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THE LENGTH OF THE SHORTEST CLOSED GEODESIC IN A CLOSED RIEMANNIAN 3-MANIFOLD WITH NONNEGATIVE RICCI CURVATURE

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Abstract. In this note we discuss the problem of finding an upper bound on the length of the shortest closed geodesic in a closed Riemannian 3-manifold in terms of the volume. More precisely, we show that there exists a positive universal constant C such that, for every Riemannian 3-manifold (M^3, g) with $Ric_g \geq 0$, at least one of the following assertions holds: (i). $Sys_g(M) \leq CVol_g(M)^{\frac{1}{3}}$, where $Sys_g(M)$ denotes the length of the shortest closed geodesic in M^3; (ii). M^3 is diffeomorphic to $\mathbb{S}^3$ and there exists a closed minimal surface Σ_0 embedded in M^3, with index 1, and $A_g(\Sigma_0) \leq CVol_g(M)^{\frac{2}{3}}$. This gives a partial answer to the problem proposed in Gromov's paper written in 1983.

1. Introduction

Given an n-dimensional closed Riemannian manifold (M^n, g), it is an important problem to estimate the upper bound of the length of the shortest closed geodesic in (M^n, g) in terms of the volume. In his seminal paper [6], Gromov proved that there exists an uniform constant $C_1(n)$ depending only on n such that for every essential manifold M^n, we have

$$Sys_{1_g}(M) \leq C_1(n)(Vol_g(M))^{\frac{1}{n}}, \tag{1.1}$$

where $Sys_{1_g}(M)$ is the least length of the noncontractible loop γ in M and the constant $C_1(n)$ can be taken as $6(n+1)n\sqrt[n]{(n+1)!}$. Here the manifold M is said to be essential if there is an aspherical space L and a map $f : M \to L$ such that f represents a nonzero homology class in $H_n(L, G)$, with $G = \mathbb{Z}$ if L is orientable and $G = \mathbb{Z}_2$ if L is nonorientable.

Inequality (1.1) is often called the systolic inequality. Denote the length of the shortest nontrivial geodesic on M by $Sys_g(M)$. In any dimension, the shortest loop in every nontrivial homotopy class is a closed geodesic; then clearly we have $Sys_g(M) \leq Sys_{1_g}(M)$. In general, there is the following question (see [6, p. 135] and [4, p. 115]).

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Question 1.1. For an n-dimensional closed manifold M^n, is there a constant $C(M)$ such that for every Riemannian metric g on M, we have

$$Sys_g(M) \leq C(M)(Vol_g(M))^{\frac{1}{n}}? \tag{1.2}$$

Moreover, the stronger question is whether the constant $C(M) = C(n)$ depends only on the dimension n.

This question is still open for many manifolds, e.g., for $M = \mathbb{S}^n, n \geq 3$. But it has a positive answer for all essential manifolds (see [6]) and for all closed surfaces (see [2, 3, 6, 8, 14]), with the universal constants $C(n)$ depending only on n. In particular, Loewner showed in 1949 (cf. [14]) that for every Riemannian metric on a torus $\mathbb{T}^2$,

$$Sys_{1_g}(\mathbb{T}^2) \leq 2^{\frac{1}{2}}3^{-\frac{1}{4}}(A_g(\mathbb{T}^2))^{\frac{1}{2}}, \tag{1.3}$$

with equality holding if and only if it is a flat equilateral torus. Gromov [6] proved that every closed surface (M^2, g) which is not a 2-sphere satisfies

$$Sys_{1_g}(M) \leq \sqrt{\frac{\pi}{2}}(A_g(M))^{\frac{1}{2}}; \tag{1.4}$$

while for the case of the 2-sphere, Croke [3] proved that every Riemannian metric g on the 2-sphere $\mathbb{S}^2$ satisfies the estimate

$$Sys_g(\mathbb{S}^2) \leq 31(A_g(\mathbb{S}^2))^{\frac{1}{2}}. \tag{1.5}$$

The inequality (1.5) is not sharp and has been improved in [12], [15] and [16]. In particular, the best known estimate was obtained by Rotman in [15],

$$Sys_g(\mathbb{S}^2) \leq 4\sqrt{2}(A_g(\mathbb{S}^2))^{\frac{1}{2}}, \tag{1.6}$$

which is still not sharp, and it was suggested by Calabi and Croke that the sharp estimate should be $Sys_g(\mathbb{S}^2) \leq (12)^{\frac{1}{4}}(A_g(\mathbb{S}^2))^{\frac{1}{2}}$. If one considers the convex hypersurfaces in $\mathbb{R}^{n+1}$, then Question 1.1 also has a positive answer which was shown independently by Treibergs [19] and Croke [3]. We refer the readers to the survey [4] for more information on this problem.

In a recent paper, Sabourau [17] considered a generalization of the above results about geodesics on closed surface in terms of minimal hypersurfaces and proved the following result.

Theorem 1.1 ([17]). *Let (M^n, g) be a closed Riemannian n-manifold with nonnegative Ricci curvature. There exists an embedded closed minimal hypersurface Σ in M with a singular set of Hausdorff dimension at most $n-8$ such that $A_g(\Sigma) \leq C_2(n)Vol_g(M)^{\frac{n-1}{n}}$, where $C_2(n)$ is an explicit positive constant depending only on n.*

Inspired by Sabourau's theorem, in this paper we consider Question 1.1 for a three-dimensional closed manifold with nonnegative Ricci curvature. Our main result is the following.

Theorem 1.2. *There exists a positive universal constant C such that, for every closed Riemannian 3-manifold (M^3, g) with $Ric_g \geq 0$, at least one of the following two assertions holds:*

(1). *$Sys_g(M) \leq CVol_g(M)^{\frac{1}{3}}$, where $Sys_g(M)$ denotes the length of the shortest closed geodesic in M^3.*

(2). *M^3 is diffeomorphic to $\mathbb{S}^3$ and there exists a closed minimal surface Σ_0 embedded in M^3, with index 1, and $A_g(\Sigma_0) \leq CVol_g(M)^{\frac{2}{3}}$.*

As a direct consequence, we obtain the following result.

Corollary 1.1. *There exists a universal positive constant C such that, for every closed Riemannian 3-manifold (M^3, g) with $Ric_g \geq 0$ and either (i). $\pi_1(M) \neq 0$; or (ii). $\pi_1(M) = 0$ and there is no index 1 compact minimal surface in (M^3, g), we have*

$$Sys_g(M) \leq CVol_g(M)^{\frac{1}{3}}, \tag{1.7}$$

where $Sys_g(M)$ denotes the length of the shortest closed geodesic in M^3.

Remark 1.1. There exist manifolds with nonnegative Ricci curvature and no closed minimal surface with index 1. For instance, there is no orientable closed minimal surface in $\mathbb{S}^2 \times \mathbb{S}^1(r)$ with index 1, when $r > 1$. For more about this, see Theorem 4 in [20].

Remark 1.2. After our paper was submitted awaiting a decision, a preprint by Liokumovich-Zhou [9] appeared on the arXiv. They proved that if a closed 3-manifold (M^3, g) has positive Ricci curvature and is not homeomorphic to a 3-sphere, then M contains a noncontractible closed geodesic of length at most $CVol_g(M)^{1/3}$, i.e.,

$$Sys_{1_g}(M) \leq CVol_g(M)^{\frac{1}{3}}. \tag{1.8}$$

The inequality (1.8) is slightly stronger than (1.7), as we usually have $Sys_g(M) \leq Sys_{1_g}(M)$. However, they need to impose a stronger assumption "$Ric_g > 0$". The proof of (1.8) in [9] is more involved by constructing a sweepout by 1-cycles of length at most $CVol_g(M)^{1/3}$.

2. Proof of Theorem 1.2

Proof. Given a closed Riemannian manifold (M^3, g) with $Ric_g \geq 0$, from Hamilton's results [7] we obtain that the universal cover of M^3 is either $\mathbb{S}^3$ or $\mathbb{R}^3$ or $\mathbb{S}^2 \times \mathbb{R}$. We consider the three cases separately.

If the universal cover of M^3 is $\mathbb{R}^3$, then M^3 is essential. Then, our theorem follows from Gromov's result (1.1),

$$Sys_g(M) \leq Sys_{1_g}(M) \leq C_1(3)Vol_g(M)^{\frac{1}{3}}.$$

If the universal cover of M^3 is $\mathbb{S}^2 \times \mathbb{R}$, then M^3 is either diffeomorphic to $\mathbb{P}^3 \sharp \mathbb{P}^3$, $\mathbb{P}^2 \times \mathbb{S}^1$, $\mathbb{S}^2 \times \mathbb{S}^1$ or $\mathbb{S}^2 \rtimes \mathbb{S}^1$. Here, $\mathbb{S}^2 \rtimes \mathbb{S}^1$ denotes the nonorientable $\mathbb{S}^2$-bundle over $\mathbb{S}^1$, or equivalently the mapping torus of the antipodal map on $\mathbb{S}^2$. Since $\mathbb{P}^3$ and $\mathbb{P}^2$ are essential, we get that $\mathbb{P}^3 \sharp \mathbb{P}^3$ and $\mathbb{P}^2 \times \mathbb{S}^1$ are essential, respectively. In these cases, we again use Gromov's result. Hence, we consider the case where M^3 is diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$. We can see that $\mathbb{S}^2 \times \mathbb{S}^1$ is not essential. To justify this, note that the map $f : M \to L$ factors through the classifying map $M \to K(\pi_1(M), 1)$ and, for $M = \mathbb{S}^2 \times \mathbb{S}^1$, $K(\pi_1(M), 1) = K(\mathbb{Z}, 1) = \mathbb{Z}$. Hence, $f_*[M] = 0$ since $H^3(\mathbb{S}^1; \mathbb{Z})$ is trivial. Then, in this case, we cannot use Gromov's result. However, since the Ricci curvature is nonnegative, from the results in [11, §8], we can assume that (M^3, g) is $\mathbb{S}^2 \times \mathbb{S}^1$ with a product metric, i.e., the metric g is a product metric on $\mathbb{S}^2 \times \mathbb{S}^1$. The strategy here is to employ a kind of systole inequality for surfaces, proved by Loewner and Rotman. We consider the following two cases separately.

If $(A_g(\mathbb{S}^2))^{\frac{1}{2}} \le \lambda L_g(\mathbb{S}^1)$, where $\lambda > 0$ is a positive constant to be determined later and $A_g(\mathbb{S}^2), L_g(\mathbb{S}^1)$ denote the area of the surface $\mathbb{S}^2$ and length of $\mathbb{S}^1$ with respect to the induced metrics from g respectively, let γ denote the shortest closed geodesic in $\mathbb{S}^2$. By Rotman's result (1.6), we have

$$L_g(\gamma) \le 4\sqrt{2}(A_g(\mathbb{S}^2))^{\frac{1}{2}}, \tag{2.1}$$

where $L_g(\gamma)$ denotes the length of γ. Using $(A_g(\mathbb{S}^2))^{\frac{1}{2}} \le \lambda L_g(\mathbb{S}^1)$, we have

$$\begin{aligned} L_g(\gamma) \le& 4\sqrt{2}(A_g(\mathbb{S}^2))^{\frac{1}{2}} = 4\sqrt{2}(A_g(\mathbb{S}^2))^{\frac{1}{3}}(A_g(\mathbb{S}^2))^{\frac{1}{6}} \\ \le& 4\sqrt{2}\lambda^{\frac{1}{3}}(A_g(\mathbb{S}^2))^{\frac{1}{3}}(L_g(\mathbb{S}^1))^{\frac{1}{3}} \\ =& 4\sqrt{2}\lambda^{\frac{1}{3}}(Vol_g(\mathbb{S}^2 \times \mathbb{S}^2))^{\frac{1}{3}}. \end{aligned}$$

If $(A_g(\mathbb{S}^2))^{\frac{1}{2}} \ge \lambda L_g(\mathbb{S}^1)$, we consider the torus $T^2 = \gamma \times \mathbb{S}^1$ embedded in $\mathbb{S}^2 \times \mathbb{S}^1$. From (1.3) we can find a closed geodesic β in T^2 such that

$$L_g(\beta) \le 2^{\frac{1}{2}}3^{-\frac{1}{4}}(A_g(T^2))^{\frac{1}{2}}, \tag{2.2}$$

where $L_g(\beta)$ denotes the length of β in T^2 with the induced metric. Note that the surface T^2 is totally geodesic in $\mathbb{S}^2 \times \mathbb{S}^1$, so β is also a geodesic in $\mathbb{S}^2 \times \mathbb{S}^1$ and has the same length $L_g(\beta)$. Then,

$$\begin{aligned} L_g(\beta) &\le 2^{\frac{1}{2}}3^{-\frac{1}{4}}(A_g(T^2))^{\frac{1}{2}} = 2^{\frac{1}{2}}3^{-\frac{1}{4}}\left(L_g(\gamma)L_g(\mathbb{S}^1)\right)^{\frac{1}{2}} \\ &\le 2^{\frac{7}{4}}3^{-\frac{1}{4}}(A_g(\mathbb{S}^2))^{\frac{1}{4}}(L_g(\mathbb{S}^1))^{\frac{1}{2}} \\ &= 2^{\frac{7}{4}}3^{-\frac{1}{4}}(A_g(\mathbb{S}^2))^{\frac{1}{4}}(L_g(\mathbb{S}^1))^{\frac{1}{3}}(L_g(\mathbb{S}^1))^{\frac{1}{6}} \\ &\le 2^{\frac{7}{4}}3^{-\frac{1}{4}}\lambda^{-\frac{1}{6}}(A_g(\mathbb{S}^2))^{\frac{1}{4}}(L_g(\mathbb{S}^1))^{\frac{1}{3}}(A_g(\mathbb{S}^2))^{\frac{1}{12}} \\ &= 2^{\frac{7}{4}}3^{-\frac{1}{4}}\lambda^{-\frac{1}{6}}(Vol_g(\mathbb{S}^2 \times \mathbb{S}^1))^{\frac{1}{3}}, \end{aligned}$$

where in the second inequality we used (2.1) and in the last inequality we used $L_g(\mathbb{S}^1) \le \lambda^{-1}(A_g(\mathbb{S}^2))^{\frac{1}{2}}$. Combining the above two cases, we obtain that if M^3 is diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$, then there exists a geodesic γ such that

$$L_g(\gamma) \le C(Vol_g(\mathbb{S}^2 \times \mathbb{S}^1))^{\frac{1}{3}}, \tag{2.3}$$

where $C = \max\{4\sqrt{2}\lambda^{\frac{1}{3}}, 2^{\frac{7}{4}}3^{-\frac{1}{4}}\lambda^{-\frac{1}{6}}\}$. Clearly, we can choose $\lambda = 2^{-\frac{3}{2}}3^{-\frac{1}{2}}$ to obtain the optimal $C = 2/3^{\frac{1}{6}}$. The nonorientable case $\mathbb{S}^2 \rtimes \mathbb{S}^1$ now follows by passing to the oriented double covering $\mathbb{S}^2 \times \mathbb{S}^1$.

Finally we assume that the universal cover of M^3 is $\mathbb{S}^3$. In this case, if M^3 is not simply connected, then, as it is well known, M^3 is essential and we can again use Gromov's inequality (1.1). Then we assume that $M^3 = \mathbb{S}^3$. From Sabourau's result [17], there exists a closed embedded minimal surface Σ in $\mathbb{S}^3$ such that

$$A_g(\Sigma) \le C_2(3)(Vol_g(\mathbb{S}^3))^{\frac{2}{3}}.$$

Note that Σ is orientable, since $M^3 = \mathbb{S}^3$ and Σ is an embedded closed minimal surface. Consider the set $\mathcal{E}$ of all closed minimal surfaces in $\mathbb{S}^3$. Let Σ_0 be the element in $\mathcal{E}$ with the least area. By Sabourau's inequality,

$$A_g(\Sigma_0) \le C_2(3)(Vol_g(\mathbb{S}^3))^{\frac{2}{3}}.$$

If Σ_0 is stable, then from the second variation of the area functional, we have

$$\int_{\Sigma_0}(Ric_g(\nu,\nu) + |A_{\Sigma_0}|^2)d\mu_{\Sigma_0} \le 0, \tag{2.4}$$

where ν is a unit normal vector on Σ_0 and $|A_{\Sigma_0}|^2$ is the squared norm of the second fundamental form of Σ_0. Hence, from the inequality above and $Ric_g \geq 0$, Σ_0 is totally geodesic: $|A_{\Sigma_0}|^2 = 0$. Also, it follows from the proof of [18, Theorem 5.1] that Σ_0 is a sphere or a torus, since the scalar curvature $R_g \geq 0$ and Σ_0 is an orientable closed stable minimal surface. In fact, since Σ_0 is an orientable closed stable minimal surface in M^3, it follows from (2.4) and the Gauss equation that

$$\begin{aligned} 0 \ \leq \ \int_{\Sigma_0} \frac{R_g}{2} d\mu_{\Sigma_0} &= \int_{\Sigma_0} (Ric_g(\nu,\nu) + \frac{1}{2}|A_{\Sigma_0}|^2 + K) d\mu_{\Sigma_0} \\ &\leq \int_{\Sigma_0} (Ric_g(\nu,\nu) + |A_{\Sigma_0}|^2) d\mu_{\Sigma_0} + 2\pi(1 - g(\Sigma_0)) \\ &\leq 2\pi(1 - g(\Sigma_0)), \end{aligned}$$

where $K, g(\Sigma_0)$ are the Gauss curvature and genus of Σ_0 respectively. Then the genus of Σ_0 must be 0 or 1, and therefore Σ_0 is a sphere or a torus. Applying Rotman's inequality (1.6) if Σ_0 is a sphere; or Loewner's inequality (1.3) if it is a torus, we obtain a closed geodesic γ in $\mathbb{S}^3$ such that

$$L_g(\gamma) \leq C Vol_g(\mathbb{S}^3)^{\frac{1}{3}}, \tag{2.5}$$

where $C = \max\{4\sqrt{2}\sqrt{C_2(3)}, 2^{\frac{1}{2}} 3^{-\frac{1}{4}} \sqrt{C_2(3)}\}$. If there exists Σ in $\mathcal{E}$ such that $\Sigma \cap \Sigma_0 = \emptyset$, then by [5, Lemma 3], we obtain again that Σ_0 is totally geodesic. Then we can again apply Rotman's inequality (1.6) and Gromov's inequality (1.1) to find a closed geodesic γ in $\mathbb{S}^3$ such that

$$L_g(\gamma) \leq C Vol_g(\mathbb{S}^3)^{\frac{1}{3}}, \tag{2.6}$$

where $C = \max\{4\sqrt{2}\sqrt{C_2(3)}, \sqrt{\frac{\pi}{2} C_2(3)}\}$. Now, assume that Σ_0 is not stable and every element Σ in $\mathcal{E}$ satisfies $\Sigma \cap \Sigma_0 \neq \emptyset$. Hence, as was done by Marques-Neves in [10], we obtain that Σ_0 has index 1. For the convenience of the readers we present a proof of this. First, we get that $\mathbb{S}^3 \setminus \Sigma_0$ is the disjoint union of two handlebodies: N_1 and N_2. Since Σ_0 is not stable, we can take a positive eigenfunction $\varphi \in C^\infty(\Sigma_0)$ for the lowest eigenvalue $\lambda < 0$ of the Jacobi operator L_{Σ_0}, and consider a vector field X on $\mathbb{S}^3$ such that $X = \varphi\nu$ on Σ_0, where ν is a unit normal vector pointing into N_1. Let $(F_t)_{t\in\mathbb{R}}$ denote the flow generated by the vector field X. Hence, we obtain

$$\frac{\partial}{\partial t} \langle H(\Sigma_t), \nu_t \rangle \big|_{t=0} = L_{\Sigma_0}\varphi = -\lambda\varphi > 0.$$

Thus, there exists $\varepsilon > 0$ small enough such that:

(i) $\Sigma_t = F_t(\Sigma_0)$ is contained in N_1 (in N_2) for all $0 < t < \varepsilon$ (for all $-\varepsilon < t < 0$);
(ii) the mean curvature vector $H(\Sigma_t)$ points into N_1 (into N_2) for all $0 < t < \varepsilon$ (for all $-\varepsilon < t < 0$);
(iii) $A_g(\Sigma_t) < A_g(\Sigma_0)$ for all $0 < |t| < \varepsilon$.

Now, using Theorem 2.1 in [10] and the fact that every element in $\mathcal{E}$ intersects Σ_0, we can find a sweepout $(\Sigma_t)_{t\in[-1,1]}$ of $\mathbb{S}^3$ satisfying the following properties:

(a) $A_g(\Sigma_t) < A_g(\Sigma_0)$ for all $t \in [-1,1] \setminus \{0\}$;

(b) $(\Sigma_t)_{t\in[-1,1]}$ is smooth around $t = 0$.

Consider then the function $f(t) = A_g(\Sigma_t)$. Hence,

$$f''(0) = \int_{\Sigma_0} -\varphi L_{\Sigma_0}\varphi d\Sigma_0 = \lambda \int_{\Sigma_0} \varphi^2 d\Sigma_0 < 0\,.$$

To finish the proof we apply Proposition 3.1 in [10]. □

3. Further remarks

In this paper, we considered the closed 3-manifold with nonnegative Ricci curvature. For the general case, let M^3 be a closed 3-manifold. We say that M^3 is prime if a connected sum decomposition $M = P\sharp Q$ implies $P = \mathbb{S}^3$ or $Q = \mathbb{S}^3$, and it is said to be irreducible if every embedded 2-sphere bounds a 3-ball. A very well-known result is that a closed prime 3-manifold is either an irreducible manifold, $\mathbb{S}^2 \times \mathbb{S}^1$ or $\mathbb{S}^2 \rtimes \mathbb{S}^1$. From the resolution of the Geometrization Conjecture given by G. Perelman, we have that a closed irreducible manifold is either an aspherical manifold, $\mathbb{P}^2 \times \mathbb{S}^1$ or an elliptic space. Also, Gromov [6] observed that if a closed n-manifold M is a connected sum $M = Q\sharp R$ and Q is essential, then M is also essential. Then, for the prime decomposition of a compact 3-manifold, every closed 3-manifold different from either $\mathbb{S}^3$, or $\sharp_k(\mathbb{S}^2 \times \mathbb{S}^1)$ or $\sharp_l(\mathbb{S}^2 \rtimes \mathbb{S}^1)$ is essential. Then, in this case, we can apply Gromov's result.

For the higher dimensional case, suppose M^n $(n \geq 4)$ is locally conformally flat with $Ric_g \geq 0$. In this case, it follows from [13] that M^n is diffeomorphic either to a quotient of $\mathbb{S}^n$, a quotient of $\mathbb{R}^n$ or a quotient of the cylinder $\mathbb{S}^{n-1} \times \mathbb{R}$. If M^n is not simply connected and it is a quotient of the sphere $\mathbb{S}^n$, then M^n is essential. Also, if M^n is a quotient of the space $\mathbb{R}^n$, M^n is essential. Again, we have to deal with the cases of the sphere $\mathbb{S}^n$ and the quotient of $\mathbb{S}^{n-1} \times \mathbb{R}$.

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