeed $\ker \varphi$ is coherent as the kernel of a morphism between coherent sheaves. This yields the following long exact sequence:

$$\cdots \to \mathrm{H}^0(X, \mathcal{O}_X^N) \to \mathrm{H}^0(X, \mathfrak{F}) \to \mathrm{H}^1(X, \ker \varphi) \to \cdots .$$

By Theorem B of Cartan we have $\mathrm{H}^1(X, \ker \varphi) = 0$. Thus the leftmost map is surjective, and therefore every global section $\nu \in \mathrm{H}^0(X, \mathfrak{F}) = \mathfrak{F}(X)$ is of the desired form. $\square$

Recall that a domain $Y \subset X$ is called Runge if all holomorphic functions on Y can be approximated uniformly on compacts $K \subset Y$ by global holomorphic functions on X.

Lemma 2.3. *Let $Y \subset X$ be a domain of X which is Runge and Stein. If the elements $(s_i)_p$ generate the stalks $\mathfrak{F}_p$ for all points $p \in Y$, then every global section $\nu \in \mathfrak{F}(X)$ can be uniformly approximated on compacts $K \subset Y$ by global sections of the form $\sum f_i s_i$ for some global holomorphic functions $f_i \in \mathcal{O}_X(X)$.*

Proof. Let $\nu|_Y \in \mathfrak{F}(Y)$ be the restriction of ν to Y. By Lemma 2.2 we have $\nu|_Y = \sum g_i s_i|_Y$ for some holomorphic functions $g_i \in \mathcal{O}_X(Y)$ on Y. Since Y is a Runge domain we may approximate the functions g_i by global functions $f_i \in \mathcal{O}_X(X)$ uniformly on compacts $K \subset Y$. Thus the global section ν can be approximated by sections $\sum f_i s_i$ uniformly on compacts $K \subset Y$. $\square$

2.2. **Criterion for the (volume) density property.** The following definition is due to [6], but adapted to the holomorphic case.

Definition 2.4. Let $p \in X$. A set $U \subset \mathrm{T}_p X$ is called a generating set for $\mathrm{T}_p X$ if the orbit of U of the induced action of the stablilizer $\mathrm{Aut}_{\mathrm{hol}}(X)_p$ contains a basis of $\mathrm{T}_p X$.

If X has a volume form ω, then a set $U \subset \mathrm{T}_p X \wedge \mathrm{T}_p X$ is called an ω-generating set for $\mathrm{T}_p X \wedge \mathrm{T}_p X$ if the orbit of U of the induced action of the stabilizer $\mathrm{Aut}^{\omega}_{\mathrm{hol}}(X)_p$ contains a basis of $\mathrm{T}_p X \wedge \mathrm{T}_p X$.

The next proposition is a powerful criterion for the density property. It is a generalization of Theorem 2 in [6] and the proof is similar.

Proposition 2.5. *Let X be a Stein manifold such that $\mathrm{Aut}_{\mathrm{hol}}(X)$ acts transitively on X. Assume that there are complete vector fields $\nu_1, \ldots, \nu_N \in \mathrm{CVF}_{\mathrm{hol}}(X)$ which generate a submodule that is contained in the closure of $\mathrm{Lie}(\mathrm{CVF}_{\mathrm{hol}}(X))$ and assume that there is a point $p \in X$ such that the tangent vectors $\nu_i[p] \in \mathrm{T}_p X$ are a generating set for the tangent space $\mathrm{T}_p X$. Then X has the density property.*

Proof. We may assume that the vectors $\nu_i[p]$ contain a basis of $\mathrm{T}_p X$. Indeed, the vectors $\nu_i[p]$ are a generating set of $\mathrm{T}_p X$. Thus after adding some pullbacks of some vector fields ν_i by automorphisms in $\mathrm{Aut}_{\mathrm{hol}}(X)_p$ we get the desired basis of $\mathrm{T}_p X$.

Let $A \subsetneq X$ be the analytic subset of points where the vectors $\nu_i[a]$ do not span the whole tangent space $\mathrm{T}_a X$.

Let $\bigcup K_i = X$ be an exhaustion by $\mathcal{O}_X$-convex compacts. For any $K = K_i$, let Y be a neighborhood of K which is Stein and Runge, and moreover, such that the closure of Y is compact. The existence of such a Y is, for example, shown in [5]: Theorem 5.1.6 implies the existence of a Stein neighborhood and Theorem 5.2.8 shows that it is Runge.

After adding finitely many complete vector fields to $\nu_1, \ldots, \nu_N$ we get that $Y \cap A = \emptyset$. Indeed, since the closure of Y is compact, $Y \cap A$ is a finite union of irreducible analytic subsets. Let $A_0 \subset A$ be an irreducible component of maximal dimension. Pick any $a \in A_0$ and $\phi \in \mathrm{Aut}_{\mathrm{hol}}(X)$ such that $\phi(a) \in Y \setminus A$. Since the vectors $\nu_i[\phi(a)]$ span the tangent space $\mathrm{T}_{\phi(a)} X$ the vectors $(\phi^* \nu_i)[a]$ span the tangent space $\mathrm{T}_a X$. Thus after adding some of the pullbacks to $\nu_1, \ldots, \nu_N$ the component $A_0 \cap Y$ is replaced by finitely many components of lower dimension. Repeating the same procedure, inductively we get after finitely many steps a list of complete vector fields $\nu_1, \ldots, \nu_N$ such that $A \cap Y = \emptyset$.

Let $\mathfrak{F}$ be the tangent sheaf. It is coherent since X is Stein. The fact that the vectors $\nu_i[a]$ span $\mathrm{T}_a X$ for all $a \in Y$ translates to the fact that the elements $\nu_i + \mathfrak{m}_a \mathfrak{F}$ span the vector space $\mathfrak{F} / \mathfrak{m}_a \mathfrak{F}$ for all $a \in Y$, where $\mathfrak{m}_a$ is the maximal ideal of a. Thus by Lemma 2.1 the assumption of Lemma 2.3 holds. Therefore every vector field on X can be approximated uniformly on K by elements of the form $\sum f_i \nu_i$ for some holomorphic functions $f_i \in \mathcal{O}_X(X)$. By assumption the submodule generated by $\nu_1, \ldots, \nu_N$ is contained in the closure of $\mathrm{Lie}(\mathrm{CVF}_{\mathrm{hol}}(X))$ (note that this property still holds after enlarging the list $\nu_1, \ldots, \nu_N$ in the procedure above). Therefore every holomorphic vector field is in the closure of $\mathrm{Lie}(\mathrm{CVF}_{\mathrm{hol}}(X))$. □

For the volume case the proof can be found in [12]. For completeness we indicate a proof here. We introduce the following isomorphisms. The article [9] is a good reference for these methods. Note that the reference is written only for the algebraic case, however all isomorphisms exist identically in the holomorphic case. Let ω be a holomorphic volume form. For $0 \le i \le n$, let $\mathcal{C}_i(X)$ be the vector space of holomorphic i-forms on X. Moreover, let $\mathcal{Z}_i(X) \subset \mathcal{C}_i(X)$ be the vector space of closed i-forms, and let $\mathcal{B}_i(X) \subset \mathcal{Z}_i(X)$ be the vector space of exact i-forms. Then we have the isomorphism

$$\Phi : \mathrm{VF}_{\mathrm{hol}}(X) \xrightarrow{\sim} \mathcal{C}_{n-1}(X), \quad \nu \mapsto i_\nu \omega,$$

and in the same spirit we also have the isomorphism Ψ induced by

$$\Psi : \mathrm{VF}_{\mathrm{hol}}(X) \wedge \mathrm{VF}_{\mathrm{hol}}(X) \xrightarrow{\sim} \mathcal{C}_{n-2}(X), \quad \nu \wedge \mu \mapsto i_\nu i_\mu \omega.$$

These are isomorphisms since ω is non-degenerate in every point on X. By definition $\mathrm{VF}^\omega_{\mathrm{hol}}(X)$ consists of those vector fields ν such that $L_\nu \omega = \mathrm{d} i_\nu \omega = 0$ holds and thus the isomorphism Φ restricts to an isomorphism

$$\Theta = \Phi|_{\mathrm{VF}^\omega_{\mathrm{hol}}(X)} : \mathrm{VF}^\omega_{\mathrm{hol}}(X) \xrightarrow{\sim} \mathcal{Z}_{n-1}(X).$$

Moreover, we consider the outer differential

$$D : \mathcal{C}_{n-2}(X) \twoheadrightarrow \mathcal{B}_{n-1}(X)$$

of $(n-2)$-forms.

Lemma 2.6. *Let $\alpha, \beta \in \mathrm{VF}^\omega_{\mathrm{hol}}(X)$. Then $i_{[\alpha,\beta]}\omega = \mathrm{d} i_\alpha i_\beta \omega$.*

Proof. Proposition 3.1 in [9]. □

The next proposition is in some sense a version of Proposition 2.5 in the volume preserving case. It is an adaption of Theorem 1 in [9].

Proposition 2.7. *Let X be a Stein manifold such that $\mathrm{Aut}^{\omega}_{\mathrm{hol}}(X)$ acts transitively on X. Assume that every class of $\mathrm{H}^{n-1}(X,\mathbb{C}) = \mathcal{Z}_{n-1}(X)/\mathcal{B}_{n-1}(X)$ contains an element of $\Theta(\mathrm{Lie}(\mathrm{CVF}^{\omega}_{\mathrm{hol}}(X)))$. Moreover, assume that there are complete vector fields $\nu_1,\ldots,\nu_N \in \mathrm{CVF}^{\omega}_{\mathrm{hol}}(X)$ and $\mu_1,\ldots,\mu_N \in \mathrm{CVF}^{\omega}_{\mathrm{hol}}(X)$ such that the submodule of $\mathrm{VF}_{\mathrm{hol}}(X) \wedge \mathrm{VF}_{\mathrm{hol}}(X)$ generated by the elements $\nu_j \wedge \mu_j$ is contained in the closure of $\mathrm{Lie}(\mathrm{CVF}^{\omega}_{\mathrm{hol}}(X)) \wedge \mathrm{Lie}(\mathrm{CVF}^{\omega}_{\mathrm{hol}}(X))$, and assume that there is a point $p \in X$ such that the vectors $\{\nu_j[p] \wedge \mu_j[p] : i = 1,\ldots,N\}$ are an ω-generating set for the vector space $\mathrm{T}_pX \wedge \mathrm{T}_pX$. Then X has the volume density property.*

Proof. Let $K \subset X$ be a compact set. By identical arguments as in the proof of Proposition 2.5 we see that every element of $\mathrm{VF}_{\mathrm{hol}}(X)\wedge\mathrm{VF}_{\mathrm{hol}}(X)$ may be uniformly approximated on K by elements contained in $\mathrm{Lie}(\mathrm{CVF}^{\omega}_{\mathrm{hol}}(X)) \wedge \mathrm{Lie}(\mathrm{CVF}^{\omega}_{\mathrm{hol}}(X))$.

Let $\zeta \in \mathrm{VF}^{\omega}_{\mathrm{hol}}(X)$. By the first assumption we may, after subtracting an element of $\Theta(\mathrm{Lie}(\mathrm{CVF}^{\omega}_{\mathrm{hol}}(X)))$, assume that $\Theta(\zeta) \in \mathcal{B}_{n-1}(X)$. Thus $\Theta(\zeta) = D(\Psi(\gamma))$ for some $\gamma \in \mathrm{VF}_{\mathrm{hol}}(X) \wedge \mathrm{VF}_{\mathrm{hol}}(X)$. Let us approximate γ uniformly on K by elements of the form $\sum \alpha_i \wedge \beta_i \in \mathrm{Lie}(\mathrm{CVF}^{\omega}_{\mathrm{hol}}(X)) \wedge \mathrm{Lie}(\mathrm{CVF}^{\omega}_{\mathrm{hol}}(X))$ with $\alpha_i, \beta_i \in \mathrm{Lie}(\mathrm{CVF}^{\omega}_{\mathrm{hol}}(X))$. We use Lemma 2.6 to see that

$$D(\Psi(\sum \alpha_i \wedge \beta_i)) = \sum \mathrm{d}i_{\alpha_i} i_{\beta_i}\omega = \sum i_{[\alpha_i,\beta_i]}\omega \in \Theta(\mathrm{Lie}(\mathrm{CVF}^{\omega}_{\mathrm{hol}}(X))).$$

Since $\Theta(\zeta) = i_\zeta\omega$ can be approximated uniformly on K by elements of the form $\sum i_{[\alpha_i,\beta_i]}\omega = \Theta(\sum[\alpha_i,\beta_i])$, the vector field ζ can be approximated uniformly on K by elements of the form $\sum[\alpha_i,\beta_i] \in \mathrm{Lie}(\mathrm{CVF}^{\omega}_{\mathrm{hol}}(X))$, which proves the proposition. □

2.3. **Semi-compatible and compatible pairs.** This section provides a way to find the submodules which are required in Propositions 2.5 and 2.7. The parts 2.9 - 2.12 are e.g. from [6] and [9] and are here adapted to the holomorphic case. For a vector field ν and a holomorphic function f we denote by $\nu(f)$ the holomorphic function which is obtained by applying ν as a derivation and $\ker\nu$ is the kernel of this linear map. The following lemma is well known.

Lemma 2.8. *Let ν be a complete vector field and let $f \in \ker\nu$. Then $f\nu$ is complete. Moreover, if ν is volume preserving, then so is $f\nu$. Let $g \in \mathcal{O}_X(X)$ be such that $\nu(g) \in \ker\nu$; then $g\nu$ is also complete.*

Definition 2.9. A semi-compatible pair is a pair (ν,μ) of complete vector fields such that the closure of the linear span of the product of the kernels $\ker\nu\cdot\ker\mu$ contains a non-trivial ideal $I \subset \mathcal{O}_X(X)$. We call I a compatible ideal of (ν,μ).

Note that the ideal I is not unique. In most applications $\mathcal{O}_X(X)$ itself serves as the ideal.

Lemma 2.10. *Let (ν,μ) be a semi-compatible pair of volume preserving vector fields and let I be a compatible ideal. Then the submodule of $\mathrm{VF}_{\mathrm{hol}}(X) \wedge \mathrm{VF}_{\mathrm{hol}}(X)$ given by $I\cdot(\nu\wedge\mu)$ is contained in the closure of $\mathrm{Lie}(\mathrm{CVF}^{\omega}_{\mathrm{hol}}(X)) \wedge \mathrm{Lie}(\mathrm{CVF}^{\omega}_{\mathrm{hol}}(X))$.*

Proof. Let $\tau = (\sum f_i g_i)\cdot(\nu\wedge\mu)$ with $f_i \in \ker\nu$ and $g_i \in \ker\mu$ be an arbitrary element of $\mathrm{span}\{\ker\nu\cdot\ker\mu\}\cdot(\nu\wedge\mu)$. By Lemma 2.8 we have $f_i\nu \in \mathrm{CVF}^{\omega}_{\mathrm{hol}}(X)$ for all

i and similarly $g_i\nu \in \mathrm{CVF}^{\omega}_{\mathrm{hol}}(X)$ for all i. Thus $\tau = \sum f_i\nu \wedge g_i\mu \in \mathrm{Lie}(\mathrm{CVF}^{\omega}_{\mathrm{hol}}(X)) \wedge \mathrm{Lie}(\mathrm{CVF}^{\omega}_{\mathrm{hol}}(X))$. Therefore the closure of $\mathrm{span}\{\ker \nu \cdot \ker \mu\} \cdot (\nu \wedge \mu)$ is contained in the closure of $\mathrm{Lie}(\mathrm{CVF}^{\omega}_{\mathrm{hol}}(X)) \wedge \mathrm{Lie}(\mathrm{CVF}^{\omega}_{\mathrm{hol}}(X))$, and the claim follows. □

Definition 2.11. A semi-compatible pair (ν, μ) is called a compatible pair if there is a holomorphic function $h \in \mathcal{O}_X(X)$ with $\nu(h) \in \ker \nu \setminus 0$ and $h \in \ker \mu$. We call h a compatible function of the pair (ν, μ). Note that this condition implies, together with Lemma 2.8, that $h\nu$ is a complete vector field.

Lemma 2.12. *Let (ν, μ) be a compatible pair, and let I be a compatible ideal and h a compatible function of (ν, μ). Then the submodule of* $\mathrm{VF}_{\mathrm{hol}}(X)$ *given by $I \cdot \nu(h) \cdot \mu$ is contained in the closure of* $\mathrm{Lie}(\mathrm{CVF}_{\mathrm{hol}}(X))$.

Proof. Let $f \in \ker \nu$ and $g \in \ker \mu$; then $f\nu, fh\nu, g\mu, gh\nu \in \mathrm{CVF}_{\mathrm{hol}}(X)$ by Lemma 2.8. A standard calculation shows that

$$[f\nu, gh\mu] - [fh\nu, g\mu] = fg\nu(h)\mu \in \mathrm{Lie}(\mathrm{CVF}_{\mathrm{hol}}(X)).$$

Thus an arbitrary element $\sum (f_i g_i)\nu(h)\mu \in \mathrm{span}\{\ker \nu \cdot \ker \mu\} \cdot \nu(h) \cdot \mu$ with $f_i \in \ker \nu$ and $g_i \in \ker \mu$ is contained in $\mathrm{Lie}(\mathrm{CVF}_{\mathrm{hol}}(X))$, which concludes the proof. □

Proof of Theorem 1.2. (1) Let I_i and h_i be compatible ideals and functions of the pairs (ν_i, μ_i) and pick any non-trivial $f_i \in I_i \cdot \nu_i(h_i)$ for every i. Since the set of points $p \in X$ where the vectors $\mu_i[p]$ form a generating set is open and non-empty, there is some $q \in X$ where the vector fields $f_i(q)\mu_i[q]$ are a generating set for T_qX. By Lemma 2.12 the module generated by the vectors $f_i\mu_i$ is contained in the closure of $\mathrm{Lie}(\mathrm{CVF}_{\mathrm{hol}}(X))$ and thus by Proposition 2.5 the manifold X has the density property.

(2) Proceed as in (1): Let I_i be compatible ideals of the pairs (ν_i, μ_i) and pick any non-trivial $f_i \in I_i$ for every i. Since the set of points $p \in X$, where the elements $\nu_i[p] \wedge \mu_i[p]$ are an ω-generating set, is open and non-empty there is a $q \in X$ where the vector fields $f_i(q) \cdot (\nu_i[q] \wedge \mu_i[q])$ are an ω-generating set for $\mathrm{T}_qX \wedge \mathrm{T}_qX$. By Lemma 2.10 the module generated by the elements $f_i \cdot (\nu_i \wedge \mu_i)$ is contained in the closure of $\mathrm{Lie}(\mathrm{CVF}^{\omega}_{\mathrm{hol}}(X)) \wedge \mathrm{Lie}(\mathrm{CVF}^{\omega}_{\mathrm{hol}}(X))$. Thus by Proposition 2.7 the manifold X has the volume density property (the first condition of Proposition 2.7 on the cohomology group is trivially fulfilled since $\mathrm{H}^{n-1}(X, \mathbb{C}) = 0$ by assumption). □

We conclude this section with two remarks. These two remarks are just for general information and are not used later in this article.

Remark 2.13. Clearly Theorem 1.2(2) still holds if we have that every class of $\mathrm{H}^{n-1}(X, \mathbb{C})$ contains an element of $\Theta(\mathrm{Lie}(\mathrm{CVF}^{\omega}_{\mathrm{hol}}(X)))$ as in Proposition 2.7 instead of $\mathrm{H}^{n-1}(X, \mathbb{C}) = 0$. Also, note that this condition is equivalent to the condition that every class of $\mathrm{H}^{n-1}(X, \mathbb{C})$ contains an element of $\Theta(\mathrm{CVF}^{\omega}_{\mathrm{hol}}(X))$. Indeed, by Lemma 2.6 all Lie brackets represent the trivial class of $\mathrm{H}^{n-1}(X, \mathbb{C})$.

Remark 2.14. There is another class of compatible pairs. Sometimes a semi-compatible pair (ν, μ) is also called compatible if there exists a function $h \in \mathcal{O}_X(X)$ with $\nu(h) \in \ker \nu \setminus 0$ and $\mu(h) \in \ker \mu \setminus 0$. For this version the identity $[f\nu, gh\mu] - [fh\nu, g\mu] = fg(\nu(h)\mu - \mu(h)\nu)$ implies that there would be a version of Theorem 1.2 where we allow compatible pairs of this kind such that the vectors $\nu(h)\mu - \mu(h)\nu$ take part in constructing the generating sets.

3. Transitivity of the $\mathrm{Aut}_{\mathrm{hol}}(X)$- and $\mathrm{Aut}^{\omega}_{\mathrm{hol}}(X)$-action

In this section we develop the criteria (A) and (B) for transitivity mentioned in the introduction. We will work explicitly with vector fields. Let $n \geq k \geq 0$ and $a, b \in \mathbb{C}[z_0, \ldots, z_n]$ such that $\deg_{z_i}(a) \leq 2$ and $\deg_{z_i}(b) \leq 1$ for all $i \leq k$. Let $\bar{z} = (z_0, \ldots, z_n)$ and $X = \{x^2 y = a(\bar{z}) + x b(\bar{z})\}$ with the holomorphic volume form[2] $\omega = \mathrm{d}x/x^2 \wedge \mathrm{d}z_0 \wedge \ldots \wedge \mathrm{d}z_n$. Consider the following vector fields on X:

$$v_x^i = \left(\frac{\partial a}{\partial z_i} + x\frac{\partial b}{\partial z_i}\right)\frac{\partial}{\partial y} + x^2\frac{\partial}{\partial z_i} \quad \text{and} \quad v_y^j = \left(\frac{\partial a}{\partial z_j} + x\frac{\partial b}{\partial z_j}\right)\frac{\partial}{\partial x} + (2xy - b(\bar{z}))\frac{\partial}{\partial z_j}$$

for $0 \leq i \leq n$ and $0 \leq j \leq k$ and moreover, let

$$v_z = a(\bar{z})x\frac{\partial}{\partial x} - \left(2a(\bar{z})y - xyb(\bar{z}) + b(\bar{z})^2\right)\frac{\partial}{\partial y}.$$

Lemma 3.1. *$f(\bar{z})v_z \in \mathrm{CVF}_{\mathrm{hol}}(X)$, $f(x, z_0, ..\hat{z_i}.., z_n)v_x^i \in \mathrm{CVF}^{\omega}_{\mathrm{hol}}(X)$ and $f(y, z_0, ..\hat{z_j}.., z_n)v_y^j \in \mathrm{CVF}^{\omega}_{\mathrm{hol}}(X)$ for $f : \mathbb{C}^{n+1} \to \mathbb{C}$, $0 \leq i \leq n$ and $0 \leq j \leq k$.*

Proof. The vector fields v_x^i induce a translation in the z_i component. Then the equation for y can be solved by simple integration. Thus the fields v_x^i are complete. The coefficients of v_y^j are linear in x and z_j for all $0 \leq j \leq k$ so the flow equation is a linear differential equation, and thus has a global solution. The vector field v_z is complete since we may first solve the linear and uncoupled differential equation for x. Then the differential equation for y becomes linear and uncoupled as well, and thus we have a global solution. It is left to show that the vector fields v_x^i and v_y^j are volume preserving. A standard calculation shows that

$$\begin{aligned}
i_{v_x^i}\omega &= (-1)^{i+1}\mathrm{d}x \wedge \mathrm{d}z_0 \wedge ..\widehat{\mathrm{d}z_i}.. \wedge \mathrm{d}z_n,\\
i_{v_y^j}\omega &= \frac{1}{x^2}\left(\frac{\partial a}{\partial z_j} + x\frac{\partial b}{\partial z_j}\right)\mathrm{d}z_0 \wedge \ldots \wedge \mathrm{d}z_n + \\
&\quad \frac{(-1)^{j+1}}{x^2}(2xy - b(\bar{z}))\mathrm{d}x \wedge \mathrm{d}z_0 \wedge ..\widehat{\mathrm{d}z_j}.. \wedge \mathrm{d}z_n\\
&= (-1)^j\ \mathrm{d}\left(\frac{a(\bar{z}) + xb(\bar{z})}{x^2}\right) \wedge \mathrm{d}z_0 \wedge ..\widehat{\mathrm{d}z_j}.. \wedge \mathrm{d}z_n\\
&= (-1)^j\mathrm{d}y \wedge \mathrm{d}z_0 \wedge ..\widehat{\mathrm{d}z_j}.. \wedge \mathrm{d}z_n.
\end{aligned}$$

Thus $L_{v_x^i}\omega = \mathrm{d}i_{v_x^i}\omega = 0$ and $L_{v_y^j}\omega = \mathrm{d}i_{v_y^j}\omega = 0$ which shows that they are volume preserving.

Multiplying with a kernel element doesn't affect these properties. $\square$

Lemma 3.2. *The group $\mathrm{Aut}_{\mathrm{hol}}(X)$ acts transitively on $X \setminus \{x = 0\}$. Moreover, $\mathrm{Aut}^{\omega}_{\mathrm{hol}}(X)$ acts transitively on $X \setminus \{x = 0\}$ provided that condition* (B) *from the introduction holds.*

Proof. Use the flow of the vector fields v_x^i to see that any fiber $F_c = \{x = c \neq 0\}$ is contained in some orbit of $\mathrm{Aut}_{\mathrm{hol}}(X)$. For any $c \neq 0$, there is a point $p \in F_c$ such that the fields v_z are transversal to F_c at p. Thus the orbit that contains some F_c automatically contains a neighborhood $\{|x - c| < \varepsilon\}$. Therefore all orbits are open in $\{x \neq 0\}$ and thus all fibers F_c are contained in the same open orbit.

[2] It is a general fact that on every smooth hypersurface $\{P = 0\} \subset \mathbb{C}^N$ there is a natural volume form ω which is given on the sets $\left\{\frac{\partial P}{\partial x_i} \neq 0\right\}$ by $\omega = (-1)^i\left(\frac{\partial P}{\partial x_i}\right)^{-1}\mathrm{d}x_1 \wedge \ldots \wedge \widehat{\mathrm{d}x_i} \wedge \ldots \wedge \mathrm{d}x_N$.

This proves the claim. If condition (B) holds, then we may also use the vector fields v_y^j to connect the fibers. Indeed, for every $c \neq 0$ there is a point $p \in F_c$ and a vector field v_y^j such that $v_y^j[p]$ is transversal to the fiber. Thus in this case $\mathrm{Aut}^\omega_{\mathrm{hol}}(X)$ acts transitively on $X \setminus \{x = 0\}$. $\square$

Lemma 3.3. *If* (A) *from the introduction holds, then for every point* $p \in X \cap \{x = 0\}$*, there is some* $\phi \in \mathrm{Aut}^\omega_{\mathrm{hol}}(X)$ *such that* $\phi(p) \notin \{x = 0\}$*.*

Proof. By (A) we have that for any point $p = (0, y_0, \bar{q}) \in \{x = 0\} \cap X$, at least one of the polynomials $b, \frac{\partial a}{\partial z_0}, \ldots, \frac{\partial a}{\partial z_k}$ does not vanish at $\bar{q}$. For a non-vanishing $\frac{\partial a}{\partial z_j}$, the vector field v_y^j points outwards from $\{x = 0\}$ at the point p. Thus the flow of v_y^j moves p away from $\{x = 0\}$. If all polynomials $\frac{\partial a}{\partial z_0}, \ldots, \frac{\partial a}{\partial z_k}$ vanish at $\bar{q} = (q_0, \ldots, q_n)$, then b is non-vanishing at $\bar{q}$ and moreover, there is a $j \leq k$ such that $\frac{\partial a}{\partial z_j}$ does not vanish along the curve $\{z_i = q_i \text{ for all } i \neq j\} \subset \mathbb{C}^{n+1}$. This means that v_y^j is non-vanishing at p. Assume that the orbit of p under the action of the flow of v_y^j is contained in $\{x = 0\}$. Then the set $\{x = 0, y = y_0, z_i = q_i \text{ for all } i \neq j\} \subset \mathbb{C}^{n+3}$ would be contained in X and tangent to v_y^j, which is not the case. $\square$

These two lemmas combined give the following proposition using the conditions from the introduction.

Proposition 3.4. *Assume that* (A) *holds. Then* $\mathrm{Aut}_{\mathrm{hol}}(X)$ *acts transitively on* X*. Assume that additionally* (B) *holds. Then* $\mathrm{Aut}^\omega_{\mathrm{hol}}(X)$ *acts transitively on* X*.*

Corollary 3.5. *The volume preserving automorphisms act transitively on the Koras-Russell cubic* $C = \{x^2 y + x + z_0^2 + z_1^3 = 0\}$*.*

Proof. We have $\frac{\partial a}{\partial z_0} = -2z_0$ and $b(z_0, z_1) = -1$. Thus it is easy to see that (A) and (B) hold, and thus the vector fields v_x^0, v_x^1 and v_y^0 induce a transitive action on C. $\square$

Remark 3.6. The transitivity of the action by automorphisms a priori need not be achieved by the vector fields v_x^i, v_y^i, v_z only (which is equivalent to condition (A) from the introduction). There could be further automorphisms. For example, depending on a and b there could be an automorphism of the form $(x, y, \bar{z}) \mapsto (x, \gamma y, \lambda(\bar{z}))$ where λ is an automorphism of $\mathbb{C}^{n+1}$ with the property that $a(\lambda(\bar{z})) + xb(\lambda(\bar{z})) = \gamma(a(\bar{z}) + xb(\bar{z}))$ for some $\gamma \in \mathbb{C}^*$. A similar statement holds for transitivity by volume preserving automorphisms.

4. The Main Theorem for $n > 0$

The proof of the Main Theorem is done using compatible pairs and applying Theorem 1.2. Actually, only one compatible pair is needed as we will see later on. Let $n > 0$, $n \geq k \geq 0$ and $a, b \in \mathbb{C}[z_0, \ldots, z_n]$ such that $\deg_{z_i}(a) \leq 2$ and $\deg_{z_i}(b) \leq 1$ for all $i \leq k$. Moreover, assume that not both of $\deg_{z_0}(a)$ and $\deg_{z_0}(b)$ are equal to zero. Let $\bar{z} = (z_0, \ldots, z_n)$ and $X = \{x^2 y = a(\bar{z}) + xb(\bar{z})\}$ with the holomorphic volume form $\omega = \mathrm{d}x/x^2 \wedge \mathrm{d}z_0 \wedge \ldots \wedge \mathrm{d}z_n$.

Consider, again, the following vector fields on X:

$$v_x^i = \left(\frac{\partial a}{\partial z_i} + x \frac{\partial b}{\partial z_i} \right) \frac{\partial}{\partial y} + x^2 \frac{\partial}{\partial z_i} \quad \text{and} \quad v_y^j = \left(\frac{\partial a}{\partial z_j} + x \frac{\partial b}{\partial z_j} \right) \frac{\partial}{\partial x} + (2xy - b(\bar{z})) \frac{\partial}{\partial z_j}$$

for $0 \leq i \leq n$ and $0 \leq j \leq k$.

Lemma 4.1. *Let $0 \leq i \leq n$ and $0 \leq j \leq k$. Then (v_x^i, v_y^j) are compatible pairs for $i \neq j$.*

First we show that (v_x^i, v_y^j) are semi-compatible pairs. Indeed, the kernel of v_x^i contains the functions depending on $x, z_0, ..\hat{z}_i.., z_n$ and the kernel of v_y^j contains the functions depending on $y, z_0, ..\hat{z}_j.., z_n$; thus the closure of $\operatorname{span}\{\ker v_x^i \cdot \ker v_y^j\}$ is equal to $\mathcal{O}_X(X)$ and in particular contains an ideal.

For (v_x^i, v_y^j) being a compatible pair, we need a function $h \in \ker v_y^j$ such that $v_x^i(h) \in \ker v_x^i \setminus 0$. The function $h = z_i$ does the job.

Lemma 4.2. *For a generic point $p \in X$, the vector $v_y^0[p]$ is a generating set for T_pX and $v_x^n[p] \wedge v_y^0[p]$ is an ω-generating set for $\mathrm{T}_pX \wedge \mathrm{T}_pX$. Note that the first statement is also true for $n = 0$.*

Proof. Let $p = (x_0, y_0, \bar{q})$ where $x_0 \neq 0$ and $\bar{q} = (q_0, \ldots, q_n)$ such that $\frac{\partial a(\bar{q})}{\partial z_0} + x_0 \frac{\partial b(\bar{q})}{\partial z_0} \neq 0$.

For a complete vector field $\nu \in \mathrm{CVF}_{\mathrm{hol}}(X)$ and kernel element $f \in \ker \nu$ with $f(p) = 0$ we get an induced action (by the time-1 map of $f\nu$) on T_pX given by $v \mapsto v + v(f)\nu[p]$. Let $\nu_i = v_x^i$ and $f_i = x - x_0$ for $0 \leq i \leq n$. Thus the orbit of $v_y^0[p]$ under the $\mathrm{Aut}^\omega_{\mathrm{hol}}(X)_p$-action contains the vectors $v_y^0[p] + \left(\frac{\partial a(\bar{q})}{\partial z_0} + x_0 \frac{\partial b(\bar{q})}{\partial z_0}\right) v_x^i[p]$. Therefore the orbit contains $n + 2$ independent vectors and therefore a basis for T_pX.

Since $f \in \ker v_x^n$, we have that $v_x^n[p] \mapsto v_x^n[p]$ under the actions given by $(x - x_0)v_x^i$. So, in particular, similarly we have that the orbit of $v_x^n[p] \wedge v_y^0[p]$ contains a basis for $v_x^n[p] \wedge \mathrm{T}_pX$. Consider now the actions given by the vector fields $(z_n - q_n)v_x^i$ for $i \leq n - 1$. We get the actions $v_x^n[p] \mapsto v_x^n[p] + x_0^2 v_x^i[p]$, and thus we see that the orbit of the $\mathrm{Aut}^\omega_{\mathrm{hol}}(X)_p$-action contains a basis for $v_x^i[p] \wedge \mathrm{T}_pX$ for all $i \leq n$. Together they build then a basis for $\mathrm{T}_pX \wedge \mathrm{T}_pX$. ☐

Proof of the Main Theorem for $n > 0$. By Lemma 4.1 there exists a point $p \in X$ and compatible pairs (ν_i, μ_i) such that the vectors $\mu_i[p]$ are a generating set for T_pX and the elements $\nu_i[p] \wedge \mu_i[p]$ are an ω-generating set for $\mathrm{T}_pX \wedge \mathrm{T}_pX$; thus by Theorem 1.2 the claim is proven. Proposition 3.4 proves the "in particular" part. ☐

Remark 4.3. We never used that a and b are polynomials. In fact, the Main Theorem also holds if a and b are polynomial in z_0 and analytic in $z_1, \ldots, z_n$.

Remark 4.4. The condition that $\mathrm{H}^{n+1}(X, \mathbb{C}) = 0$ in the Main Theorem could be omitted in the case when every class of $\mathrm{H}^{n+1}(X, \mathbb{C})$ contains an element of $\Theta(\mathrm{Lie}(\mathrm{CVF}^\omega_{\mathrm{hol}}(X)))$ as in Proposition 2.7. However this cannot be achieved by the complete vector fields $f \cdot v_x^i$ and $g \cdot v_y^j$ since they are all mapped to the zero class by Θ (see the calculation in the proof of Lemma 3.1). So the existence of other complete volume preserving vector fields would be required.

5. The Main Theorem for $n = 0$

For $n = 0$ there is no compatible pair, but still the Main Theorem holds in this case. The proof is done by direct calculation. Let $a, b \in \mathbb{C}$, and let $X_{a,b} = \{x^2 y = z^2 - b + ax\}$. Note that $X_{a,b}$ is smooth if and only if a and b are not both equal to zero. Also, the conditions (A) and (B) from the introduction hold automatically if $X_{a,b}$ is smooth. Therefore the volume preserving automorphisms act transitively on $X_{a,b}$. The following proposition is mostly taken from [15], and among others it shows that $X_{a,b}$ is algebraically isomorphic to $X_{1,0}$, $X_{0,1}$ or $X_{1,1}$. Moreover, it shows that $X_{0,1}$ and $X_{1,1}$ are biholomorphic.

Proposition 5.1. (a) *There are isomorphisms*

(i) $X_{a,1} \cong_{\text{alg}} X_{a,b}$ *for* $a \in \mathbb{C}$ *and* $b \in \mathbb{C}^*$,
(ii) $X_{a,b} \cong_{\text{alg}} X_{1,b}$ *for* $a \in \mathbb{C}^*$ *and* $b \in \mathbb{C}$, *and*
(iii) $X_{0,1} \cong_{\text{hol}} X_{a,1}$ *for* $a \in \mathbb{C}$,

where $\cong_{\text{alg}}$ *means isomorphic as algebraic surfaces and* $\cong_{\text{hol}}$ *isomorphic as complex manifolds.*

(b) *Let* $X = \{x^2y = p(x,z)\}$ *be a smooth hypersurface with* $p \in \mathbb{C}[x,z]$ *and* $\deg_z p(0,z) \leq 2$; *then*

(i) *if* $\deg_z p(0,z) = 0$, *then* $X \cong_{\text{alg}} \mathbb{C}^* \times \mathbb{C}$,
(ii) *if* $\deg_z p(0,z) = 1$, *then* $X \cong_{\text{alg}} \mathbb{C}^2$,
(iii) *if* $p(0,z)$ *has a zero with multiplicity* 2, *then* $X \cong_{\text{alg}} X_{1,0}$,
(iv) *if* $p(0,z)$ *has two different zeros, then there is a unique* $a \in \{0,1\}$ *such that* $X \cong_{\text{alg}} X_{a,1}$.

Proof. Theorem 9 from [15] gives the isomorphisms in (a)(i). The biholomorphic map in (a)(ii) is given by

$$(x,y,z) \mapsto \left(x, e^{-ax}y + \frac{e^{-ax} + ax - 1}{x^2}, e^{-\frac{a}{2}x}z\right).$$

Theorem 5 from [15] states that X is algebraically isomorphic to $x^2y = s(z) + xt(z)$ for some $s, t \in \mathbb{C}[z]$ with $\deg t \leq \deg s - 2$. Following the given algorithm we see that $s(z) = p(0,z)$ which is then, after a linear change in the z-coordinate, given by (i) 1, (ii) z, (iii) z^2 or (iv) $z^2 - 1$. The isomorphisms in (b) are then easily found and the uniqueness in (b)(iv) follows from Theorem 9 from [15]. □

In spite of Proposition 5.1(a) we will start working on $X_{a,b}$ for general $a, b \in \mathbb{C}$. It turns out that this is more convenient for most arguments.

Remark 5.2. Every function $f \in \mathbb{C}[X_{a,b}]$ can be written uniquely as

$$f(x,y,z) = \sum_{i=1}^{\infty} x^i a_i(z) + \sum_{i=1}^{\infty} xy^i b_i(z) + \sum_{i=1}^{\infty} y^i c_i(z) + d(z);$$

indeed replace every x^2y by $z^2 - b + ax$. Alternatively f can be written uniquely as

$$f(x,y,z) = f_1(x,y) + zf_2(x,y);$$

indeed replace every z^2 by $x^2y + b - ax$.

We will use a tool in order to prove the (volume) density property: the algebraic (volume) density property, which was already mentioned in the introduction. Note that the algebraic (volume) density property; implies the (volume) density property; see [7].

Definition 5.3. Let X be an affine algebraic manifold. If the Lie algebra $\text{Lie}(\text{CVF}_{\text{alg}}(X))$ generated by complete algebraic vector fields $\text{CVF}_{\text{alg}}(X)$ on X is equal to the Lie algebra of all algebraic vector fields $\text{VF}_{\text{alg}}(X)$ on X, then X has the algebraic density property.

Let X be an affine algebraic manifold equipped with an algebraic volume form ω. If the Lie algebra $\mathrm{Lie}(\mathrm{CVF}^{\omega}_{\mathrm{alg}}(X))$ generated by complete volume preserving algebraic vector fields $\mathrm{CVF}^{\omega}_{\mathrm{alg}}(X)$ on X is equal to the Lie algebra of all volume preserving algebraic vector fields $\mathrm{VF}^{\omega}_{\mathrm{hol}}(X)$ on X, then X has the algebraic volume density property.

5.1. **The volume density property.** Let $a, b \in \mathbb{C}$ not both be equal to zero. For proving the volume density property for $X_{a,b} = \{x^2y = z^2 - b + ax\}$ with respect to $\omega = \mathrm{d}x/x^2 \wedge \mathrm{d}z$, we will need the following two vector fields:

$$v_x = 2z\frac{\partial}{\partial y} + x^2\frac{\partial}{\partial z}, \quad v_y = 2z\frac{\partial}{\partial x} + (2xy - a)\frac{\partial}{\partial z}.$$

Lemma 3.1 with Lemma 2.8 translates into

Lemma 5.4. $x^k v_x \in \mathrm{CVF}^{\omega}_{\mathrm{alg}}(X_{a,b})$ *and* $y^k v_y \in \mathrm{CVF}^{\omega}_{\mathrm{alg}}(X_{a,b})$ *for* $k \geq 0$.

Lemma 5.5. *Let* $v \in \mathrm{VF}^{\omega}_{\mathrm{alg}}(X_{a,b})$. *Then the 1-form* $i_v\omega$ *is exact and* $i_v\omega = \mathrm{d}f$ *defines a bijection between algebraic volume preserving vector fields and algebraic functions modulo constants.*

The functions corresponding to $x^k v_x$ *and* $y^k v_y$ *are given by the equations*

$$(k+1)i_{(x^k v_x)}\omega = -\mathrm{d}x^{k+1} \text{ and } (k+1)i_{(y^k v_k)}\omega = \mathrm{d}y^{k+1}.$$

Proof. The correspondence is given by the isomorphism Θ composed with the morphism D from Section 2 using the fact that $\mathrm{H}^1(X_{a,b}, \mathbb{C}) = 0$. The same correspondence was also used in [13]. The triviality of $\mathrm{H}^1(X_{a,b}, \mathbb{C})$ follows from the fact that $X_{a,b}$ is an affine modification of $\mathbb{C}^2$ along the divisor $2 \cdot \{x = 0\}$ with center at the ideal $(x^2, z^2 - b + ax)$ using the notation from [10]. If $b \neq 0$, then Proposition 3.1 from [10] shows that $X_{a,b}$ is simply connected (since $\mathbb{C}^2$ is simply connected). If $b = 0$, then Theorem 3.1 from [10] shows in a similar way that $\mathrm{H}^1(X_{a,0}, \mathbb{C}) = \mathrm{H}^1(\mathbb{C}^2, \mathbb{C})$, and thus is trivial. For the two identities we make the calculations

$$\begin{aligned}
(k+1)i_{(x^k v_x)}\omega &= (k+1)x^k i_{v_x}\omega = -(k+1)x^k \mathrm{d}x = -\mathrm{d}x^{k+1}, \\
(k+1)i_{(y^k v_y)}\omega &= (k+1)y^k i_{v_y}\omega = (k+1)y^k\left(\frac{2z}{x^2}\mathrm{d}z - \frac{2xy-a}{x^2}\mathrm{d}x\right) \\
&= (k+1)y^k \mathrm{d}\left(\frac{z^2 - b + ax}{x^2}\right) = (k+1)y^k \mathrm{d}y = \mathrm{d}y^{k+1}.
\end{aligned}$$

□

Recall that for a vector field ν and a algebraic function f we denote by $\nu(f)$ the algebraic function which is obtained by applying ν as a derivation.

Lemma 5.6. *Let* $v_1, v_2 \in \mathrm{VF}^{\omega}_{\mathrm{alg}}(X_{a,b})$ *and* $i_{v_1}\omega = \mathrm{d}f$; *then* $i_{[v_2,v_1]}\omega = \mathrm{d}v_2(f)$. *In particular, if* f *corresponds to a vector field in* $\mathrm{Lie}(\mathrm{CVF}^{\omega}_{\mathrm{alg}}(X_{a,b}))$, *then* $x^k v_x(f)$ *and* $y^k v_y(f)$ *correspond to a vector field in* $\mathrm{Lie}(\mathrm{CVF}^{\omega}_{\mathrm{alg}}(X_{a,b}))$.

Proof. The identity $i_{[v_2,v_1]}\omega = \mathrm{d}v_2(f)$ is shown in Lemma 3.2 in [13]. □

Lemma 5.7. *Let $i, j, k \geq 0$. Then*

$$\begin{aligned}
&(1) & v_x(y^{j+1}) &= 2(j+1)y^j z, \\
&(2) & v_x(y^{j+1}z^{k+1}) &= y^j z^k(2(j+1)z^2 + (k+1)(z^2 - b + ax)), \\
&(3) & y^j v_y(z^{k+1}) &= (k+1)y^j z^k(2xy - a), \\
&(4) & v_y(x^{i+1}) &= 2(i+1)x^i z, \\
&(5) & v_y(x^{i+1}z^{k+1}) &= x^i z^k \left(2(i+1)z^2 + (k+1)(2z^2 - 2b + ax)\right).
\end{aligned}$$

Proof. The lemma is proven by straightforward calculations. □

Proposition 5.8. *Smooth surfaces $X_{a,b} = \{x^2y = z^2 - b + ax\}$ have the algebraic volume density property with respect to $\omega = \mathrm{d}x/x^2 \wedge \mathrm{d}z$.*

Proof. Let L be the set of functions that corresponds to the Lie algebra of complete volume preserving vector fields. By Lemma 5.5 we already have $x^i \in L$ and $y^i \in L$ for $i \geq 0$. We need to show that all functions on X (modulo constants) are contained in L. By Remark 5.2 it is enough to show that (a) $x^i z^{k+1} \in L$, (b) $xy^{j+1}z^k \in L$ and (c) $y^{j+1}z^{k+1} \in L$ for all $i, j, k \geq 0$.

First we show (a) $x^i z^{k+1} \in L$*:* The statement (a) is also true for $k = -1$ by Lemma 5.5. Lemma 5.6 shows that $v_y(x^{i+1}) \in L$. Therefore by (4) we get $2(i+1)x^i z \in L$ and thus $x^i z \in L$ for $i \geq 0$ which is the statement for $k = 0$. Let us assume that the statement is true for $k-1$ and for k. Then, by Lemma 5.6 we have $v_y(x^{i+1}z^k) \in L$. By the induction assumption and (5) we have also $x^i z^{k+1} \in L$, which concludes the proof of (a) $x^i z^{k+1} \in L$ inductively for all $i, k \geq 0$.

The next step is to show (b) $xy^{j+1}z^k \in L$ *and* (c) $y^{j+1}z^{k+1}$ *for* $k = 0$*:* Note that (c) holds also for $k = -1$ by Lemma 5.5. By Lemma 5.6 we have $v_x(y^{j+2}) \in L$, and thus by (1) $y^{j+1}z \in L$, which proves statement (c) for $j \geq 0$ and $k = 0$. By the same lemma we have $y^j v_y(z) \in L$. Thus by (3) and (c) we have $xy^{j+1} \in L$ proving statement (b) for $j \geq 0$ and $k = 0$.

The last step is to show (b) $xy^{j+1}z^k \in L$ *and* (c) $y^{j+1}z^{k+1}$ *for arbitrary* k*:* Let us assume that (b) and (c) hold for k and, moreover, (c) holds for $k-1$. By Lemma 5.6 and the induction assumption we have $v_x(y^{j+1}z^{k+1}) \in L$ for all $j \geq 0$. Thus by the induction assumption and (2) we also have $y^j z^{k+2} \in L$ for all $j \geq 0$, which is statement (c) for $k+1$. Similarly, we have $y^j v_y(z^{k+2}) \in L$, and thus by (3) and the induction assumption we get $xy^{j+1}z^{k+1} \in L$ for all $j \geq 0$. This is statement (b) for $k+1$. Thus by induction over k the statements (b) and (c) are shown. □

Theorem 5.9. *Let $X = \{x^2y = p(x,z)\}$ with $p \in \mathbb{C}[x,z]$ and $\deg(p(0,z)) \leq 0$ be a smooth surface. Then X has the algebraic volume density property for the volume form $\mathrm{d}x/x^2 \wedge dz$.*

Proof. If $\deg(p(0,z)) \in \{1,2\}$, then by Proposition 5.1 the surface X is algebraically isomorphic to some surface $X_{a,b}$ or to $\mathbb{C}^2$. Since on these surfaces there are no non-constant invertible algebraic functions, two different algebraic volume forms differ only by multiplication with a constant. So the isomorphism induces a natural bijection between algebraic volume preserving vector fields on X and $X_{a,b}$ (resp. $\mathbb{C}^2$). Thus the algebraic volume density property is preserved under algebraic isomorphisms. Therefore Proposition 5.8 and the well-known fact that $\mathbb{C}^2$ has the algebraic volume density property conclude this case.

If $\deg(p(0,z)) = 0$, then by Proposition 5.1 the surface X is algebraically isomorphic to $\mathbb{C}^* \times \mathbb{C}$. For any algebraic volume form η on $\mathbb{C}^* \times \mathbb{C}$ there is an algebraic automorphism such that the pullback of η is equal to $\eta_0 = \mathrm{d}u/u \wedge \mathrm{d}v$. Indeed for an arbitrary $\eta = au^k \mathrm{d}u \wedge \mathrm{d}v$ with $a \in \mathbb{C}^*$ and $k \in \mathbb{Z}$, the pullback of η by $(u,v) \mapsto (u, a^{-1}u^{-k-1}v)$ is η_0. Apply Theorem 1 of [9] to the semi-compatible pair $(u \cdot \partial/\partial u, \partial/\partial v)$ to see that $\mathbb{C}^* \times \mathbb{C}$ with η_0 has the algebraic volume density property and thus $\mathbb{C}^* \times \mathbb{C}$ has the algebraic volume density property for all algebraic volume forms. □

5.2. **The density property.** We will show the density property for the surfaces $X_{1,b} = \{x^2y = z^2 - b + x\}$ with $b \in \mathbb{C}$. We will use the following vector fields:

$$v_x = 2z\frac{\partial}{\partial y} + x^2\frac{\partial}{\partial z}, \quad v_y = 2z\frac{\partial}{\partial x} + (2xy-1)\frac{\partial}{\partial z},$$
$$v_z = z^2x\frac{\partial}{\partial x} - \left(2(z^2-b)y - xy + 1\right)\frac{\partial}{\partial y}.$$

Definition 5.10. For an algebraic vector field ν, the ω-divergence $\mathrm{div}_\omega \nu$ is the algebraic function given by $\mathrm{d}i_\nu\omega = (\mathrm{div}_\omega \nu) \cdot \omega$.

Lemma 5.11. *We have* $x^k v_x, y^k v_y \in \mathrm{CVF}^\omega_{\mathrm{alg}}(X_{1,b})$ *and* $zx^k v_x, z^k v_z \in \mathrm{CVF}_{\mathrm{alg}}(X_{1,b})$ *for* $k \geq 0$. *Moreover,* $\mathrm{div}_\omega zx^k v_x = x^{k+2}$ *and* $\mathrm{div}_\omega z^k v_z = -z^{k+2}$.

Proof. Lemma 2.8 and Lemma 5.4 give $x^k v_x, y^k v_y \in \mathrm{CVF}^\omega_{\mathrm{alg}}(X_0)$ and $zx^k v_x \in \mathrm{CVF}_{\mathrm{alg}}(X_{1,b})$. For the statement about the divergence, we make the calculations

$$\mathrm{d}(i_{zx^k v_x}\omega) = -\mathrm{d}(zx^k \mathrm{d}x) = x^k \mathrm{d}x \wedge \mathrm{d}z = x^{k+2}\omega,$$
$$\mathrm{d}(i_{z^k v_z}\omega) = \mathrm{d}\left(\frac{z^{k+2}}{x}\mathrm{d}z\right) = \frac{-z^{k+2}}{x^2}\mathrm{d}x \wedge \mathrm{d}z = -z^{k+2}\omega,$$

and thus prove the statement. □

The following lemma is well known.

Lemma 5.12. *For* $v_1, v_2 \in \mathrm{VF}_{\mathrm{alg}}(X_{1,b})$*, we have*

$$\mathrm{div}_\omega[v_1, v_2] = v_1(\mathrm{div}_\omega v_2) - v_2(\mathrm{div}_\omega v_1).$$

In particular we have

$$\mathrm{div}_\omega[x^k v_x, v_2] = x^k v_x(\mathrm{div}_\omega v_2) \quad \textit{and} \quad \mathrm{div}_\omega[y^k v_y, v_2] = y^k v_y(\mathrm{div}_\omega v_2)$$

for any $k \geq 0$. *Moreover, for* $f \in \mathbb{C}[X_{1,b}]$,

$$\mathrm{div}_\omega fv = f\mathrm{div}_\omega v + v(f).$$

Lemma 5.13. *Let* $f \in \mathbb{C}[X_{1,b}]$. *Then* $fv_y \in \mathrm{Lie}(\mathrm{CVF}_{\mathrm{alg}}(X_{1,b}))$.

Proof. Let $E \subset \mathbb{C}[X_{1,b}]$ be given by $E = \{\mathrm{div}_\omega \nu : \nu \in \mathrm{Lie}(\mathrm{CVF}_{\mathrm{alg}}(X_{1,b}))\}$. It is enough to show that for every $f \in \mathbb{C}[X_{1,b}]$, we have $\mathrm{div}_\omega(fv_y) \in E$. Indeed, then $fv_y - \nu \in \mathrm{VF}^\omega_{\mathrm{alg}}(X_{1,b})$ for some $\nu \in \mathrm{Lie}(\mathrm{CVF}_{\mathrm{alg}}(X_{1,b}))$. Thus by Proposition 5.8 we have $fv_y \in \mathrm{Lie}(\mathrm{CVF}_{\mathrm{alg}}(X_{1,b}))$.

By Lemma 5.11 we have $x^{i+2} \in E$ and $z^{i+2} \in E$ for all $i \geq 0$. Thus by Lemma 5.12 we get $v_y(x^{i+2}) = 2(i+2)x^{i+1}z \in E$, and therefore $v_y(xz) = 2z^2 + 2x^2y - x = 4z^2 - 2b + x \in E$. Since $z^2 \in E$ we get $x - 2b \in E$, and thus $v_y(x - 2b) = 2z \in E$. Altogether we have $x^iz \in E$, $x^{i+2} \in E$ and $x - 2b \in E$ for all $i \geq 0$.

Let $f = x^iy^jz^k$ for $i, j \geq 0$ and $k \in \{0, 1\}$. If $f \neq xy^j$, then we have

$$\mathrm{div}_\omega(fv_y) = y^jv_y(x^iz^k) \in E$$

by Lemma 5.12, because $x^i z^k \in E$ by the above. If $f = xy^j$, then

$$\mathrm{div}_\omega(fv_y) = y^j v_y(x) = y^j v_y(x - 2b) \in E$$

by the same arguments. Thus the lemma is proven since any algebraic function is a sum of such monomials by Remark 5.2. □

Proposition 5.14. *The surface $X_{1,b}$ has the density property.*

Proof. By Lemma 4.2 the tangent vectors of v_y are a generating set for the tangent space $\mathrm{T}_q X_{1,b}$ at a generic point $q \in X_{1,b}$. By Lemma 5.13 the $\mathbb{C}[X_{1,b}]$-submodule generated by v_y is contained in $\mathrm{Lie}(\mathrm{CVF}_{\mathrm{alg}}(X_{1,b}))$. Thus the $\mathcal{O}_{X_{1,b}}(X_{1,b})$-submodule generated by v_y is contained in the closure of $\mathrm{Lie}(\mathrm{CVF}_{\mathrm{alg}}(X_{1,b}))$. Therefore, by Proposition 2.5 the surface $X_{1,b}$ has the density property. □

Theorem 5.15. *Let $X = \{x^2 y = p(x,z)\}$ with $p \in \mathbb{C}[x,z]$ and $\deg(p(0,z)) \leq 2$ be a smooth surface. Then X has the density property.*

Proof. By Proposition 5.1 the surface X is biholomorphic to $\mathbb{C}^* \times \mathbb{C}$, $\mathbb{C}^2$, $X_{1,0}$ or $X_{1,1}$. It is well known that the first two have the density property. The two other surfaces have the density property by Proposition 5.14. □

References

[1] Erik Andersén and László Lempert, *On the group of holomorphic automorphisms of* $\mathbf{C}^n$, Invent. Math. **110** (1992), no. 2, 371–388, DOI 10.1007/BF01231337. MR1185588 (93i:32038)

[2] A. Dubouloz, L. Moser-Jauslin, and P.-M. Poloni, *Inequivalent embeddings of the Koras-Russell cubic 3-fold*, Michigan Math. J. **59** (2010), no. 3, 679–694, DOI 10.1307/mmj/1291213961. MR2745757 (2011m:14103)

[3] Franc Forstnerič, *Stein manifolds and holomorphic mappings: The homotopy principle in complex analysis*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 56, Springer, Heidelberg, 2011. MR2975791

[4] Franc Forstnerič and Jean-Pierre Rosay, *Approximation of biholomorphic mappings by automorphisms of* $\mathbf{C}^n$, Invent. Math. **112** (1993), no. 2, 323–349, DOI 10.1007/BF01232438. MR1213106 (94f:32032)

[5] Lars Hörmander, *An introduction to complex analysis in several variables*, 3rd ed., North-Holland Mathematical Library, vol. 7, North-Holland Publishing Co., Amsterdam, 1990. MR1045639 (91a:32001)

[6] Shulim Kaliman and Frank Kutzschebauch, *Criteria for the density property of complex manifolds*, Invent. Math. **172** (2008), no. 1, 71–87, DOI 10.1007/s00222-007-0094-6. MR2385667 (2008k:32066)

[7] Shulim Kaliman and Frank Kutzschebauch, *Algebraic volume density property of affine algebraic manifolds*, Invent. Math. **181** (2010), no. 3, 605–647, DOI 10.1007/s00222-010-0255-x. MR2660454 (2011k:32031)

[8] Shulim Kaliman and Frank Kutzschebauch, *On the present state of the Andersén-Lempert theory*, Affine algebraic geometry, CRM Proc. Lecture Notes, vol. 54, Amer. Math. Soc., Providence, RI, 2011, pp. 85–122. MR2768636

[9] S. Kaliman and F. Kutzschebauch, *On algebraic volume density property*, ArXiv e-prints, Jan. 2012. arXiv1201.4769, to appear in Transfom. Groups.

[10] Sh. Kaliman and M. Zaidenberg, *Affine modifications and affine hypersurfaces with a very transitive automorphism group*, Transform. Groups **4** (1999), no. 1, 53–95, DOI 10.1007/BF01236662. MR1669174 (2000f:14099)

[11] Frank Kutzschebauch, *Flexibility properties in complex analysis and affine algebraic geometry*, Automorphisms in birational and affine geometry, Springer Proc. Math. Stat., vol. 79, Springer, Cham, 2014, pp. 387–405, DOI 10.1007/978-3-319-05681-4_22. MR3229363

[12] F. Kutzschebauch and S. Kaliman, *On Density and Volume Density Property*, *Complex Analysis and Geometry*, 2014, Springer Proceedings in Mathematics & Statistics, KSCV10, Gyeongju, Korea.

[13] F. Kutzschebauch and M. Leuenberger, *Lie algebra generated by locally nilpotent derivations on Danielewski surfaces*, ArXiv e-prints, Nov. 2013, arXiv1311.1075, to appear in Ann. Sc. Norm. Super. Pisa Cl. Sci. (5).

[14] L. Makar-Limanov, *On the hypersurface $x + x^2y + z^2 + t^3 = 0$ in $\mathbf{C}^4$ or a $\mathbf{C}^3$-like threefold which is not $\mathbf{C}^3$*. part B, Israel J. Math. **96** (1996), no. part B, 419–429, DOI 10.1007/BF02937314. MR1433698 (98a:14052)

[15] P.-M. Poloni, *Classification(s) of Danielewski hypersurfaces*, Transform. Groups **16** (2011), no. 2, 579–597, DOI 10.1007/s00031-011-9146-5. MR2806502 (2012e:14115)

[16] Árpád Tóth and Dror Varolin, *Holomorphic diffeomorphisms of semisimple homogeneous spaces*, Compos. Math. **142** (2006), no. 5, 1308–1326, DOI 10.1112/S0010437X06002077. MR2264667 (2007i:32022)

[17] Dror Varolin, *The density property for complex manifolds and geometric structures*, J. Geom. Anal. **11** (2001), no. 1, 135–160, DOI 10.1007/BF02921959. MR1829353 (2002g:32026)

Mathematisches Institut, Universität Bern, Sidlerstrasse 5, CH-3012 Bern, Switzerland

E-mail address: `matthias.leuenberger@bluewin.ch`