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MINIMAL SETS FOR GROUP ACTIONS ON DENDRITES

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ABSTRACT. Let G be a group acting by homeomorphisms on a dendrite X. First, we show that any minimal set M of G is either a finite orbit or a Cantor set (resp. a finite orbit) when the set of endpoints of X is closed (resp. countable). Furthermore, we prove, regardless of the type of the dendrite X, that if the action of G on X has at least two minimal sets, then necessarily it has a finite orbit (and even an orbit consisting of one or two points). Second, we explore the topological and geometrical properties of infinite minimal sets when the action of G has a finite orbit. We show in this case that any infinite minimal set M is a Cantor set which is the set of endpoints of its convex hull $[M]$ and there is no other infinite minimal set in $[M]$. On the other hand, we consider the family $\mathcal{M}$ of all minimal sets in the hyperspace 2^X (endowed with the Hausdorff metric). We prove that $\mathcal{M}$ is closed in 2^X and that the family $\mathcal{F}$ of all finite orbits (when it is non-empty) is dense in $\mathcal{M}$. As a consequence, the union of all minimal sets of G is closed in X.

1. Introduction

Let X be a compact metric space with a metric d and let G be a discrete group. By an action of G on X we mean a continuous map $\varphi : G \times X \longrightarrow X$ satisfying $\varphi(e,x) = x$ and $\varphi(g_1g_2,x) = \varphi(g_1,\varphi(g_2,x))$ for all $x \in X$, and all $g_1, g_2 \in G$ where e is the identity of G. For convenience we often use $g(x)$ to denote $\varphi(g,x)$. Obviously for each $g \in G$, the map $g : X \longrightarrow X; x \longmapsto g(x)$ is a homeomorphism of X. For any $x \in X$, the subset $G(x) = \{g(x) : g \in G\}$ is called the orbit of x under G. A subset Λ of X is called *G-invariant* if $g(\Lambda) = \Lambda$, for every $g \in G$. A closed G-invariant set Λ is *minimal* if its only G-invariant closed subsets are the empty set and Λ itself; this is equivalent to saying that each orbit in Λ is dense; for example, a finite orbit is a minimal set. When X itself is a minimal set, then we say that the action of G on X is *minimal*. One of the objectives of the theory of dynamical systems has been to characterize the topological structure of minimal sets. It is well known and easy to prove by Zorn's lemma that every group action on a compact metric space must have a minimal set. In this paper, we study minimal sets for group actions on dendrites. Recent interest in dynamics on dendrites is motivated by the fact that dendrites often appear as Julia sets in complex dynamics (see [6]). On the other hand, dendrites are examples of Peano continua with complex topological structures (e.g., [13], pp. 165–187). For continuous maps on dendrites, a full characterization of minimal sets was given by Balibrea et al. in [5]. For group actions on dendrites, several results related to sensitivity and existence of

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global fixed points have been obtained by some authors (see e.g., [18], [11], [19], [20]). Existence of minimal group actions on dendrites can occur in the study of 3-hyperbolic geometry (see [17], p. 601). These facts, among others, motivate us to explore the structure of minimal sets for group actions on dendrites.

The plan of the paper is as follows. In Section 2, we give some preliminary results which are useful for the rest of the paper. In Section 3, we deal with group actions on dendrites X with the set of endpoints closed or countable and prove that any minimal set is either a finite orbit or a Cantor set if the set of endpoints of X is closed, and is a finite orbit if the set of endpoints is countable. In Section 4, we give a dynamical criterion for the existence of finite orbit. Section 5 is devoted to minimal sets for group action admitting a finite orbit. In Section 6, we prove that, for a group action on a dendrite X, the family of all minimal sets is closed in the hyperspace 2^X under the Hausdorff metric and the union of all minimal sets is closed in X.

2. Preliminaries and some results

In this section, we recall some basic properties of dendrites. A continuum is a compact connected metric space. An arc is any space homeomorphic to the compact interval $[0, 1]$. A topological space is arcwise connected if any two of its points can be joined by an arc. We use the terminologies from Nadler [13].

By a *dendrite* X, we mean a locally connected continuum containing no homeomorphic copy to a circle. Every sub-continuum of a dendrite is a dendrite ([13], Theorem 10.10) and every connected subset of X is arcwise connected ([13], Proposition 10.9). In addition, any two distinct points x, y of a dendrite X can be joined by a unique arc with endpoints x and y, which is denoted by $[x, y]$. We denote $[x, y) = [x, y] \setminus \{y\}$ (resp. $(x, y] = [x, y] \setminus \{x\}$ and $(x, y) = [x, y] \setminus \{x, y\}$). A point $x \in X$ is called an *endpoint* if $X \setminus \{x\}$ is connected. It is called a *branch point* if $X \setminus \{x\}$ has more than two connected components. The number of connected components of $X \setminus \{x\}$ is called the *order* of x. We denote by $E(X)$ and $B(X)$ the sets of endpoints and branch points of X respectively. A point $x \in X \setminus E(X)$ is called a *cut point*. The set of cut points of X is dense in X. Following [2], Corollary 3.6, for any dendrite X, we have $B(X)$ is discrete whenever $E(X)$ is closed. By ([13], Theorem 10.23), $B(X)$ is at most countable.

Let X be a dendrite. For a subset A of X, we denote by $\overline{A}$ the closure of A and by $\mathrm{diam}(A)$ the diameter of A. We call *the convex hull* of A, denoted by $[A]$, the intersection of all sub-continuums of X containing A. For a dendrite $A \subset X$, define the *first point map* (or retraction) $r_A : X \to A$ by letting $r_A(x) = x$, if $x \in A$, and by letting $r_A(x)$ be a unique point $r_A(x) \in A$ such that $r_A(x)$ is a point of any arc in X from x to any point of A, if $x \notin A$ (see [13], 10.26, p. 176). Note that the map r_A is monotone (i.e. the preimage by r_A of any connected subset of A is a connected subset of X) and it is constant on each connected component of $X \setminus A$.

Lemma 2.1 ([3], Lemma 2.3)**.** *Let X be a dendrite and let $(C_i)_{i\in\mathbb{N}}$ be a sequence of pairwise disjoint connected subsets of X. Then* $\lim\limits_{n\to+\infty} \mathrm{diam}(C_n) = 0$.

Lemma 2.2 ([12], Lemma 2.2)**.** *Let $[a, b]$ be an arc in a dendrite (X, d) and let $w \in [a, b)$. There is $\delta > 0$ such that if $v \in X$ with $d(v, b) \leq \delta$, then $[v, a] \supset [w, a]$.*

Lemma 2.3. *Let X be a dendrite and let U and V be two disjoint non-empty connected subsets of X. Then $\overline{U} \cap \overline{V}$ contains at most one point.*

Proof. Suppose that $\overline{U} \cap \overline{V}$ contains two distinct points a, b. Then the arc $[a, b] \subset \overline{U} \cap \overline{V}$. Let $w \in (a, b)$ and $w' \in (w, b)$. Then by Lemma 2.2, there is $\delta > 0$ and $u, u' \in U$ with $d(u, a) < \delta$ and $d(u', b) < \delta$ such that $[u, b] \supset [w, b]$ and $[u', a] \supset [w', a]$. Hence $[w, w'] \subset [u, u'] \subset U$. In the same way, there are $v, v' \in V$ such that $[w, w'] \subset [v, v'] \subset V$. Therefore $[w, w'] \subset U \cap V$, and so U and V are not disjoint, a contradiction. $\square$

Lemma 2.4 ([18])**.** *Let X be a dendrite with metric d. Then for any $\varepsilon > 0$ there is $0 < \delta < \varepsilon$ such that if $d(x, y) < \delta$, then* $\operatorname{diam}([x, y]) < \varepsilon$.

Lemma 2.5 ([15])**.** *Let X be a dendrite with a countable set of endpoints. Then every sub-dendrite of X has a countable set of endpoints.*

Lemma 2.6 ([2])**.** *Let X be a dendrite with a closed set of endpoints. Then we have:*

(i) $\overline{B(X)} \subset B(X) \cup E(X)$.
(ii) *Every sub-dendrite of X has a closed set of endpoints.*

From [2], Theorem 3.3, we easily deduce the following lemma.

Lemma 2.7. *The order of every branch point of a dendrite with a closed set of endpoints is finite.*

The following lemma is trivial (see also [14]).

Lemma 2.8. *Let X be a dendrite and let $f : X \to X$ be a homeomorphism. Then:*

(i) $f(B(X)) = B(X)$.
(ii) $f(E(X)) = E(X)$.

Lemma 2.9 (Interval case)**.** *Let G be a group acting on the closed interval I and let $M \subset I$ be a minimal set of G. Then M is a finite orbit (in fact a single point or two points).*

Proof. Let $J = [M]$ be the convex hull of M. It is an interval with $\gamma(J) = \{a, b\} \subset M$. For every $g \in G$, $g(J) = J$ and hence $g(a) \in \{a, b\}$. Thus $G(a)$ is finite and so $M = G(a)$. $\square$

3. Minimal sets for group actions on dendrites with set of endpoints closed or countable

The aim of this section is to prove the following.

Theorem 3.1. *Let a group G act on a dendrite X and let M be a minimal set of G. Then M is*

(i) *a finite orbit, if $E(X)$ is countable,*
(ii) *a finite orbit, or a Cantor set included into $E(X)$, if $E(X)$ is closed.*

Proof. Let $D = [M]$ be the convex hull of M. Then D is a G-invariant sub-dendrite of X and we have $E(D) \subset M$.

(i) Assume that $E(X)$ is countable. By Lemma 2.5, $E(D)$ is countable. By [7], Theorem 5.5, $E(D)$ has an isolated point e. As $\overline{G(e)} = M$ and $G(e) \subset E(D)$, then e is isolated in M and hence $M = G(e)$ is finite.

(ii) Assume that $E(X)$ is closed. Then by Lemma 2.6, $E(D)$ is closed. By Lemma 2.8, $E(D)$ is G-invariant. As $E(D) \subset M$, then by the minimality of M, we get $E(D) = M$. So M is totally disconnected. If M is infinite, then it has no isolated point and so it is a Cantor set. We show in this case that $M \subset E(X)$. Otherwise, $M \cap E(X) = \emptyset$ and by Lemma 2.6, D contains only finitely many branch points of X. As all branch points of X have finite order, so D is a tree and hence $M = E(D)$ is finite, a contradiction. □

4. A dynamical criterion for the existence of finite orbit

The question of whether there is a global fixed point or a finite orbit for a group G acting on a given topological space X is of wide interest. Some authors have proved the existence of a global fixed point by assuming algebraic conditions on G. For instance, Isbell [9] showed the existence of a global fixed point for the action of a commutative group G on a dendrite X. Shi and Sun [19] proved that the same holds for a nilpotent group G acting on a uniquely arcwise connected continuum X. Later, Shi and Zhou [20] proved the existence of finite orbits with at most two points for solvable continuous groups G acting on a uniquely arcwise connected continuum X. Here we give, in the following theorem, a dynamical criterion for the existence of a finite orbit.

Theorem 4.1. *Let X be a dendrite and let a group G act on X. If the action admits at least two minimal sets, then it has a finite orbit.*

Proposition 4.2. *Let T be a tree and let a group G act on T. Then there exists a finite orbit containing at most two points.*

The proof of Proposition 4.2 is by induction on the number of endpoints of T.

Proof of Theorem 4.1. Suppose that M_1 and M_2 are two infinite minimal sets of G. Then $M_1 \cap M_2 = \emptyset$. So we have that $\delta := \inf\{d(x_1, x_2) : x_1 \in M_1,\ x_2 \in M_2\} > 0$. Since X is connected, $X \neq M_1 \cup M_2$. There exists a connected component C of $X \setminus (M_1 \cup M_2)$ such that its boundary $\overline{C} \setminus C$ contains a point a from M_1 and another point b from M_2; indeed, take two points $c_1 \in M_1$ and $c_2 \in M_2$ and let $(a_1, c_2]$ be the connected component of $[c_1, c_2] \setminus M_1$ containing c_2. So $a_1 \in M_1$. Now let $[a_1, a_2)$ be the connected component of $[a_1, c_2] \setminus M_2$ containing a_1. So $a_2 \in M_2$. It suffices to take C containing (a_1, a_2). For any $g \in G$, $g(C)$ is also a connected component of $X \setminus (M_1 \cup M_2)$ with $\operatorname{diam}(g(C)) \geq \delta$ (since $d(g(a), g(b)) \geq \delta$). By Lemma 2.1, the set $G(C) = \{g(C) : g \in G\}$ is finite, so write $G(C) = \{C_1, \dots, C_k\}$. Let us show that $k = 1$; that is, C is G-invariant. Indeed, suppose that $k > 1$. Since $C_1, \dots, C_k$ are pairwise open connected sets, so by Lemma 2.3, $\overline{C_1}, \dots, \overline{C_k}$ is a family of sub-dendrites which intersects in at most one point. We denote by D the convex hull of $\bigcup_{i=1}^k C_i$. Then D is G-invariant and so is $D \setminus \bigcup_{i=1}^k C_i$. Moreover, $D \setminus \bigcup_{i=1}^k C_i$ is a finite union of points in $M_1 \cup M_2$ and/or arcs with endpoints in $M_1 \cup M_2$. In either case, from the minimality of M_1 and M_2, one of them, say M_1, is included in $D \setminus \bigcup_{i=1}^k C_i$. This implies that M_1 is finite, which is a contradiction. We conclude that $k = 1$. Since $a \in M_1 \cap \overline{C}$ and $b \in M_2 \cap \overline{C}$, and as C is G-invariant, so by minimality of both M_1 and M_2, we get $M_1 \cup M_2 \subset \overline{C} \setminus C$. This implies that $M_1 \cup M_2 \subset \operatorname{End}(\overline{C})$. Let H be the convex hull of $M_1 \cup M_2$. It is easy to see that H is a G-invariant sub-dendrite of $\overline{C}$ with a set of endpoints $M_1 \cup M_2$. Let H_1 (resp. H_2) be the convex hull of M_1 (resp. M_2). Again, H_1 (resp. H_2) is a G-invariant sub-dendrite of H with a set of endpoints M_1 (resp. M_2). We distinguish two cases.

Case 1. $H_1 \cap H_2 \neq \emptyset$. In this case, $H_1 \cap H_2$ will be a G-invariant sub-dendrite of H containing no endpoint of H. Since H is a dendrite with a closed set of endpoints, so by Lemmas 2.6 and 2.7, $H_1 \cap H_2$ has a finite set of branch points with finite order. Therefore, $H_1 \cap H_2$ is a tree and by Proposition 4.2, it contains a finite orbit.

Case 2. $H_1 \cap H_2 = \emptyset$. In this case, $H = H_1 \cup H_2 \cup (u, v)$, where $u = r_{H_1}(y)$ for some $y \in H_2$. Recall that u is well defined since H_2 is included in a connected component of $X \setminus H_1$, hence r_{H_1} is constant on H_2. Similarly $v = r_{H_2}(z)$ for some $z \in H_1$. Obviously, the arc $[u, v]$ is G-invariant and so it contains a finite orbit. The proof is done. □

Corollary 4.3. *If the action of a group G on a dendrite X has no finite orbits, then it has a unique minimal set.*

5. Minimal sets for group actions on dendrites with a finite orbit

The aim of this section is to give some geometrical properties of infinite minimal subsets when finite orbits occur.

Proposition 5.1. *Let X be a dendrite and let a group G act on X. If finite orbits exist, then at least one of them consists of one or two points.*

Proof. Let Λ be a finite orbit and let $T := [\Lambda]$ be its convex hull. Then T is a tree and G-invariant. By Proposition 4.2, there is a finite orbit in T with ≤ 2 points. □

Remark 5.2. The hypothesis "existence of a finite orbit" can be reduced to the hypothesis "existence of a global fixed point" without affecting the global dynamical behavior of the system. Indeed, by Lemma 5.1, we assume that there is a finite orbit $\{\gamma_1, \gamma_2\}$ with $\gamma_1 \neq \gamma_2$. Then it is easy to see that the arc $[\gamma_1, \gamma_2]$ is G-invariant. By collapsing this arc to a point γ, this point will be a global fixed point for the new generated action which is a little bit modified from the old one.

The following lemma describes how points from a minimal set are located; it can be interpreted as a geometrical structure.

Lemma 5.3. *Let X be a dendrite and let a group G act on X. Assume that the action has a finite orbit. Then there is no arc in X containing three points from a same minimal set.*

Proof. By Proposition 5.1, there is a finite orbit with ≤ 2 points, say $\{\gamma_1, \gamma_2\}$. Clearly, the set arc $[\gamma_1, \gamma_2]$ is G-invariant. Suppose the lemma were false. Then we could find an arc containing three points from a minimal set M. Then we have $M \cap [\gamma_1, \gamma_2] = \emptyset$. Necessarily, there are two distinct points $a, b \in M$ in the same connected component of $X \setminus [\gamma_1, \gamma_2]$ and so $r_{[\gamma_1,\gamma_2]}([a, b]) = \{r_{[\gamma_1,\gamma_2]}(a)\} := \{\gamma\}$. It follows that the points γ, a, b are colinear (i.e. lie on a common arc). Without loss of generality, we can assume that $b \in (\gamma, a)$. Let $[\gamma, c)$ be the component of $[\gamma, b] \setminus M$ containing γ. Choose a neighborhood U of a such that for any $x \in U$, $[\gamma, x] \supsetneq [\gamma, b]$. Since $c \in M$ and $a \in M = \overline{G(c)}$, so there is $g \in G$ such that $g(c) \in U$. It follows that $g([\gamma, c]) = [g(\gamma), g(c)] \supset [\gamma, g(c)] \supsetneq [\gamma, b]$. Hence $g^{-1}(b) \in (\gamma, c)$, which is a contradiction with the fact that $M \cap [\gamma, c) = \emptyset$. □

Gathering all the results above we obtain the following characterization of minimal sets for group actions with a finite orbit on dendrites.

Theorem 5.4. *Let X be a dendrite, let a group G act on X, and let M be a minimal set of G. Let $[M]$ be the convex hull of M. Assume that the action of G has a finite orbit. Then*

(i) *M is the set of endpoints of $[M]$.*
(ii) *For any point $a \in X$ in a finite orbit, the point $r_{[M]}(a)$ is in a finite orbit.*
(iii) *$[M]$ contains at least one finite orbit consisting of one or two points.*
(iv) *If M is infinite, then it is the only infinite minimal subset in $[M]$.*

Proof. Assertion (i) follows immediately from Lemma 5.3 since $[M] = \bigcup_{x,y\in M}[x, y]$. Assertion (ii): Set $b = r_{[M]}(a)$. If $a \in [M]$, then it is clear that $b = a$. So assume that $a \notin [M]$ and suppose that the orbit $G(b)$ is infinite. As $G(b) \subset [M]$ and the orbit $G(a)$ is finite and disjoint from $[M]$, so there is $g \in G$ such that $r_{[M]}(g(a)) \neq g(b)$. Hence the point $c := r_{[M]}(g(a)) \in (g(a), g(b))$ and so $g^{-1}(c) \in (a, b)$. It follows that $g^{-1}(c) \notin [M]$. Nevertheless, $c \in [M]$ and so is $g^{-1}(c)$, a contradiction. Assertion (iii) follows from assertion (ii) since there is a point a in a finite orbit by hypothesis and so $r_{[M]}(a) \in [M]$ is in a finite orbit. Assertion (iv): Since by (i), M is the set $E([M])$ of endpoints of $[M]$, it follows that $E([M])$ is closed. Moreover we have that $[M]$ is a G-invariant sub-dendrite of X. If there is a minimal set N included in $[M]$ and distinct from M, then N is not included in $M = E([M])$. So by Theorem 3.1, N is finite. □

Corollary 5.5. *Let X be a dendrite, let a group G act on X, and let M be a minimal set of G. Assume that the action has a finite orbit. Then M is either a finite orbit or a Cantor set.*

Remark 5.6. A Cantor set can be realized as a minimal set even for a group generated by a single homeomorphism acting on a dendrite. Indeed, Efremova and Makhrova [8] constructed a homeomorphism g of the Gehman dendrite X having an infinite ω-limit set E. By [16], E is a minimal Cantor set, so it is a minimal set of the group generated by g. Notice that in this example, E is the only infinite minimal set, which is moreover the set of endpoints of X as was described in Theorem 5.4.

Following [20], a solvable group acting continuously on a dendrite (or more generally on a uniquely arcwise connected continuum) has either a global fixed point or a finite orbit consisting of one or two points.

Corollary 5.7. *If G is solvable, then any minimal set of G is either finite or a Cantor set.*

Proposition 5.8. *Let M be a minimal set for an action of a group G on a dendrite X and let D be a G-invariant sub-dendrite of X. If M is not included into D, then $r_D(M)$ is a finite orbit.*

Proof. By the minimality of M, $M \cap D = \emptyset$. By the continuity of the mapping r_D, the set $r_D(M)$ is closed. Let $a', b' \in M$, $a = r_D(a')$ and $b = r_D(b')$. Suppose that C_a (resp. C_b) is the connected component of $X \setminus D$ containing a' (resp. b'). Then C_b is an open neighborhood of b' and by the minimality of M, there is $g \in G$ such that $g(a') \in C_b$. Therefore, $g(C_a) = C_b$ and $g(a) = b$. Consequently, the set $r_D(M)$ is a closed orbit. In addition, the number of components of $X \setminus D$ is at most countable. This implies that $r_D(M)$ is also at most countable and then it is a finite set. □

Corollary 5.9. *Let M and N be two different minimal sets for an action of a group G on a dendrite X. Then $r_{[M]}(N)$ is a finite orbit.*

Proof. If N intersects $[M]$, then by the fact that $[M]$ is G-invariant, we get $N \subset [M]$. Hence if M is finite, then its convex hull $[M]$ is a tree and so N is also finite. Assume that M is infinite. By Theorem 4.1, finite orbits exist. Then by Theorem 5.4, there is only one infinite minimal set in $[M]$ which is M itself. As $N \neq M$, so N is finite. If now N is disjoint from $[M]$, then by Proposition 5.8, $r_{[M]}(N)$ is a finite set. $\square$

6. The family of minimal sets in the hyperspace

Let (X, d) be a compact metric space. The Hausdorff distance $d_H(A, B)$ between two subsets A and B is defined as follows:

$$d_H(A, B) := \max\left(\sup_{a\in A} d(a, B),\ \sup_{b\in B} d(b, A)\right),$$

where for every $x \in X$, $d(x, C) := \inf_{c\in C} d(x, c)$ for any subset C of X. We denote by 2^X the hyperspace of all closed subsets of X. It is well known that 2^X endowed with the Hausdorff metric d_H is a compact metric space (see [13]).

Here X is a dendrite with a metric d and G is a group acting by homeomorphisms on X. We denote by $\mathcal{M}$ the family of all minimal sets of G and $\mathcal{F}$ the family of all finite orbits.

6.1. **Finite orbits in the convex hull of an infinite minimal set.** In this subsection, we will assume that the action of G on X has a finite orbit. Our aim is to prove that any infinite minimal set is in fact a limit of finite orbits.

Theorem 6.1. *The family $\mathcal{F}$ of finite orbits is dense in $\mathcal{M}$ with respect to the Hausdorff metric.*

Let M be a minimal set of G. We denote by $D := [M]$. Any retraction map mentioned in this subsection is considered as a map defined on D into itself except in Lemma 6.4. By Theorem 5.4, there is a finite orbit in D with ≤ 2 points.

Lemma 6.2. *Let $\Gamma = \{\gamma_1, \dots, \gamma_k\}$ be a finite orbit in D disjoint from M and let $T = [\Gamma]$ be its convex hull. Then $D \setminus T$ has $k(p-1)$ components $C_1^1, \dots, C_1^{p-1}; \dots; C_k^1, \dots, C_k^{p-1}$, where $p \geq 2$ is the order of γ_1 and for any $i = 1, \dots, k$ and $j = 1, \dots, p-1$, $r_T(C_i^j) = \{\gamma_i\}$.*

Proof. By Theorem 5.4, T is a tree and $E(T) = \Gamma$. As the points $\gamma_1, \dots, \gamma_k$ belong to the same orbit, they have the same order $p \geq 2$. For each $i = 1, \dots, k$, we denote by $C_i^1, \dots, C_i^{p-1}$ the connected components of $D \setminus T$ such that $\overline{C_i^j} \cap T = \{\gamma_i\}$. So C_i^j is open in D, $r_T(C_i^j) = \{\gamma_i\}$ and for any $(i, j) \in \{1, \dots, k\} \times \{1, \dots, p-1\}$, we have $C_i^j \cap M \neq \emptyset$. Thus for any (i, j); $(i', j') \in \{1, \dots, k\} \times \{1, \dots, p-1\}$, there is $g \in G$ such that $g(C_i^j) = C_{i'}^{j'}$. Hence for any $(i, j) \in \{1, \dots, k\} \times \{1, \dots, p-1\}$, we have $G(C_i^j) = (C_1^1 \cup \cdots \cup C_1^{p-1}) \cup \cdots \cup (C_k^1 \cup \cdots \cup C_k^{p-1})$. It remains to show that $C_1^1, \dots, C_1^{p-1}; \dots; C_k^1, \dots, C_k^{p-1}$ are the only connected components of $D \setminus T$. Indeed, if C is a connected component of $D \setminus T$, then $C \cap M \neq \emptyset$. So let $c \in C \cap M$. As $C_1^1 \cap M \neq \emptyset$, there is $a \in M \cap C_1^1$. Hence $c \in \overline{G(a)} \subset \overline{G(C_1^1)} \cap M \subset G(C_i^j)$ and so $c \in C_{i'}^{j'}$ for some i', j'. Therefore $C = C_{i'}^{j'}$. $\square$

Proposition 6.3. *Let M be a minimal set of G. If M is infinite, then there exists a sequence $(T_n)_{n\in\mathbb{N}}$ of sub-trees in D satisfying the following properties:*

(i) $T_n \subset T_{n+1}$, $\forall n \in \mathbb{N}$.
(ii) $E(T_n) \subset B(D)$ *is a finite orbit,* $\forall n \in \mathbb{N}$.
(iii) $D = \overline{\bigcup_{n\in\mathbb{N}} T_n}$.

Proof. First, we construct T_1. By Theorem 5.4, there is a finite orbit in D with ≤ 2 points, noted $\{\gamma_1, \gamma_2\}$. By Lemma 6.2, we can choose $T_1 = [\beta_1, \beta_2]$ to be the arc in D containing $[\gamma_1, \gamma_2]$ and such that $T_1 \cap B(D) = \{\beta_1, \beta_2\}$. If $\gamma_1 = \gamma_2$, take $T_1 = \{\gamma_1\}$. When $\gamma_1 \neq \gamma_2$, it is easy to show that T_1 is G-invariant and that $\{\beta_1, \beta_2\}$ is a finite orbit. Now we construct, by induction, the other trees T_n, $n \geq 2$. Suppose that T_n was already constructed and set $E(T_n) = \{\beta_1, \dots, \beta_{k_n}\}$. As the points $\beta_1, \dots, \beta_{k_n}$ lie in the same orbit, they have the same order $p_n \geq 3$. Moreover, as $B(D)$ is discrete, then for each $i = 1, \dots, k_n$, there exist $p_n - 1$ branch points $\beta_{i,1}, \dots, \beta_{i,p_n-1}$ of D such that $[\beta_i, \beta_{ij}] \cap B(D) = \{\beta_i, \beta_{i,j}\}$ and $[\beta_i, \beta_{i,j}] \cap T_n = \{\beta_i\}$ for each $j = 1, \dots, p_n - 1$. We let

$$T_{n+1} = \bigcup_{1\leq i\leq k_n,\ 1\leq j\leq p_n-1} [\beta_i, \beta_{i,j}] \cup T_n.$$

So $T_n \subset T_{n+1}$. Also $E(T_{n+1}) = \{\beta_{i,j} : 1 \leq i \leq k_n,\ 1 \leq j \leq p_n - 1\}$ is G-invariant, so each point $\beta_{i,j}$ has a finite orbit $\Gamma_{i,j}$. We shall prove that $\Gamma_{i,j} = E(T_{n+1})$: Suppose that $\Gamma_{i,j} \subsetneq E(T_{n+1})$. So there is some $\beta_{i',j'}$ with $r_{[\Gamma_{i,j}]}(b_{i',j'}) \notin E([\Gamma_{i,j}])$. Hence the connected component of $D \setminus [\Gamma_{i,j}]$ containing $\beta_{i',j'}$ is different from any connected component of $D \setminus [\Gamma_{i,j}]$ containing in its closure a point from the orbit of $\beta_{i,j}$, which is a contradiction with Lemma 6.2. As a result, $\Gamma_{i,j} = E(T_{n+1})$ and condition (ii) is fulfilled.

Now, we are going to prove that $D = \overline{\bigcup_{n\in\mathbb{N}} T_n}$: Let $Z = \overline{\bigcup_{n\in\mathbb{N}} T_n}$. Then Z is a sub-dendrite of D with a closed set of endpoints, so $E(Z)$ is G-invariant. By construction, $(E(T_n))_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint subsets of $B(D)$. Hence $\bigcup_{n\in\mathbb{N}} E(T_n)$ is infinite which implies that $E(Z) \cap E(D) \neq \emptyset$. Recall that $M = E(D)$. Thus $M = E(D) = E(Z)$, equivalently $Z = D$. □

Lemma 6.4. *Let X be a dendrite with a closed set of endpoints $E(X)$ and let $(X_n)_{n\in\mathbb{N}}$ be a sequence of trees in X satisfying the following properties:*

(a) $X = \overline{\bigcup_{n\in\mathbb{N}} X_n}$.
(b) $X_n \subset X_{n+1}$, $\forall n \in \mathbb{N}$.
(c) *For any connected component C of $X \setminus X_n$, $r_{X_n}(C) = \{e_n\} \subset E(X_n)$.*

Then $\lim_{n\to+\infty} E(X_n) = E(X)$ *(in the Hausdorff metric).*

Proof. Let $\varepsilon > 0$. By Lemma 2.4, there exists $0 < \delta < \varepsilon$ such that if $d(x, y) < \delta$, then $\operatorname{diam}([x, y]) < \varepsilon$. By (a), it is easy to see that $\lim_{n\to+\infty} X_n = X$. Then there is $n_0 > 0$ such that $d_H(X_n, X) < \delta$ for $n \geq n_0$. Take $n \geq n_0$ and let $e \in E(X)$. We shall prove that $d(e, E(X_n)) < \varepsilon$: If $e \in E(X_n)$, then clearly $d(e, E(X_n)) = 0$. Otherwise, the point e belongs to some connected component of $X \setminus X_n$. Thus by (b), $e_n := r_{X_n}(e) \in E(X_n)$ and so $e_n \in [e, x]$ for any $x \in X_n$. Let $x \in X_n$ be such that $d(e, x) < \delta$. Then we have $d(e, e_n) \leq \operatorname{diam}([e, x]) < \varepsilon$ and therefore $d(e, E(X_n)) < \varepsilon$. Now take $e_n \in E(X_n)$. We shall prove that $d(e_n, E(X)) < \varepsilon$: If $e_n \in E(X)$, then clearly $d(e_n, E(X)) = 0$. If $e_n \notin E(X)$, then let C be a connected component of $X \setminus X_n$ such that $\overline{C} \cap X_n = \{e_n\}$. Then for any $x \in X_n$

and $e \in C \cap E(X)$, we have $e_n \in [e, x]$. So take any point $e \in C \cap E(X)$ and let $x \in X_n$ be such that $d(e, x) < \delta$. Then $\mathrm{diam}([e, x]) < \varepsilon$ and so $d(e, e_n) < \varepsilon$. It follows that $d(e_n, E(X)) < \varepsilon$. As a result, we conclude that $d_H(E(X_n), E(X)) < \varepsilon$ for any $n \geq n_0$ and therefore $\lim\limits_{n\to+\infty} d_H(E(X_n), E(X)) = 0$. □

Proposition 6.5. *Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of trees as in Proposition* 6.3. *Then* $\lim\limits_{n\to+\infty} E(T_n) = M$ *(in the Hausdorff metric).*

Proof. By Lemma 6.2, for any $n \in \mathbb{N}$ and for any connected component C of $D \setminus T_n$, $r_{T_n}(C) = \{e_n\} \subset E(T_n)$. Hence Proposition 6.5 follows immediately from Proposition 6.3 and Lemma 6.4. □

Proof of Theorem 6.1. Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of trees as in Proposition 6.3. Since the $(E(T_n))_n$ is a sequence of finite orbits, so $(E(T_n))_n \subset \mathcal{F}$ and therefore the theorem follows directly from Proposition 6.5.

Definition 6.6. Let G be a group acting on a metric space (X, d). We say that the action of G on X is *distal* if $\inf\limits_{g\in G}(d(g(a), g(b))) > 0$ for every $(a, b) \in X \times X$ with $a \neq b$.

As a consequence we have

Corollary 6.7. *If G has a finite orbit, then the action of G on each minimal set M is distal.*

Proof. It is plain that the action of G on M is distal when M is finite. Suppose that M is infinite and let $a, b \in M$ with $a \neq b$. From Lemma 6.2 and Proposition 6.5, the arc $[a, b]$ contains two distinct points from the finite orbit $E(T_n)$, for some $n \in \mathbb{N}$. It follows that (a, b) is a distal pair, $\inf\limits_{g\in G}(d(g(a), g(b))) > 0$. □

It is easy to see that equicontinuity implies distality. It was shown by Auslander et al. [4] that the converse is true for finitely generated groups acting on zero-dimensional compact metric spaces. As a consequence, we obtain the following.

Corollary 6.8. *If G is a finitely generated group acting on a dendrite X having a finite orbit, then the action of G on each minimal set M is equicontinuous.*

Proof. Indeed, in this case, any minimal set is either finite or a Cantor set (Corollary 5.5). So it is of zero dimension and we conclude by [4], Corollary 1.9. □

6.2. **The family of minimal sets in the hyperspace.** The aim of this subsection is to prove that the family $\mathcal{M}$ of all minimal sets for the action G on X is closed in 2^X. In other words, any convergent sequence of minimal sets in 2^X converges to a minimal set.

Theorem 6.9. *For any group G acting on a dendrite X, the family $\mathcal{M}$ of all minimal sets of G is closed in 2^X.*

Proof. If there is only one minimal set in X, the theorem is obvious. So assume that there are at least two minimal sets. Then the action of G on X has a finite orbit. Let $(M_n)_n$ be a sequence of minimal sets of G that converges to M in the Hausdorff metric. By Theorem 6.1, one can assume that every M_n is finite and hence its convex hull $[M_n]$ is a tree. By compactness of the hyperspace $(2^X, d_H)$,

M is a non-empty closed set. Furthermore, we easily check that M is G-invariant. Now we shall prove that M is minimal.

We denote $D = [M]$. Then D is a G-invariant sub-dendrite of X. For each $n \in \mathbb{N}$, we let $N_n = r_D(M_n)$. Then by Proposition 5.8, N_n is a finite orbit included into D and therefore $[N_n] \subset D$.

Claim 1. $\lim_{n\to+\infty} N_n = M$ in Hausdorff metric.

Let $\varepsilon > 0$ and $0 < \delta < \varepsilon$ be as in Lemma 2.4. There exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$, $d_H(M_n, M) < \delta$. Fix $n \geq n_0$. For any $a \in M_n$, there is $b \in M$ such that $d(a, b) < \delta$. By Lemma 2.4, $\operatorname{diam}([a, b]) < \varepsilon$. As $r_D(a) := c \in [a, b]$, then $d(c, b) < \varepsilon$. It follows that for any $c \in N_n$, $d(c, M) < \varepsilon$. Similarly for any $b \in M$, there is $a \in M_n$ such that $d(a, b) < \delta$. So by Lemma 2.4, $\operatorname{diam}([a, b]) < \varepsilon$. As $c := r_D(a) \in [a, b]$, then $d(c, b) < \varepsilon$. It follows that for any $b \in M$, $d(b, N_n) < \varepsilon$. As a result, $d_H(N_n, M) < \varepsilon$ for any $n \geq n_0$. This proves Claim 1.

If $(N_n)_n$ has a constant subsequence, then $N_n = M$ for some n and so M is minimal. Otherwise, for each $n \in \mathbb{N}$, there is $m > n$ such that $N_n \cap N_m = \emptyset$ and in this case, we have the following.

Claim 2. There is a sequence $(K_n)_n$ of finite orbits with $[K_n] \subset D \setminus E(D)$ for each n and such that $\lim_{n\to+\infty} K_n = M$.

If $[N_n] \subset D \setminus E(D)$ for infinitely many n, then we choose $(K_n)_n$ as a subsequence of $(N_n)_n$. Otherwise, there is $n_0 \in \mathbb{N}$ such that for each $n > n_0$, $N_n \cap E(D) \neq \emptyset$ and hence $N_n \subset E(D)$. We let $K_n = r_{[N_n]}(N_m)$. By Corollary 5.9, K_n is a finite orbit. As $N_m \cap N_n = \emptyset$ and $(N_m \cup N_n) \subset E(D)$, so $N_m \subset D \setminus [N_n]$ and then $K_n \subset D \setminus E(D)$. We shall prove that $\lim_{n\to+\infty} K_n = M$: Indeed, let $\varepsilon > 0$ and let $0 < \delta < \varepsilon$ be as in Lemma 2.4. Let $m_0 \in \mathbb{N}$ be such that for any $n \geq m_0$, $d_H(N_n, M) < \frac{\delta}{2}$. So $d_H(N_n, N_m) < \delta$ for $m > n$. As in the proof of Claim 1, we have, by Lemma 2.4, $d_H(K_n, N_n) < \varepsilon$ and hence $d_H(K_n, M) < 2\varepsilon$. It follows that $\lim_{n\to+\infty} K_n = M$.

Claim 3. There is a subsequence $(L_n)_n$ of $(K_n)_n$ such that for each n, $[L_n] \subset [L_{n+1}] \setminus E([L_{n+1}])$ and $\lim_{n\to+\infty} L_n = M$.

Let $n \in \mathbb{N}$. By Claim 2, we have $[K_n] \subset D \setminus E(D)$. Let $c \in K_n$ and $a, b \in E(D)$ with $a \neq b$ such that $c \in [a, b]$. By Lemma 2.2, there is $\mu > 0$ such that if $x \in B(a, \mu)$ and $y \in B(b, \mu)$, then $c \in (x, y)$, where $B(z, \mu)$ is the open ball of radius μ centered at z. Take $m > n$ such that $d_H(K_m, M) < \mu$. As $a, b \in M$, there are $x \in B(a, \mu) \cap K_n$ and $y \in B(b, \mu) \cap K_n$. So $c \in (x, y)$ and thus c is a cut point of $[K_m]$. Hence $G(c) = K_n \subset [K_m] \setminus E([K_m])$.

Claim 4. $D \setminus E(D) = \bigcup_{n\in\mathbb{N}} [L_n]$. In particular, $\overline{\bigcup_{n\in\mathbb{N}} [L_n]} = D$.

First, we have the inclusion $\bigcup_{n\in\mathbb{N}} [L_n] \subset D$. Let $c \in D \setminus E(D)$ and let $a, b \in E(D)$ such that $c \in [a, b]$. Similarly as in the proof of Claim 3, let $\mu > 0$ be such that if $x \in B(a, \mu)$ and $y \in B(b, \mu)$, then $c \in (x, y)$. Take n such that $d_H(L_n, M) < \mu$. Then $c \in [x, y]$ for some $x, y \in L_n$ and hence $c \in [L_n]$. Therefore, $D \setminus E(D) \subset \bigcup_{n\in\mathbb{N}} [L_n]$. Now suppose that $D \setminus E(D) \subsetneq \bigcup_{n\in\mathbb{N}} [L_n]$. Then there exists $c \in [L_n] \cap E(D)$ for some n. Thus $c \in [L_{n+1}] \cap E(D) \subset E([L_{n+1}])$, and this contradicts Claim 3.

Claim 5. For each $n \in \mathbb{N}$ and for any connected component C of $D \setminus [L_n]$, $r_{[L_n]}(C) = \{e_n\} \subset L_n = E([L_n])$.

Indeed, let C be a connected component of $D \setminus [L_n]$. As $C \cap (D \setminus E(D)) \neq \emptyset$, then by Claim 4, there is $m > n$ such that $[L_m] \cap C \neq \emptyset$. Take $a \in [L_m] \cap C$. Then a belongs to a connected component of $[L_m] \setminus [L_n]$. By Lemma 6.2, $(r_{[L_n]})_{/[L_m]}(a) = \{e_n\} \subset E(L_n)$, where $(r_{[L_n]})_{/[L_m]} : [L_m] \longrightarrow [L_n]$ is the restriction of $r_{[L_n]}$ on $[L_m]$. Hence $r_{[L_n]}(C) = (r_{[L_n]})_{/[L_m]}(a) = \{e_n\} \subset L_n = E([L_n])$.

Claim 6. For each n, $[L_n] \setminus L_n$ is open in D.

By Claim 5, for any connected component C of $D \setminus [L_n]$, $\overline{C} = C \cup \{e\}$, where $e \in L_n = E([L_n])$. The order in D of any point $e \in L_n$ is the same as in the tree $[L_{n+1}]$ and so it is finite; indeed, otherwise, there is some connected component Q of $X \setminus [L_{n+1}]$ such that $r_{[L_{n+1}]}(Q) = \{e\}$. So by Claim 5, $e \in E([L_{n+1}])$; however, by Claim 3, e is a cut point of $[L_{n+1}]$, a contradiction. We conclude that the number of connected components of $D \setminus [L_n]$ is finite. Remark that for any $e \in L_n$, there is a connected component of $D \setminus [L_n]$, where $r_{[L_n]}(C) = \{e\}$. So $D \setminus ([L_n] \setminus L_n)$ is closed in D since it is the union of the closure of finitely many connected components of $D \setminus [L_n]$. Therefore $[L_n] \setminus L_n$ is open in D.

Claim 7. $E(D)$ is closed in D.

Let $(e_n)_n \subset E(D)$ be a sequence that converges to a point e. If $e \in D \setminus E(D)$, then by Claim 4, $e \in [L_m]$ for some $m \in \mathbb{N}$. Hence $e \in [L_{m+1}] \setminus E([L_{m+1}]) = [L_{m+1}] \setminus L_{m+1}$. By Claim 6, $[L_{m+1}]$ is a neighborhood of e in D. Then $e_n \in [L_{m+1}]$ for some n which implies by Claim 4 that $e_n \in D \setminus E(D)$, a contradiction.

Claim 8. $E(D) = M$.

We apply Lemma 6.4 for $X_n = [L_n]$ and $X = D$ since all conditions are fulfilled by Claims 3, 4, 5 and 7. So we get $\lim\limits_{n \to +\infty} L_n = \lim\limits_{n \to +\infty} E([L_n]) = E(D)$ and hence $E(D) = M$.

Claim 9. For any $a, b \in M$ and for any $\varepsilon > 0$, there is $g \in G$ such that $d(g(a), b) < 2\varepsilon$.

Let $a, b \in M$. By Claims 4 and 8, $a, b \notin [L_n]$ for every n. Let $\varepsilon > 0$ and $0 < \delta < \varepsilon$ be as in Lemma 2.4. Let n be such that $d_H(L_n, M) < \delta$. Suppose that C_a (resp. C_b) is the connected component of $D \setminus [L_n]$ containing a (resp. b). Set $\{a_n\} = r_{[L_n]}(C_a)$ and $\{b_n\} = r_{[L_n]}(C_b)$. As $a_n, b_n \in L_n$ and L_n is a finite orbit, there is $g \in G$ such that $g(a_n) = b_n$ and so $g(C_a)$ is a connected component of $D \setminus [L_n]$ with $r_{[L_n]}(g(C_a)) = \{b_n\}$. We prove that for any connected component C of $D \setminus [L_n]$, if $\{e_n\} = r_{[L_n]}(C)$, then $d(e, e_n) < \varepsilon$ for any $e \in M \cap C$: Indeed, as $d_H(L_n, M) < \delta$, then for any $e \in M \cap C$, there is $x_n \in L_n$ such that $d(e, x_n) < \delta$. So by Lemma 2.4, $\mathrm{diam}([e, x_n]) < \varepsilon$. Since the point $e_n := r_{[L_n]}(e) \in [e, x_n]$, so $d(e, e_n) < \varepsilon$.

We conclude that $d(g(a), b) \leq d(g(a), b_n) + d(b_n, b) < 2\varepsilon$.

As ε was chosen arbitrarily, we conclude that $G(a)$ is dense in M for each $a \in M$. Equivalently, M is a minimal set of G. $\square$

We deduce from Theorem 6.9 the following corollary.

Corollary 6.10. *Let G be a group acting on a dendrite X. Then the union of all minimal sets of G is closed in X.*

Remark 6.11. When the space X is not a dendrite, the conclusion of Corollary 6.10 fails to be true even for a single homeomorphism.

In [1], Aarts et al. proved that for any compact manifold X of dimension greater than one, there exists a Devaney chaotic homeomorphism f on X; that is, f is transitive and the set of its periodic points is dense in X. In this case, the union Y of all minimal sets of f is not closed in X. Indeed, suppose that Y is closed in X. So any transitive point will belong to a minimal set and hence X itself is a minimal set. This leads to a contradiction since there are periodic points.

Also when X is a Cantor space, the conclusion of Corollary 6.10 is not true by taking the shift homeomorphism on X which is Devaney chaotic.

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