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# TWISTS OVER ÉTALE GROUPOIDS AND TWISTED VECTOR BUNDLES

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ABSTRACT. Inspired by recent papers on twisted $K$-theory, we consider in this article the question of when a twist $\mathcal{R}$ over a locally compact Hausdorff groupoid $\mathcal{G}$ (with unit space a CW-complex) admits a twisted vector bundle, and we relate this question to the Brauer group of $\mathcal{G}$. We show that the twists which admit twisted vector bundles give rise to a subgroup of the Brauer group of $\mathcal{G}$. When $\mathcal{G}$ is an étale groupoid, we establish conditions (involving the classifying space $B\mathcal{G}$ of $\mathcal{G}$) which imply that a torsion twist $\mathcal{R}$ over $\mathcal{G}$ admits a twisted vector bundle.

## 1. INTRODUCTION

$C^*$-algebras associated to dynamical systems have provided motivation and examples for a wide array of topics in $C^*$-algebra theory: representation theory, ideal structure, $K$-theory, classification, and connections with mathematical physics, to name a few. In many of these cases, a complete understanding of the theory has required expanding the notion of a dynamical system to allow for partial actions and twisted actions, as well as actions of group-like objects such as semigroups or groupoids.

For example, the $C^*$-algebras $C^*(\mathcal{G};\mathcal{R})$ associated to a groupoid $\mathcal{G}$ and a twist $\mathcal{R}$ over $\mathcal{G}$ (hereafter referred to as *twisted groupoid $C^*$-algebras*) provide important insights into mathematical physics as well as the structure of other $C^*$-algebras. First, the collection of twists $\mathcal{R}$ over a groupoid $\mathcal{G}$ is intimately related to the cohomology of $\mathcal{G}$; cf. [15, 16, 26]. Another structural result is due to Kumjian [14] and Renault [24]: groupoid twists classify Cartan pairs. Finally, the papers [2, 4, 27] establish that twisted groupoid $C^*$-algebras classify $D$-brane charges in many flavors of string theory.

We also note, following [22, 27], that groupoid twists constitute an example of Fell bundles. Indeed, Fell bundles provide a universal framework for studying all of the generalized dynamical systems mentioned above.

In several recent papers (cf. [3, 10, 27]) on twisted groupoid $C^*$-algebras, the $K$-theory groups of these $C^*$-algebras have received a good deal of attention. Of particular interest is the question of when $K_0(C^*(\mathcal{G};\mathcal{R}))$ can be completely understood in terms of $\mathcal{G}$-equivariant vector bundles. Phillips established in Chapter 9 of [23] that $\mathcal{G}$-equivariant vector bundles may not suffice to describe $K_0(C^*(\mathcal{G};\mathcal{R}))$, even when $\mathcal{G} = M \rtimes G$ is a transformation group and $\mathcal{R}$ is trivial. Vector bundles

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provide a highly desirable geometric perspective on $K_0(C^*(\mathcal{G};\mathcal{R}))$, however, and so conditions are sought (cf. [1, 2, 5, 9, 10, 18]) under which $K_0(C^*(\mathcal{G};\mathcal{R}))$ is generated by $\mathcal{G}$-equivariant vector bundles.

In Theorem 5.28 of [27], Tu, Xu, and Laurent-Gengoux study this question for proper Lie groupoids $\mathcal{G}$. They establish, in this context, sufficient conditions for the $K$-theory group $K_0(C^*(\mathcal{G},\mathcal{R}))$ associated to a twist $\mathcal{R}$ over $\mathcal{G}$ to be generated by $(\mathcal{R},\mathcal{G})$-twisted vector bundles over the unit space of $\mathcal{G}$ (see Definition 2.4 below). A necessary condition is that $\mathcal{R}$ be a torsion element of the Brauer group of $\mathcal{G}$. Conjecture 5.7 on p. 888 of [27] states that, if $\mathcal{G}$ is a proper Lie groupoid acting cocompactly on its unit space, then this condition is also sufficient.

Conjecture 5.7 of [27] has not yet been disproved, but it has only been proven true in certain special cases; cf. [5, 10, 18] when $\mathcal{R} = \mathcal{G} \times \mathbb{T}$ is the trivial twist, [2] for nontrivial twists $\mathcal{R}$ over manifolds $M$, and [1, 9, 19] for nontrivial twists over representable orbifolds $G \ltimes M$, where $G$ is a discrete group acting properly on a compact space $M$.

In hopes of shedding more light on this conjecture, we present an equivalent formulation in Conjecture 3.5 below, using the Brauer group of $\mathcal{G}$ as defined in [16]. Our reformulated conjecture relies on our result (Proposition 3.4) that, for any locally compact Hausdorff groupoid $\mathcal{G}$ whose unit space is a CW-complex, the collection of twists $\mathcal{R}$ over $\mathcal{G}$ which admit twisted vector bundles gives rise to a subgroup $Tw_\tau(\mathcal{G})$ of the Brauer group $\mathrm{Br}(\mathcal{G})$.

We note that Theorem 3.2 of [13] also establishes a link between twisted vector bundles and the Brauer group, but Karoubi's approach in [13] differs substantially from ours, and does not address the group structure of $Tw_\tau(\mathcal{G})$.

In the second part of the paper, we address the question of when a torsion twist $\mathcal{R}$ over an étale groupoid $\mathcal{G}$ admits a twisted vector bundle. The existence of such vector bundles is necessary (but not sufficient) in order for $K_0(C^*(\mathcal{G};\mathcal{R}))$ to be generated by twisted vector bundles.

Theorem 4.6 below establishes that if the classifying space $B\mathcal{G}$ is a compact CW-complex and if a certain principal $PU(n)$-bundle $P$ lifts to a $U(n)$-principal bundle $\tilde{P}$, then up to Morita equivalence, the torsion twist $\mathcal{R}$ admits a twisted vector bundle. To our knowledge, the connection between classifying spaces and twisted vector bundles has not been explored previously in the literature; we are optimistic that Theorem 4.6 will lead to new insights into the $K$-theory of twisted groupoid $C^*$-algebras.

1.1. **Structure of the paper.** We begin in Section 2 by reviewing the basic concepts we will rely on throughout this paper: locally compact Hausdorff groupoids, twists over such groupoids, groupoid vector bundles, and twisted vector bundles. In Section 3 we show that, for any locally compact Hausdorff groupoid $\mathcal{G}$ whose unit space is a CW-complex, the collection of twists over $\mathcal{G}$ which admit twisted vector bundles gives rise to a subgroup of the Brauer group of $\mathcal{G}$, and we use this to present an alternate formulation of Conjecture 5.7 from [27]. Finally, in Section 4 we consider torsion twists for étale groupoids. We establish, in Theorem 4.6, sufficient conditions for a torsion twist $\mathcal{R}$ over an étale groupoid $\mathcal{G}$ to admit a twisted vector bundle, and we present examples showing that the hypotheses of Theorem 4.6 are satisfied in many cases of interest.

## 2. Definitions

Recall that a *groupoid* is a small category with inverses. Throughout this note, $\mathcal{G}$ will denote (the space of arrows of) a groupoid with unit space $\mathcal{G}^{(0)}$, with source, range (or target), and unit maps

$$s, r : \mathcal{G} \longrightarrow \mathcal{G}^{(0)},\ u : \mathcal{G}^{(0)} \longrightarrow \mathcal{G}.$$

As usual we denote the set of composable elements of $\mathcal{G}$ by $\mathcal{G}^{(2)}$, where

$$\mathcal{G}^{(2)} = \mathcal{G} \times_{s,\mathcal{G}^{(0)},t} \mathcal{G} = \{(g_1, g_2) \in \mathcal{G} \times \mathcal{G} \mid s(g_1) = r(g_2)\}.$$

In this paper, we will primarily be concerned with *locally compact Hausdorff groupoids*. These are groupoids $\mathcal{G}$ such that the spaces $\mathcal{G}^{(0)}, \mathcal{G}, \mathcal{G}^{(2)}$ have locally compact Hausdorff topologies with respect to which the maps $s, r : \mathcal{G} \to \mathcal{G}^{(0)}$, the multiplication $\mathcal{G}^{(2)} \to \mathcal{G}$, and the inverse map $\mathcal{G} \to \mathcal{G}$ are continuous. Conjecture 3.5 below makes reference to *Lie groupoids*, which are locally compact Hausdorff groupoids such that the spaces $\mathcal{G}^{(0)}, \mathcal{G}, \mathcal{G}^{(2)}$ are smooth manifolds and all of the structure maps between them are smooth.

Theorem 4.6 deals with *étale groupoids*, which are locally compact Hausdorff groupoids $\mathcal{G}$ for which $r, s$ are local homeomorphisms. For example, if a discrete group $\Gamma$ acts on a CW-complex $M$, the associated transformation group $\Gamma \ltimes M$ is an étale groupoid.

**Definition 2.1.** Let $\mathcal{G}_1, \mathcal{G}_2$ be two locally compact Hausdorff groupoids with unit spaces $\mathcal{G}_1^{(0)}, \mathcal{G}_2^{(0)}$ respectively. A morphism $f : \mathcal{G}_1 \to \mathcal{G}_2$ consists of a pair of continuous maps $f = (f_0, f_1)$, with

$$f_0 : \mathcal{G}_1^{(0)} \to \mathcal{G}_2^{(0)},\ f_1 : \mathcal{G}_1 \to \mathcal{G}_2,$$

such that, if we denote by $s_{\mathcal{G}_j}$ and $r_{\mathcal{G}_j}$ the source and range maps of $\mathcal{G}_j$, $j = 1, 2$, we have

$$s_{\mathcal{G}_2} \circ f_1 \ = \ f_0 \circ s_{\mathcal{G}_1}, \text{ and } r_{\mathcal{G}_2} \circ f_1 \ = \ f_0 \circ r_{\mathcal{G}_1}.$$

The notion of a twist or $\mathbb{T}$-central extension of a groupoid $\mathcal{G}$ was originally developed (cf. [14, 21, 27]) to provide a "second cohomology group" for groupoids. Groupoid twists and their associated twisted vector bundles (see Definition 2.4 below) are the groupoid analogues of group 2-cocycles and projective representations.

**Definition 2.2.** Let $\mathcal{G}$ be a locally compact Hausdorff groupoid with unit space $\mathcal{G}^{(0)}$. A $\mathbb{T}$-*central extension (or "twist")* of $\mathcal{G}$ consists of:

(1) A locally compact Hausdorff groupoid $\mathcal{R}$ with unit space $\mathcal{G}^{(0)}$, together with a morphism of locally compact Hausdorff groupoids

$$(id, \pi) : \mathcal{R} \to \mathcal{G},$$

which restricts to the identity on $\mathcal{G}^{(0)}$.

(2) A left $\mathbb{T}$–action on $\mathcal{R}$, with respect to which $\mathcal{R}$ is a left principal $\mathbb{T}$-bundle over $\mathcal{G}$.

(3) These two structures are compatible in the sense that

$$(z_1 r_1)(z_2 r_2) = z_1 z_2 (r_1 r_2), \forall\ z_1, z_2 \in \mathbb{T},\ \forall (r_1, r_2) \in \mathcal{R}^{(2)} = \mathcal{R} \times_{s,\mathcal{G}^{(0)},r} \mathcal{R}.$$

We write $Tw(\mathcal{G})$ for the set of twists over $\mathcal{G}$.

These conditions (1)-(3) imply the exactness of the sequence of groupoids

$$\mathcal{G}^{(0)} \to \mathcal{G}^{(0)} \times \mathbb{T} \to \mathcal{R} \to \mathcal{G} \rightrightarrows \mathcal{G}^{(0)},$$

which highlights the parallel between twists over a groupoid $\mathcal{G}$ and extensions of $\mathcal{G}$ by $\mathbb{T}$ (or elements of the second cohomology group $H^2(\mathcal{G}, \mathbb{T})$).

If $\mathcal{R}_1, \mathcal{R}_2 \in Tw(\mathcal{G})$, we can form their Baer sum

$$\mathcal{R}_1 + \mathcal{R}_2 := \{(r_1, r_2) \in \mathcal{R}_1 \times \mathcal{R}_2 : \pi_1(r_1) = \pi_2(r_2)\}/\sim,$$

where $(r_1, r_2) \sim (zr_1, \overline{z}r_2)$ for all $z \in \mathbb{T}$. Define an action of $\mathbb{T}$ on $\mathcal{R}_1 + \mathcal{R}_2$ by $z \cdot [(r_1, r_2)] = [(zr_1, r_2)] = [(r_1, zr_2)]$, and observe that with this action, $\mathcal{R}_1 + \mathcal{R}_2$ becomes a twist over $\mathcal{G}$.

With this operation, $Tw(\mathcal{G})$ becomes a group; the identity element is the trivial extension $\mathcal{G} \times \mathbb{T}$, and the inverse of a twist $\mathcal{R}$ is the twist $\overline{\mathcal{R}}$. As groupoids, $\mathcal{R} = \overline{\mathcal{R}}$; however, the action of $\mathbb{T}$ on $\overline{\mathcal{R}}$ is the conjugate of the action on $\mathcal{R}$. To be precise, if $r \in \mathcal{R}$, denote by $\overline{r}$ the corresponding element of $\overline{\mathcal{R}}$. Then

$$z \cdot \overline{r} = \overline{\overline{z} \cdot r}.$$

In this note, we will consider actions of groupoids $\mathcal{G}$ and twists $\mathcal{R}$ over $\mathcal{G}$ on a variety of spaces. We make this concept precise as follows.

**Definition 2.3.** Let $\mathcal{G}$ be a locally compact Hausdorff groupoid with unit space $\mathcal{G}^{(0)}$. A *$\mathcal{G}$-space* is a locally trivial fiber bundle $J : P \to \mathcal{G}^{(0)}$ such that, setting

$$\mathcal{G} * P = \{(g, p) \in \mathcal{G} \times P : s(g) = J(p)\}$$

and equipping $\mathcal{G} * P$ with the subspace topology inherited from $\mathcal{G} \times P$, we have a continuous map $\sigma : \mathcal{G} * P \to P$ satisfying:

- $\sigma(J(p), p) = p$ for all $p \in P$.
- $J(\sigma(g, p)) = r(g)$ for all $(g, p) \in \mathcal{G} * P$.
- If $(g, h) \in \mathcal{G}^{(2)}$ and $(h, p) \in \mathcal{G} * P$, then $\sigma(g, \sigma(h, p)) = \sigma(gh, p)$.

We will often write $g \cdot p$ for $\sigma(g, p) \in P$.

Note that, as a consequence of the above definition, the map $\sigma_g : P_{s(g)} \to P_{r(g)}$ given by $p \mapsto \sigma(g, p)$ must be a homeomorphism, for all $g \in \mathcal{G}$.

**Definition 2.4.** (1) Let $\mathcal{G}$ be a locally compact Hausdorff groupoid with unit space $\mathcal{G}^{(0)}$, where $\mathcal{G}^{(0)}$ is a CW-complex. A *$\mathcal{G}$–vector bundle* is a vector bundle $J : E \to \mathcal{G}^{(0)}$ which is a $\mathcal{G}$-space in the sense of Definition 2.3.

(2) Let

$$\mathcal{G}^{(0)} \to \mathbb{T} \times \mathcal{G}^{(0)} \xrightarrow{i} \mathcal{R} \xrightarrow{j} \mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$$

be a $\mathbb{T}$-central extension of locally compact Hausdorff groupoids. By a *$(\mathcal{G}, \mathcal{R})$–twisted vector bundle*, we mean an $\mathcal{R}$-vector bundle $J : E \to \mathcal{G}^{(0)}$ such that, whenever $z \in \mathbb{T},\ r \in \mathcal{R}, e \in E$ such that $s(r) = J(e)$, we have

$$(z \cdot r) \cdot e = z(r \cdot e). \tag{1}$$

Here, the action on the right-hand side of the equation is simply scalar multiplication (identifying $\mathbb{T}$ with the unit circle of $\mathbb{C}$).

(3) An equivalent characterization of $(\mathcal{R}, \mathcal{G})$-twisted vector bundles is the following:

An $\mathcal{R}$-vector bundle $E \to \mathcal{G}^{(0)}$ is an $(\mathcal{R}, \mathcal{G})$-twisted vector bundle if and only if the subgroupoid $\ker j \cong \mathcal{G}^{(0)} \times \mathbb{T}$ of $\mathcal{R}$ acts on $E$ by scalar multiplication, where $\mathbb{T}$ is identified with the unit circle of $\mathbb{C}$.

In Proposition 3.4, we will establish a connection between the twists over $\mathcal{G}$ which admit twisted vector bundles and the Brauer group of $\mathcal{G}$, as introduced in [16]. Thus, we review here a few facts about the Brauer group and its connection to $Tw(\mathcal{G})$.

**Definition 2.5.** Let $\mathcal{G}$ be a locally compact Hausdorff groupoid. As in Definition 8.1 of [16], we will denote by $Br_0(\mathcal{G})$ the group of Morita equivalence classes of $\mathcal{G}$-spaces $\mathcal{A}$ such that $\mathcal{A} = \mathcal{G}^{(0)} \times \mathcal{K}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. We denote the class in $Br_0(\mathcal{G})$ of $\mathcal{A}$ by $[\mathcal{A}, \alpha]$, where $\alpha$ is the action of $\mathcal{G}$ on $\mathcal{A}$.

Also, let $\mathcal{E}(\mathcal{G})$ be the quotient of $Tw(\mathcal{G})$ by Morita equivalence or, equivalently, the quotient by the subgroup $W$ of elements which are Morita equivalent to the trivial twist. See Definition 3.1 and Corollary 7.3 of [16] for details.

Theorem 8.3 of [16] establishes that

$$Br_0(\mathcal{G}) \cong \mathcal{E}(\mathcal{G}) = Tw(\mathcal{G})/W.$$

## 3. Twisted vector bundles and the Brauer group

Let $\mathcal{G}$ be a locally compact Hausdorff groupoid with unit space a CW-complex. In this section, we will show that the subset $Tw_\tau(\mathcal{G})$ of twists over $\mathcal{G}$ which admit twisted vector bundles gives a subgroup of $Br_0(\mathcal{G})$.

**Definition 3.1.** For a locally compact Hausdorff groupoid $\mathcal{G}$, let $Br_\tau(\mathcal{G})$ be the subgroup of $Br_0(\mathcal{G})$ consisting of Morita equivalence classes $[\mathcal{A}, \alpha]$ of elementary $\mathcal{G}$-bundles $\mathcal{A} = \mathcal{G}^{(0)} \times \mathcal{K}(\mathcal{H})$ with zero Dixmier-Douady invariant, such that $\mathcal{H}$ is finite dimensional.

When, in addition, the unit space of $\mathcal{G}$ is a CW-complex, we denote by $Tw_\tau(\mathcal{G})$ the subset of $Tw(\mathcal{G})$ consisting of twists $\mathcal{R}$ over $\mathcal{G}$ that admit a twisted vector bundle.

**Proposition 3.2.** *Let $\mathcal{G}$ be a locally compact groupoid whose unit space is a CW-complex. Then $Tw_\tau(\mathcal{G})$ is a subgroup of $Tw(\mathcal{G})$.*

*Proof.* (1) (Closure under operation) Given two twists

$$\mathcal{G}^{(0)} \to \mathcal{G}^{(0)} \times \mathbb{T} \xrightarrow{i_1} \mathcal{R}_1 \xrightarrow{j_1} \mathcal{G} \rightrightarrows \mathcal{G}^{(0)}, \qquad \mathcal{G}^{(0)} \to \mathcal{G}^{(0)} \times \mathbb{T} \xrightarrow{i_2} \mathcal{R}_2 \xrightarrow{j_2} \mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$$

that admit twisted vector bundles $E_1$ and $E_2$ respectively, it is straightforward to show that

$$E_1 * E_2 := \{(e_1, e_2) \in E_1 \oplus E_2 | \; J_1(e_1) = J_2(e_2)\}/\sim$$

is a twisted vector bundle for the Baer sum $\mathcal{R}_1 + \mathcal{R}_2$. The action of $\mathcal{R}_1 + \mathcal{R}_2$ on $E_1 * E_2$ is given by

$$[(r_1, r_2)] \cdot [(e_1, e_2)] = [(r_1 \cdot e_1, r_2 \cdot e_2)].$$

(2) (Neutral Element) The neutral element of $Tw(\mathcal{G})$ is $\mathcal{G} \times \mathbb{T}$. Note that $\mathcal{G} \times \mathbb{T}$ admits a twisted vector bundle $E$ – namely, $E = \mathcal{G}^{(0)} \times \mathbb{C}$, with the action $(g, z) \cdot (s(g), v) = (r(g), zv)$.

(3) (Inverses) We must show that, if $\mathcal{R} \in Tw_\tau(\mathcal{G})$, then $\overline{\mathcal{R}} \in Tw_\tau(\mathcal{G})$.

For $\mathcal{R} \in Tw_\tau(\mathcal{G})$, let $E \to \mathcal{G}^{(0)}$ be a $(\mathcal{R}, \mathcal{G})$-twisted vector bundle. Write $\overline{E}$ for the conjugate vector bundle – that is, $\overline{E} = E$ as sets, and the additive operation on $\overline{E}$ agrees with that on $E$ (in symbols, $\overline{e + f} = \overline{e} + \overline{f}$), but the $\mathbb{C}$ action on $\overline{E}$ is the conjugate of the action on $E$: $z \cdot \overline{e} = \overline{\overline{z} \cdot e}$. Define an action of $\overline{\mathcal{R}}$ on $\overline{E}$ by $\overline{r} \cdot \overline{e} = \overline{r \cdot e}$. This action makes $\overline{E}$ into an $\overline{\mathcal{R}}$-vector bundle since $E$ is an $\mathcal{R}$-vector bundle. Moreover, for any $z \in \mathbb{T}$, we have

$$\begin{aligned}(z \cdot \overline{r}) \cdot \overline{e} &= \overline{\overline{z} \cdot r} \cdot \overline{e} = \overline{(\overline{z}r) \cdot e} \\ &= \overline{\overline{z}(r \cdot e)} = z\overline{r \cdot e} \\ &= z(\overline{r} \cdot \overline{e}).\end{aligned}$$

Thus, $\mathbb{T}$ acts by scalars on $\overline{E}$, and so $\overline{E}$ is a $(\overline{W}, \mathcal{G})$-twisted vector bundle. □

*Remark* 3.3. Recall from Proposition 5.5 of [27] that if a twist $\mathcal{R}$ over $\mathcal{G}$ admits a twisted vector bundle, then $\mathcal{R}$ must be torsion. Thus, $Tw_\tau(\mathcal{G})$ is a subgroup of $Tw^{tor}(\mathcal{G})$, the torsion subgroup of $Tw(\mathcal{G})$.

3.1. **The image of $Tw_\tau(\mathcal{G})$ in $Br_0(\mathcal{G})$.** Recall that if $\mathcal{G}$ is a locally compact Hausdorff groupoid with unit space $\mathcal{G}^{(0)}$, then $Br_0(\mathcal{G})$ consists of Morita equivalence classes of $\mathcal{G}$-spaces of the form $\mathcal{A} = \mathcal{G}^{(0)} \times \mathcal{K}(\mathcal{H})$.

In Section 8 of [16], the authors construct an isomorphism $\Theta : Br_0(\mathcal{G}) \to Tw(\mathcal{G})/W$, where $W$ is the subgroup of $Tw(\mathcal{G})$ consisting of elements which are Morita equivalent to the trivial twist. We will use this isomorphism to study the subgroup of $Br_0(\mathcal{G})$ corresponding to $Tw_\tau(\mathcal{G})$.

Proposition 8.7 of [16] describes a homomorphism $\theta : Tw(\mathcal{G}) \to Br_0(\mathcal{G})$ which induces the inverse of $\Theta$.

**Proposition 3.4.** *Suppose $\mathcal{G}$ is a locally compact Hausdorff groupoid whose unit space $\mathcal{G}^{(0)}$ is a connected CW-complex, and suppose $\mathcal{R} \in Tw_\tau(\mathcal{G})$. Then there exists a finite-dimensional $\mathcal{G}$-vector bundle $V \to \mathcal{G}^{(0)}$ such that $\theta(\mathcal{R}) = [\mathrm{Aut}(V), \alpha]$, where $\alpha$ is induced by the action of $\mathcal{G}$ on $V$.*

*Moreover, if $[\mathcal{A}, \alpha'] \in Br_0(\mathcal{G})$ and $(\mathcal{A}, \alpha')$ is Morita equivalent to $(\mathcal{M}_n, \alpha)$ where $\mathcal{M}_n$ is an $M_n(\mathbb{C})$-bundle over $\mathcal{G}^{(0)}$, then $[\mathcal{A}, \alpha'] = [\mathcal{M}_n, \alpha]$ lies in $\theta(Tw_\tau(\mathcal{G}))$.*

*In other words, $Br_\tau(\mathcal{G}) \cong Tw_\tau(\mathcal{G})$.*

*Proof.* If $\mathcal{R} \in Tw_\tau(\mathcal{G})$ and $V$ is a $(\mathcal{R}, \mathcal{G})$-twisted vector bundle, write $j : \mathcal{R} \to \mathcal{G}$ for the projection of $\mathcal{R}$ onto $\mathcal{G}$ and write $\sigma : \mathcal{R} * V \to V$ for the action of $\mathcal{R}$ on $V$. Define $\alpha : \mathcal{G} * \mathrm{Aut}(V) \to \mathrm{Aut}(V)$ by

$$(\alpha(g, A)(v) = \sigma(\eta, A(\sigma(\eta^{-1}, v))),$$

where $v \in V_{r(g)}$ and $\eta \in j^{-1}(g)$.

Note that $\alpha(g, A)$ does not depend on our choice of $\eta \in j^{-1}(g)$: If $\eta, \eta' \in j^{-1}(g)$, the fact that $\mathcal{R}$ is a principal $\mathbb{T}$-bundle over $\mathcal{G}$ implies that $\eta = z\eta'$ for some $z \in \mathbb{T}$. Since $V$ is a $(\mathcal{G}, \mathcal{R})$-twisted vector bundle, $\sigma(\eta, v) = z\sigma(\eta', v)$, and consequently

$$\sigma(\eta, A(\sigma(\eta^{-1}, v))) = \sigma(\eta', A(\sigma((\eta')^{-1}, v))).$$

Now, Lemma 8.8 of [16] establishes that $[\mathrm{Aut}(V), \alpha] = \theta(\mathcal{R})$.

For the second statement, suppose $\alpha$ is an action of $\mathcal{G}$ on a bundle $\mathcal{M}_n$ of $n$-dimensional matrix algebras over $\mathcal{G}^{(0)}$. Then Theorem 8.3 of [16] explains how to construct the twist $\Theta([\mathcal{M}_n, \alpha])$, using a pullback construction. To be precise,

$$\Theta([\mathcal{M}_n, \alpha]) = \{(g, U) \in \mathcal{G} \times U_n(\mathbb{C}) : \alpha_g = \mathrm{Ad}\, U\} =: \mathcal{R}(\alpha).$$

We will construct a $(\mathcal{G}, \mathcal{R}(\alpha))$-twisted vector bundle, proving that $[\mathcal{M}_n, \alpha] \in \theta(Tw_\tau(\mathcal{G}))$.

The $\mathbb{T}$-action on $\mathcal{R}(\alpha)$ which makes it into a twist over $\mathcal{G}$ is given by

$$z \cdot (g, U) = (g, z \cdot U).$$

Consider the sub-bundle $\mathcal{GL}_n$ of $\mathcal{M}_n$ obtained by considering only the invertible elements of $M_n(\mathbb{C})$ in each fiber of $\mathcal{M}_n$. Notice that $GL_n(\mathbb{C})$ acts on $\mathcal{GL}_n$ by right multiplication in each fiber, and that this action is continuous, and free and transitive in each fiber, and hence makes $\mathcal{GL}_n$ into a principal $GL_n$ bundle. We consequently obtain an associated vector bundle over $\mathcal{G}^{(0)}$,

$$V = \mathcal{GL}_n \times_{GL_n(\mathbb{C})} \mathbb{C}^n.$$

Moreover, $\mathcal{R}(\alpha)$ acts on $V$,

$$(g, U) \cdot [A, v] = [\alpha_g(A), Uv].$$

To see that this action is well defined, take $G \in GL_n(\mathbb{C})$ and calculate

$$\begin{aligned}[][\alpha_g(AG), U(G^{-1}v)] = [UAGU^{-1}, UG^{-1}v] = [UA, v] = [UAU^{-1}, Uv] \\ [\alpha_g(A), v] = [UAU^{-1}, Uv].\end{aligned}$$

Moreover,

$$\begin{aligned}(z \cdot (g, U)) \cdot [A, v] &= [\alpha_g(A), zU(v)] \\ &= z \cdot [\alpha_g(A), Uv] = z \cdot ((g, U) \cdot [A, v]),\end{aligned}$$

so $V$ is an $(\mathcal{R}(\alpha), \mathcal{G})$-twisted vector bundle. Thus, $\mathcal{R}(\alpha) \in Tw_\tau(\mathcal{G})$ whenever $[\alpha, \mathcal{M}] \in Br_0(\mathcal{G})$. □

Proposition 3.4 thus establishes that twists $\mathcal{R}$ over $\mathcal{G}$ which admit twisted vector bundles correspond to $C^*$-bundles over $\mathcal{G}^{(0)}$ with finite-dimensional fibers. Phrased in this way, the parallel between Proposition 3.4 and Theorem 3.2 of [13] becomes evident. However, the two proofs take very different approaches. Moreover, Karoubi does not address the group structure of $Tw_\tau(\mathcal{G})$ in Theorem 3.2 of [13].

Proposition 3.4 also allows us to rephrase Conjecture 5.7 of [27] in terms of the Brauer group, as follows. Recall that, in its original form, Conjecture 5.7 of [27] asserts that all torsion elements of $Tw(\mathcal{G})$ should admit twisted vector bundles, if $\mathcal{G}$ is proper and the quotient $\mathcal{G}^{(0)}/\mathcal{G}$ is compact.

**Conjecture 3.5** ([27], Conjecture 5.7)**.** *Let $\mathcal{G}$ be a proper Lie groupoid such that the quotient $\mathcal{G}^{(0)}/\mathcal{G}$ is compact, and let $[\mathcal{A}, \alpha] \in Br_0(\mathcal{G})$ be a torsion element. Then $[\mathcal{A}, \alpha] = [\mathcal{M}, \alpha']$ for some finite-dimensional matrix algebra bundle $\mathcal{M}$ over $\mathcal{G}^{(0)}$ and an action $\alpha'$ of $\mathcal{G}$ on $\mathcal{M}$.*

## 4. Twisted vector bundles for étale groupoids

In this section, we consider torsion twists over étale groupoids $\mathcal{G}$. We establish in Theorem 4.6 sufficient conditions for a torsion twist $\mathcal{R}$ over $\mathcal{G}$ to admit (up to Morita equivalence) a twisted vector bundle, and we describe examples meeting these conditions in Section 4.1. The conditions of Theorem 4.6 are phrased in terms of the classifying space $B\mathcal{G}$ and in terms of a principal bundle $P$ associated to $\mathcal{R}$. Using $B\mathcal{G}$ to study twisted vector bundles appears to be a new approach; this perspective was inspired by Moerdijk's result in [20] identifying $H^*(\mathcal{G}, \mathscr{S})$ and $H^*(B\mathcal{G}, \tilde{\mathscr{S}})$ for an abelian $\mathcal{G}$-sheaf $\mathscr{S}$, and the Serre-Grothendieck Theorem (cf. Theorem 1.6 of [11]) relating $H^1(M, PU(n))$ and $H^2(M, \mathbb{T})$ for $M$ a CW-complex.

We begin with some preliminary definitions and results.

**Definition 4.1.** Let $\mathcal{G}$ be a topological groupoid. The *simplicial space associated to* $\mathcal{G}$ is

$$\mathcal{G}_\bullet = \{\mathcal{G}^{(k)}, \epsilon_j^k, \eta_k^j\}_{0\le j\le k\in\mathbb{N}},$$

where $\mathcal{G}^{(k)}$ is the space of composable $n$-tuples in $\mathcal{G}$, $\epsilon_j^k : \mathcal{G}^{(k)} \to \mathcal{G}^{(k-1)}$, and $\eta_k^j : \mathcal{G}^{(k)} \to \mathcal{G}^{(k+1)}$ are given as follows:

$$\begin{aligned}
\epsilon_0^k(g_1,\ldots,g_k) &= (g_2,\ldots,g_k),\\
\epsilon_i^k(g_1,\ldots,g_k) &= (g_1,\ldots,g_ig_{i+1},\ldots,g_k) \text{ if } 1\le i\le k-1,\\
\epsilon_k^k(g_1,\ldots,g_k) &= (g_1,\ldots,g_{k-1}).
\end{aligned}$$

If $k=1$, we have $\epsilon_0^1(g) = s(g)$, $\epsilon_1^1(g) = r(g)$.

The degeneracy maps $\eta_i^k$ are given for $k\ge 1$ by

$$\begin{aligned}
\eta_i^k(g_1,\ldots,g_k) &= (g_1,\ldots,g_i,s(g_i),g_{i+1},\ldots,g_k) \text{ if } i\ge 1,\\
\eta_0^k(g_1,\ldots,g_k) &= (r(g_1),g_1,\ldots,g_k).
\end{aligned}$$

When $k=0$, the map $\eta_0^0 : \mathcal{G}^{(0)} \to \mathcal{G}^{(1)}$ is just the standard inclusion of $\mathcal{G}^{(0)}$ into $\mathcal{G}^{(1)} = \mathcal{G}$.

For the definition of a general simplicial space, see e.g. [17], Section 2.1.

**Definition 4.2** (cf. [20, 29])**.** Let $\mathcal{G}$ be a topological groupoid. A *classifying space* $B\mathcal{G}$ for $\mathcal{G}$ is any space which can be realized as a quotient $B\mathcal{G} = E\mathcal{G}/\mathcal{G}$ of a weakly contractible space $E\mathcal{G}$ by a free action of $\mathcal{G}$. When we need an explicit model for $B\mathcal{G}$, we will use the geometric realization $|\mathcal{G}_\bullet|$ of the simplicial space associated to $\mathcal{G}$,

$$B\mathcal{G} = |\mathcal{G}_\bullet| = \left(\bigsqcup_{k\ge 0} \mathcal{G}^{(k)}\times\Delta^k\right)/\sim,$$

where $\Delta^k$ denotes the standard $k$-simplex.[1] The equivalence relation $\sim$ is defined by $(p, \delta_i^{k-1}v) \sim (\epsilon_i^k p, v)$ for $p\in\mathcal{G}^{(k)}, v\in\Delta^{k-1}$, where $\delta_i^{k-1} : \Delta^{k-1}\to\Delta^k$ is the $i$th degeneracy map, gluing $\Delta^{k-1}$ to the $i$th face of $\Delta^k$, and $\epsilon_i^k : \mathcal{G}^{(k)}\to\mathcal{G}^{(k-1)}$ is the $i$th face map. In other words, we have $\delta_0^0(\emptyset) = 0, \delta_1^0(\emptyset) = 1$, and if $k>1$, then

$$\delta_i^{k-1}(t_1,\ldots,t_{k-1}) = \begin{cases} (0,t_1,\ldots,t_{k-1}) & \text{if } i=0,\\ (t_1,\ldots,t_i,t_i,t_{i+1},\ldots,t_k) & \text{if } 1\le i\le k-1,\\ (t_1,\ldots,t_{k-1},1) & \text{if } i=k. \end{cases}$$

The topology on this model of $B\mathcal{G}$ is the inductive limit topology induced by the natural topologies on $\mathcal{G}^{(n)}, \Delta^n$.

**Definition 4.3** ([17], Definition 2.2)**.** Let $X_\bullet$ be a simplicial space and let $G$ be a topological group. A *principal $G$-bundle over $X_\bullet$* is a simplicial space $P_\bullet$ such that, for each $k\ge 0$, $P_k$ is a principal $G$-bundle over $X_k$, and the face and degeneracy maps in $P_\bullet$ are morphisms of principal bundles.

*Remark* 4.4. Combining [27], Definition 2.1, and Proposition 2.4 of [17], we see that principal $G$-bundles over $\mathcal{G}_\bullet$ are equivalent to generalized morphisms $\mathcal{G}\to G$.

[1] For $k>0$, $\Delta^k$ can be realized as a subset of $\mathbb{R}^k$, namely,

$$\Delta^k = \{(t_1,\ldots,t_k) : 0\le t_1\le t_2\le\cdots\le t_k\le 1\}.$$

If $k=0$, $\Delta^k$ consists of one point, and we will denote $\Delta^0 = \emptyset$.

**Proposition 4.5.** *Let $\mathcal{G}$ be an étale groupoid. Suppose that the classifying space $B\mathcal{G}$ is (homotopy equivalent to) a compact CW-complex. If $\mathcal{R} \to \mathcal{G}$ is a twist of order $n$, then $\mathcal{R}$ gives rise to a principal $PU(n)$-bundle $P \to \mathcal{G}^{(0)}$. Moreover, $P$ admits a left action of $\mathcal{G}$ which commutes with the right action of $PU(n)$.*

*Proof.* For any étale groupoid $\mathcal{G}$, and any twist $\mathcal{R} \to \mathcal{G}$, Proposition 11.3, Corollary 7.3, and Theorem 8.3 of [16] combine to tell us that $\mathcal{R}$ determines an element of $H^2(\mathcal{G}, \mathcal{S}^1)$, where $\mathcal{S}^1$ denotes the sheaf of circle-valued functions on $\mathcal{G}^{(0)}$. The main Theorem of [20] tells us that we then obtain an associated element $[\mathcal{R}]$ of $H^2(B\mathcal{G}, \mathcal{S}^1) \cong H^3(B\mathcal{G}, \mathbb{Z})$. All of the maps $\mathrm{Tw}(\mathcal{G}) \to H^2(\mathcal{G}, \mathcal{S}^1) \cong H^3(B\mathcal{G}, \mathbb{Z})$ are group homomorphisms, so if $\mathcal{R}$ is a torsion twist of order $n$, then $n \cdot [\mathcal{R}] = 0$ also in $H^3(B\mathcal{G}, \mathbb{Z})$.

Now, suppose that $B\mathcal{G}$ is a compact CW-complex and that $\mathcal{R}$ is a torsion twist of order $n$. The Serre-Grothendieck Theorem (cf. [11], Theorem 1.6, [8], Theorem 8, or [19], Theorem 7.2.11) tells us that $\mathcal{R}$ gives rise to a principal $PU(n)$ bundle $Q$ over $B\mathcal{G}$.

Note that, for each $k \in \mathbb{N}$, the map $\varphi_k : \mathcal{G}^{(k)} \to B\mathcal{G}$ given by $(g_1, \ldots, g_k) \mapsto [(g_1, \ldots, g_k), (0, \ldots, 0)]$ is continuous. Moreover, the equivalence relation which defines $B\mathcal{G}$ ensures that the maps $\varphi_k$ commute with the face and degeneracy maps $\epsilon_i^k, \eta_i^k$,

$$\forall\ i,\ \varphi_k \circ \eta_i^{k-1} = \varphi_{k-1} \text{ and } \varphi_{k-1} \circ \epsilon_i^k = \varphi_k.$$

Principal $PU(n)$-bundles over a space $X$ are classified by homotopy classes of maps $X \to BPU(n)$, so the maps $\varphi_k$ allow us to pull back our principal $PU(n)$-bundle $Q$ over $B\mathcal{G}$ to a principal $PU(n)$-bundle $P_k$ over $\mathcal{G}^{(k)}$ for each $k \geq 0$. Since the maps $\varphi_k$ commute with the face and degeneracy maps for $\mathcal{G}_\bullet$, the maps $\eta_i^k, \epsilon_i^k$ induce morphisms of principal bundles which make $P_\bullet$ into a principal $PU(n)$-bundle over $\mathcal{G}_\bullet$ in the sense of Definition 4.3. Thus, by Proposition 2.4 of [17], we have a principal $PU(n)$-bundle $P$ over $\mathcal{G}^{(0)}$ which admits an action of $\mathcal{G}$. $\square$

In what follows, we will combine the bundle $P_\bullet$ constructed above with the canonical $\mathbb{T}$-central extension

$$\beta : \quad 1 \to \mathbb{T} \to U(n) \to PU(n) \xrightarrow{\pi} 1 \tag{2}$$

of $PU(n)$. The Leray spectral sequence for the map $BU(n) \to PU(n)$ implies that $\beta$ is a generator of $H^2(PU(n), \mathbb{T}) \cong \mathbb{Z}_n$. When $n$ is prime, an alternate proof of this fact is given in Theorem 3.6 of [28].

These preliminaries completed, we now present the main result of this section.

**Theorem 4.6.** *Let $\mathcal{G}$ be an étale groupoid. Suppose that the classifying space $B\mathcal{G}$ is (homotopy equivalent to) a compact CW-complex. Let $\mathcal{R} \to \mathcal{G}$ be a twist of order $n$ over $\mathcal{G}$ such that the associated $PU(n)$-bundle $P$ of Proposition* 4.5 *lifts to a $U(n)$-bundle $\tilde{P}$ over $\mathcal{G}^{(0)}$. Then there is a twist $\mathcal{T}$ such that $[\mathcal{T}] = [\mathcal{R}] \in H^2(\mathcal{G}, \mathcal{S}^1)$ and such that $\mathcal{T}$ admits a twisted vector bundle.*

*Proof.* Recall from [20] that for all $s \in \mathbb{N}$, the inclusion $i : \mathcal{G}_\bullet \to B\mathcal{G}$ induces an isomorphism $i_s^* : H^s(B\mathcal{G}, \mathbb{T}) \to H^s(\mathcal{G}, \mathbb{T})$, for all $s \in \mathbb{N}$. Moreover, since $i$ is continuous, it also induces a pullback homomorphism $p_1 : H^1(B\mathcal{G}, PU(n)) \to H^1(\mathcal{G}_\bullet, PU(n))$, which need not be an isomorphism since $PU(n)$ is not abelian.

Write $v : H^1(B\mathcal{G}, PU(n)) \to H^2(B\mathcal{G}, \mathbb{T})$ for the Serre map which associates to a principal $PU(n)$-bundle over $B\mathcal{G}$ its Dixmier-Douady class in $H^2(B\mathcal{G}, \mathbb{T}) \cong H^3(B\mathcal{G}, \mathbb{Z})$. The Serre-Grothendieck Theorem (cf. [8], Theorem 8, [19], Theorem 7.2.11, [11], Theorem 1.6) establishes that

$$v : H^1(B\mathcal{G}, PU(n)) \to H^3(B\mathcal{G}, \mathbb{Z})$$

is an isomorphism onto the $n$-torsion subgroup of $H^3(B\mathcal{G}, \mathbb{Z})$ which is induced by the short exact sequence $\beta$ of equation (2).

If $P$ is the principal $PU(n)$-bundle over $\mathcal{G}$ which is associated to $\mathcal{R}$ by Proposition 4.5, examining the constructions employed in the proof of Proposition 4.5 reveals that

$$P = p_1 \circ v^{-1} \circ (i_2^*)^{-1}(\mathcal{R}).$$

Recall from p. 860 of [27] that we have a natural map

$$\Phi : H^1(\mathcal{G}_\bullet, PU(n)) \times H^2(PU(n), \mathcal{S}^1) \to H^2(\mathcal{G}, \mathcal{S}^1),$$

which arises from pulling back a principal $PU(n)$-bundle over $\mathcal{G}$ along a $\mathbb{T}$-central extension of $PU(n)$. We claim that

$$\Phi(P_\bullet, \beta) = [\mathcal{R}]. \tag{3}$$

Since $\Phi$ is natural, and taking pullbacks preserves cohomology classes, (3) holds because $v$ is induced by $\beta$, and $\beta$ generates $H^2(PU(n), \mathcal{S}^1)$.

We will now use the hypothesis that $P$ admits a lift to a principal $U(n)$-bundle $\tilde{P} \to \mathcal{G}^{(0)}$ to show that $\Phi(P_\bullet, \beta)$ is represented by a twist $\mathcal{T}$ which admits a twisted vector bundle. As explained in [27], pp. 860-861, this hypothesis allows us to construct an explicit representative $\mathcal{T}$ of $\Phi(P_\bullet, \beta)$ as follows.

By hypothesis, the quotient map $\pi : U(n) \to PU(n)$ induces a bundle morphism $\tilde{\pi} : \tilde{P} \to P$. Write $\frac{P \times P}{PU(n)}$ for the gauge groupoid of the bundle $P$, and notice that, if $\rho : P \to \mathcal{G}^{(0)}$ is the projection map of the principal bundle $P$, we can define a morphism $\varphi : \mathcal{G} \to \frac{P \times P}{PU(n)}$ as follows. Given $g \in \mathcal{G}$, choose $p \in P$ with $\rho(p) = s(g)$, and define

$$\varphi(g) = [g \cdot p, p].$$

The fact that $P$ is a principal $PU(n)$-bundle implies that $\varphi(g)$ is a well-defined groupoid homomorphism.

We define the twist $\mathcal{T}$ over $\mathcal{G}$ by

$$\mathcal{T} = \{([q_1, q_2], g) \in \frac{\tilde{P} \times \tilde{P}}{U(n)} \times \mathcal{G} : [\tilde{\pi}(q_1), \tilde{\pi}(q_2)] = \varphi(g)\}.$$

We observe that

$$([q_1, q_2], g) \in \mathcal{T} \Leftrightarrow g \cdot \tilde{\pi}(q_2) = \tilde{\pi}(q_1).$$

The backward implication is evident; for the forward implication, note that

$$([q_1, q_2], g) \in \mathcal{T} \Rightarrow \tilde{\pi}(q_2) \in P_{s(g)} \Rightarrow \varphi(g) = [g \cdot \tilde{\pi}(q_2), \tilde{\pi}(q_2)].$$

But also, $([q_1, q_2], g) \in \mathcal{T} \Rightarrow \varphi(g) = [\tilde{\pi}(q_1), \tilde{\pi}(q_2)]$. Note that

$$[\tilde{\pi}(q_1), \tilde{\pi}(q_2)] = [p^1, p^2] \Leftrightarrow \exists\, u \in PU(n) \text{ s.t. } \tilde{\pi}(q_i) = p^i \cdot u \;\forall\, i;$$

consequently, $g \cdot \tilde{\pi}(q_2) = \tilde{\pi}(q_1)$ as claimed.

The groupoid structure on $\mathcal{T}$ is given by

$$s([q_1, q_2], g) = s(g), \quad r([q_1, q_2], g) = r(g);$$

if $s(g) = r(h)$, then we define the multiplication by

$$([q_1, q_2], g)([p_1, p_2], h) = ([q_1 \cdot u, p_2], gh),$$

where $u \in U(n)$ is the unique element such that $q_2 \cdot u = p_1 \in \tilde{P}$.

Proposition 2.36 of [27] establishes that $\mathcal{T}$ is a twist over $\mathcal{G}$ such that $[\mathcal{T}] = \Phi(P_\bullet, \beta)$. The action of $\mathbb{T}$ on $\mathcal{T}$ is given by

$$z \cdot ([q_1, q_2], g) = ([q_1 \cdot z, q_2], g). \tag{4}$$

By construction, $\mathcal{T}$ admits a generalized homomorphism $\mathcal{T} \to U(n)$ which is $\mathbb{T}$-equivariant. To be precise, the bundle $\tilde{P}$ admits a left action of $\mathcal{T}$: if $\tilde{p} \in \tilde{P}$ lies in the fiber over $s(g)$, and $([q_1, q_2], g) \in \mathcal{T}$, there exists a unique $u \in U(n)$ such that $q_2 \cdot u = \tilde{p}$. Thus, we define

$$([q_1, q_2], g) \cdot \tilde{p} = q_1 \cdot u.$$

One checks immediately that this action is continuous, $\mathbb{T}$-equivariant, and commutes with the right action of $U(n)$ on $\tilde{P}$. In other words, the bundle $\tilde{P}$ equipped with this action constitutes a $\mathbb{T}$-equivariant generalized morphism $\mathcal{T} \to U(n)$. Thus, Proposition 5.5 of [27] explains how to construct a $(\mathcal{G}, \mathcal{T})$-twisted vector bundle. Since $[\mathcal{T}] = \Phi(P_\bullet, \beta) = [\mathcal{R}]$, this completes the proof. □

4.1. **Examples.** In this section, we present some examples establishing that the hypotheses of Theorem 4.6 are satisfied in many cases of interest.

**Example 4.7.** Let $M$ be a compact CW-complex, and let $\alpha$ be a homeomorphism of $M$. If we set $\mathcal{G} = M \rtimes_\alpha \mathbb{Z}$, the first paragraph of [29], Section 1.4.3, tells us that $B\mathcal{G} = M \times_{\mathbb{Z}} \mathbb{R}$. Since $M$ is compact, so is $B\mathcal{G}$.

**Example 4.8** (cf. [25], p. 273)**.** Let $\mathcal{F}$ be a foliation of a manifold $M$. The holonomy groupoid $\mathcal{H}_\mathcal{F}$ of $\mathcal{F}$ is an étale groupoid; moreover, if the leaves of the foliation all have contractible holonomy coverings, $B\mathcal{H}_\mathcal{F} = M$. Examples of such foliations include the Reeb foliation of $S^3$ and the Kronecker foliation of $\mathbb{T}^n$.

In particular, if $M$ is compact, any foliation $\mathcal{F}$ of $M$ with contractible leaves has an associated holonomy groupoid $\mathcal{H}_\mathcal{F}$ with $B\mathcal{H}_\mathcal{F}$ compact.

**Example 4.9.** Let $M := \mathbb{R}P^2 \times S^4$. We will identify $\mathbb{R}P^2$ with $D^1/\sim$, where (in polar coordinates) $D^1 = \{(\rho, \theta) \in \mathbb{R}^2 : 0 \leq \theta < 2\pi, 0 \leq \rho \leq 1\}$ and $(1, \theta) \sim (1, \theta + \pi)$.

Fix $x \in \mathbb{R} \backslash \mathbb{Q}$, and consider the homeomorphism $\alpha$ of $\mathbb{R}P^2 \times S^4$ given by

$$\alpha([\rho, \theta], z) = ([\rho, \theta + (1-\rho)x], z).$$

Let $\mathcal{G} = M \rtimes_\alpha \mathbb{Z}$. Since $M$ is compact, Example 4.7 tells us that $B\mathcal{G}$ is compact as well.

By the Künneth Theorem, $\mathbb{Z}/2\mathbb{Z} \cong H^2(\mathbb{R}P^2, \mathbb{Z}) \otimes H^0(S^4, \mathbb{Z})$ is a subgroup of $H^2(M, \mathbb{Z}) \cong H^1(M, \mathbb{T})$. The groupoid $\mathcal{G}$ is an example of a Renault-Deaconu groupoid (cf. [6, 7, 12]); thus, by Theorem 2.2 of [7], twists over $\mathcal{G} = M \times_\alpha \mathbb{Z}$ are classified by $H^1(M, \mathbb{T})$. It follows that $\mathcal{G}$ admits nontrivial torsion twists.

The short exact sequence $1 \to \mathbb{T} \to U(n) \to PU(n) \to 1$ tells us that the obstruction to a principal $PU(n)$-bundle over $M$ (an element of $H^1(M, PU(n))$) lifting to a principal $U(n)$-bundle over $M$ lies in $H^2(M, \mathbb{T}) \cong H^3(M, \mathbb{Z})$. However, the Künneth Theorem tells us that

$$H^3(M, \mathbb{Z}) \cong H^3(\mathbb{R}P^2, \mathbb{Z}) \otimes H^0(S^4) \cong H^3(\mathbb{R}P^2, \mathbb{Z}) = 0.$$

In other words, every principal $PU(n)$-bundle over $M$ lifts to a principal $U(n)$-bundle over $M$, so every torsion twist over $\mathcal{G} = M \times_\alpha \mathbb{Z}$ satisfies the hypotheses of Theorem 4.6.

Furthermore, since the action of $\mathbb{Z}$ on $M$ is not proper, this example lies outside the cases (cf. [9,27]) where it was previously known that torsion twists admit twisted vector bundles.

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