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GENERALIZED POLYNOMIAL MODULES OVER THE VIRASORO ALGEBRA

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Abstract. Let $\mathcal{B}_r$ be the $(r+1)$-dimensional quotient Lie algebra of the positive part of the Virasoro algebra $\mathcal{V}$. Irreducible $\mathcal{B}_r$-modules were used to construct irreducible Whittaker modules in a work of Mazorchuk and Zhao (2014) and irreducible weight modules with infinite dimensional weight spaces over $\mathcal{V}$ in a work of Liu, Lu and Zhao (2015). In the present paper, we construct non-weight Virasoro modules $F(M, \Omega(\lambda, \beta))$ from irreducible $\mathcal{B}_r$-modules M and $(\mathcal{A}, \mathcal{V})$-modules $\Omega(\lambda, \beta)$. We give necessary and sufficient conditions for the Virasoro module $F(M, \Omega(\lambda, \beta))$ to be irreducible. Using the weighting functor introduced by J. Nilsson, we also determine necessary and sufficient conditions for two $F(M, \Omega(\lambda, \beta))$ to be isomorphic.

1. Introduction

Let $\mathcal{V}$ denote the complex *Virasoro algebra*, that is, the Lie algebra with a basis $\{\mathtt{c}, \mathtt{d}_i : i \in \mathbb{Z}\}$ and the Lie bracket defined as follows:

$$[\mathtt{d}_i, \mathtt{d}_j] = (j-i)\mathtt{d}_{i+j} + \delta_{i,-j}\frac{i^3-i}{12}\mathtt{c}; \quad [\mathtt{d}_i, \mathtt{c}] = 0, \forall\, i, j \in \mathbb{Z}.$$

The Virasoro algebra $\mathcal{V}$ is one of the most important algebras studied by physicists and mathematicians in the last few decades. It has a profound impact on mathematical and physical sciences; see [3, 9, 10, 13, 14]. The representation theory on the Virasoro algebra has attracted a lot of attention from mathematicians and physicists. The recent monograph [12] is a detailed survey of the classical part of the representation theory of $\mathcal{V}$. There are two classical families of simple weight $\mathcal{V}$-modules with all finite dimensional weight spaces: highest weight modules (completely described in [7]) and the so-called intermediate series modules. In [19] it is shown that these two families exhaust all simple weight modules with all finite dimensional weight spaces. In [21] it is even shown that the above modules exhaust all simple weight modules admitting a non-zero finite dimensional weight space. Very naturally, the next important task for the Virasoro algebra is to study simple weight modules with infinite dimensional weight spaces and non-weight simple modules.

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The first such examples of weight modules with infinite dimensional weight spaces were constructed by taking the tensor product of some highest weight modules and some intermediate series modules in [30] in 1997, while the irreducibility of these tensor products was recently solved completely in [4]. Then Conley and Martin gave another class of such examples with four parameters in [5] in 2001. In [15], a large class of an irreducible Virasoro module with infinite dimensional weight spaces was constructed from $\mathcal{B}_r$-modules, where $\mathcal{B}_r$ (see Section 2) is an $(r+1)$-dimensional quotient Lie algebra of the positive part of the Virasoro algebra; also see [17].

During the last decade, various families of non-weight simple Virasoro modules were studied. These include Whittaker modules, see [8, 16, 18, 20, 22, 25, 26, 29], $\mathbb{C}[\mathtt{d}_0]$-free modules, see [23, 27, 28], highest-weight-like modules, see [11], and irreducible modules from Weyl modules, see [18]. In particular, all Whittaker modules and even more were described in a uniform way in [22], using irreducible $\mathcal{B}_r$-modules. In [28], it was shown that if M is a $\mathcal{V}$-module which is a free $\mathbb{C}[\mathtt{d}_0]$-module of rank 1, then $M \cong \Omega(\lambda, \beta)$ for some $\beta \in \mathbb{C}$ and $\lambda \in \mathbb{C} \setminus \{0\}$. One can refer to Example 2 for the definition of $\Omega(\lambda, \beta)$.

It is easy to see that an irreducible $\mathcal{V}$-module is not a weight module if and only if it is $\mathbb{C}[\mathtt{d}_0]$-free. So it is valuable to construct non-weight $\mathcal{V}$-modules from $\Omega(\lambda, \beta)$. This is the motivation of the present paper.

Now we briefly describe the main results in the present paper. In Section 2, we first recall the definition of $(\mathcal{A}, \mathcal{V})$-modules where $\mathcal{A}$ is the Laurent polynomial algebra $\mathbb{C}[x^{\pm 1}]$. The module $\Omega(\lambda, \beta)$ is a typical example of an $(\mathcal{A}, \mathcal{V})$-module. For any irreducible module M over $\mathcal{B}_r$ and an $(\mathcal{A}, \mathcal{V})$-module W, we define a Virasoro module structure on the vector space $F(M, W) = M \otimes W$; see (2.6) and (2.8). These modules are generalizations of Virasoro modules $\mathcal{N}(M, \alpha)$ defined in [15]. In Section 3, we prove that the Virasoro module $F(M, \Omega(\lambda, \beta))$ is reducible if and only if $M \cong M_\beta$, a 1-dimensional $\mathcal{B}_r$-module determined by the parameter $\beta \in \mathbb{C}$; see Theorem 3.2. Thus we obtain a huge class of irreducible non-weight Virasoro modules. In [24], Nilsson introduced a weighting functor which maps non-weight modules to weight modules. It is written in [24] that the idea of this functor is due to O. Mathieu. Using the weighting functor, we also determine necessary and sufficient conditions for two such irreducible Virasoro modules $F(M, \Omega(\lambda, \beta))$ to be isomorphic; see Theorem 3.6.

2. Constructing new Virasoro modules

We denote by $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ the sets of all integers, non-negative integers, positive integers, real numbers and complex numbers, respectively.

First we recall the definition of $(\mathcal{A}, \mathcal{V})$-modules; see Section 3 in [6].

Definition 2.1. An $(\mathcal{A}, \mathcal{V})$-module M is a module both for the Lie algebra $\mathcal{V}$ and the commutative associative algebra $\mathcal{A}$ with compatible actions:

$$m x^{n+m} v = \mathtt{d}_n x^m v - x^m \mathtt{d}_n v, \ \ \mathtt{c} v = 0,$$

where $m, n \in \mathbb{Z}, v \in M$.

The following two examples are two interesting classes of $(\mathcal{A}, \mathcal{V})$-modules.

Example 1. For each $\alpha, \beta \in \mathbb{C}$, there is a natural $(\mathcal{A}, \mathcal{V})$-module structure on $\mathcal{A}$ as follows:

$$\mathtt{d}_m x^n = (n + \alpha + \beta m) x^{n+m}, \ x^m x^n = x^{n+m}, \ \mathbf{c} x^n = 0.$$

We denote this module by $A(\alpha,\beta)$ which is called the intermediate series module. It is well known that, as a $\mathcal{V}$-module $A(\alpha,\beta)$ is reducible if and only if $\alpha\in\mathbb{Z}$, and $\beta=0$ or 1. Mathieu showed that any irreducible uniformly bounded weight module over $\mathcal{V}$ is isomorphic to some irreducible sub-quotient of $A(\alpha,\beta)$; see [19].

Example 2. For $\lambda\in\mathbb{C}\setminus\{0\},\beta\in\mathbb{C}$, denote by $\Omega(\lambda,\beta)=\mathbb{C}[t]$ the polynomial associative algebra over $\mathbb{C}$ in indeterminate t. In [18], a class of $(\mathcal{A},\mathcal{V})$-modules is defined on $\Omega(\lambda,\beta)$ by

$$\mathtt{c}f(t)=0, \mathtt{d}_m f(t)=\lambda^m(t-\beta m)f(t-m),$$

$$x^m f(t)=\lambda^m f(t-m),$$

for all $m\in\mathbb{Z}, f(t)\in\mathbb{C}[t]$. From [18] we know that $\Omega(\lambda,\beta)$ is irreducible when it was restricted to $\mathcal{V}$ if and only if $\beta\neq 0$. When $\beta=0$, one can check that $\mathbb{C}[t]t$ is a proper Virasoro submodule of $\Omega(\lambda,0)$. It was shown that if M is a $\mathcal{V}$-module which is a free $\mathbb{C}[\mathtt{d}_0]$-module of rank 1, then $M\cong\Omega(\lambda,\beta)$ for some $\beta\in\mathbb{C}$ and $\lambda\in\mathbb{C}\setminus\{0\}$; see [28].

For an $(\mathcal{A},\mathcal{V})$-module M, we define the following operator on M:

$$\mathtt{g}(m)=x^{-m}\mathtt{d}_m,\ m\in\mathbb{Z}.$$

We can check that

$$\mathtt{g}(m)\mathtt{g}(k)v-\mathtt{g}(k)\mathtt{g}(m)v=-k\mathtt{g}(k)v+m\mathtt{g}(m)v+(k-m)\mathtt{g}(m+k)v, \tag{2.1}$$

$$\mathtt{g}(m)x^n v-x^n\mathtt{g}(m)v=nx^n v, \tag{2.2}$$

for any $m,k,n\in\mathbb{Z}$, $v\in M$.

By abuse of language, we denote $\mathcal{G}$ by the Lie algebra with the basis $\{\mathtt{g}(m): m\in\mathbb{Z}\}$ and the Lie bracket defined as follows:

$$[\mathtt{g}(m),\mathtt{g}(k)]=-k\mathtt{g}(k)+m\mathtt{g}(m)+(k-m)\mathtt{g}(m+k),\ \forall\, m,k\in\mathbb{Z}. \tag{2.3}$$

Let us define the notion of $(\mathcal{A},\mathcal{G})$-modules which appears naturally in the above construction.

Definition 2.2. An $(\mathcal{A},\mathcal{G})$-module M is a module both for the Lie algebra $\mathcal{G}$ and the commutative associative algebra $\mathcal{A}$ with compatible actions:

$$\mathtt{g}(m)x^n v-x^n\mathtt{g}(m)v=nx^n v,$$

where $m,n\in\mathbb{Z},v\in M$.

From (2.1) and (2.2), any $(\mathcal{A},\mathcal{V})$-module can be viewed as an $(\mathcal{A},\mathcal{G})$-module. Conversely, an $(\mathcal{A},\mathcal{G})$-module M can also be viewed as an $(\mathcal{A},\mathcal{V})$-module via

$$\mathtt{d}_m v=x^m\mathtt{g}(m)v,\ \mathtt{c}v=0,\ \forall\, v\in M.$$

Let M be a $\mathcal{G}$-module and W be an $(\mathcal{A},\mathcal{V})$-module. Since W is also a $\mathcal{G}$-module, considering the tensor product $M\otimes W$ of $\mathcal{G}$-modules M and W, there is a natural $(\mathcal{A},\mathcal{G})$-module structure on $M\otimes W$ as follows:

$$\mathtt{g}(m)(v\otimes w)=v\otimes(\mathtt{g}(m)w)+(\mathtt{g}(m)v)\otimes w, \tag{2.4}$$

$$x^m(v\otimes w)=v\otimes(x^m w), \tag{2.5}$$

where $m\in\mathbb{Z},v\in M,w\in W$.

From $\mathtt{d}_m = x^m \mathtt{g}(m)$, we know that the action of $\mathcal{V}$ on $M \otimes W$ is

$$\mathtt{d}_m(v \otimes w) = v \otimes (d_m w) + (\mathtt{g}(m)v) \otimes x^m w, \tag{2.6}$$

$$\mathtt{c}(v \otimes w) = 0, \tag{2.7}$$

where $m \in \mathbb{Z}, v \in M, w \in W$.

Consequently, the formulas (2.5),(2.6) and (2.7) define an $(\mathcal{A}, \mathcal{V})$-module structure on $M \otimes W$. We denote this $(\mathcal{A}, \mathcal{V})$-module by $F(M, W)$.

Denote by $\mathcal{V}_+$ the Lie subalgebra of $\mathcal{V}$ spanned by all $\mathtt{d}_i$ with $i \geq 0$. For $r \in \mathbb{Z}_+$, denote by $\mathcal{V}_+^{(r)}$ the Lie subalgebra of $\mathcal{V}$ generated by all $\mathtt{d}_i$, $i > r$, and by $\mathcal{B}_r$ the quotient algebra $\mathcal{V}_+/\mathcal{V}_+^{(r)}$. By $\bar{\mathtt{d}}_i$ we denote the image of $\mathtt{d}_i$ in $\mathcal{B}_r$. Note that $\mathcal{B}_r$ is a solvable Lie algebra of dimension $r + 1$.

Let M be a $\mathcal{G}$-module. Motivated by the notion of polynomial modules in [2], we suppose that the action of $\mathtt{g}(m)$ on M is defined by

$$\mathtt{g}(m) = \sum_{i=0}^{r} \frac{m^{i+1}\mathfrak{A}_i}{(i+1)!},$$

where $\mathfrak{A}_i \in \mathrm{End}_{\mathbb{C}}(M)$ for $i : 0 \leq i \leq r$. From the Lie bracket (2.3) of $\mathcal{G}$, we can check that

$$[\mathfrak{A}_i, \mathfrak{A}_j] = (j - i)\mathfrak{A}_{i+j}, \ \forall\, i, j : 0 \leq i, j \leq r,$$

where $\mathfrak{A}_{i+j} = 0$, when $i + j > r$. Thus M can be viewed as a module over the Lie algebra $\mathcal{B}_r$ via

$$\bar{\mathtt{d}}_i v = \mathfrak{A}_i v, \ \ \forall\, v \in M.$$

Conversely, for a module M over $\mathcal{B}_r$, we define the action of $\mathcal{G}$ on M by

$$\mathtt{g}(m)v = \sum_{i=0}^{r} \frac{m^{i+1}\bar{\mathtt{d}}_i v}{(i+1)!}, \ \forall\, v \in M. \tag{2.8}$$

Lemma 2.3. *Let M be a module over $\mathcal{B}_r$. Then M becomes a $\mathcal{G}$-module under the action* (2.8)*.*

Proof. We can compute that

$$\begin{aligned}
&\mathtt{g}(m)\mathtt{g}(k)v - \mathtt{g}(k)\mathtt{g}(m)v - m\mathtt{g}(m)v + k\mathtt{g}(k)v \\
=& \Big(\sum_{i=0}^{r} \frac{m^{i+1}}{(i+1)!}\bar{\mathtt{d}}_i\Big)\Big(\sum_{j=0}^{r} \frac{k^{j+1}}{(j+1)!}\bar{\mathtt{d}}_j\Big)v \\
&- \Big(\sum_{i=0}^{r} \frac{k^{j+1}}{(j+1)!}\bar{\mathtt{d}}_j\Big)\Big(\sum_{i=0}^{r} \frac{m^{i+1}}{(i+1)!}\bar{\mathtt{d}}_i\Big)v \\
&+ \Big(k\sum_{i=0}^{r} \frac{k^{i+1}}{(i+1)!} - m\sum_{i=0}^{r} \frac{m^{i+1}}{(i+1)!}\Big)\bar{\mathtt{d}}_i v \\
=& \sum_{i,j=0}^{r} \frac{m^{i+1}k^{j+1}}{(i+1)!(j+1)!}(j-i)\bar{\mathtt{d}}_{i+j}v + \Big(k\sum_{i=0}^{r} \frac{k^{i+1}}{(i+1)!} \\
&- m\sum_{i=0}^{r} \frac{m^{i+1}}{(i+1)!}\Big)\bar{\mathtt{d}}_i v
\end{aligned}$$

$$= \left(k\sum_{i,j=0}^{r}\frac{m^{i+1}k^j}{(i+1)!j!}\bar{\mathtt{d}}_{i+j} - m\sum_{i,j=0}^{r}\frac{m^i k^{j+1}}{i!(j+1)!}\bar{\mathtt{d}}_{i+j}\right)v$$
$$+ \Big(k\sum_{i=0}^{r}\frac{k^{i+1}}{(i+1)!} - m\sum_{i=0}^{r}\frac{m^{i+1}}{(i+1)!}\Big)\bar{\mathtt{d}}_i v$$
$$= k\sum_{i=0}^{r}\sum_{j=0}^{i}\frac{m^{i+1-j}k^j}{(i+1-j)!j!}\bar{\mathtt{d}}_i v - m\sum_{i=0}^{r}\sum_{j=0}^{i}\frac{k^{i+1-j}m^j}{(i+1-j)!j!}\bar{\mathtt{d}}_i v$$
$$+ \Big(k\sum_{i=0}^{r}\frac{k^{i+1}}{(i+1)!} - m\sum_{i=0}^{r}\frac{m^{i+1}}{(i+1)!}\Big)\bar{\mathtt{d}}_i v$$
$$= k\sum_{i=0}^{r}\sum_{j=0}^{i+1}\frac{m^{i+1-j}k^j}{(i+1-j)!j!}\bar{\mathtt{d}}_i - m\sum_{i=0}^{r}\sum_{j=0}^{i+1}\frac{k^{i+1-j}m^j}{(i+1-j)!j!}\bar{\mathtt{d}}_i v$$
$$= (k-m)\sum_{i=0}^{r}\frac{(m+k)^{i+1}}{(i+1)!}\bar{\mathtt{d}}_i v = (k-m)\mathtt{g}(m+k)v.$$

The lemma is proved. □

For a module M over $\mathcal{B}_r$, we can see that the Virasoro module $F(M,W)$ is a weight module if and only if W is a weight Virasoro module. From (2.6) and (2.8), we can see that if $W = \oplus_{\lambda\in\mathbb{C}}W_\lambda$ is a weight module over $\mathcal{V}$, then $F(M,W) = \oplus_{\lambda\in\mathbb{C}}N_\lambda$ is a weight Virasoro module with weight spaces $N_\lambda = M\otimes W_\lambda$ where

$$N_\lambda = \{v\in F(M,W) \mid d_0 v = \lambda v\}.$$

Proposition 2.4. *If M_1, M_2 are modules over $\mathcal{B}_r$, then as $(\mathcal{A},\mathcal{V})$-modules,*

$$F(M_1, F(M_2,W)) \cong F(M_1\otimes M_2, W).$$

Proof. For any $v_1\in M_1, v_2\in M_2, w\in W, m,n\in\mathbb{Z}$, we have that

$$\begin{aligned}
&\mathtt{d}_m\Big(v_1\otimes(v_2\otimes w)\Big)\\
&= v_1\otimes \mathtt{d}_m(v_2\otimes w) + (\mathtt{g}(m)v_1)\otimes(v_2\otimes x^m w)\\
&= v_1\otimes(v_2\otimes \mathtt{d}_m w + \mathtt{g}(m)v_2\otimes x^m w) + (\mathtt{g}(m)v_1)\otimes(v_2\otimes x^m w)\\
&= (v_1\otimes v_2)\otimes \mathtt{d}_m w + \Big(\mathtt{g}(m)(v_1\otimes v_2)\Big)\otimes x^m w\\
&= \mathtt{d}_m\Big((v_1\otimes v_2)\otimes w\Big),
\end{aligned}$$

and

$$x^m\Big(v_1\otimes(v_2\otimes w)\Big) = v_1\otimes v_2\otimes x^m w = x^m\Big((v_1\otimes v_2)\otimes w\Big).$$

The proposition is proved. □

Remark 2.5. From Proposition 2.4, it is very hard to determine the irreducibility of $F(M,W)$ for any irreducible $(\mathcal{A},\mathcal{V})$-module W, since it is a difficult problem to discuss the irreducibility of the tensor product of two $\mathcal{B}_r$-modules. Moreover, up to now there is no explicit classification of all irreducible $(\mathcal{A},\mathcal{V})$-modules. However, it is still valuable to research the structure of $F(M,W)$ for some interesting $(\mathcal{A},\mathcal{V})$-modules.

Let M be a module over $\mathcal{B}_r$. From Example 1, (2.6) and (2.8), the action of $\mathcal{V}$ on $F(M, A(\alpha, \beta))$ is defined by

$$\mathtt{d}_m(v \otimes x^n) = \Big((n + \alpha + \beta m)v + \sum_{i=0}^{r} \frac{m^{i+1}\bar{\mathtt{d}}_i v}{(i+1)!}\Big) \otimes x^{n+m}, \tag{2.9}$$

where $m, n \in \mathbb{Z}, v \in M$. Clearly $F(M, A(\alpha, \beta))$ is a weight module over $\mathcal{V}$, which is isomorphic to a module $\mathcal{N}(M, \alpha)$ defined in [15] where M is modified by the action of $\mathtt{d}_0$. Thus our Virasoro modules $F(M, W)$ are generalizations of the modules $\mathcal{N}(M, \alpha)$ defined in [15].

If M is a finite dimensional irreducible $\mathcal{B}_r$-module, then by Lie's Theorem, $M = \mathbb{C}v$, $\bar{\mathtt{d}}_i M = 0$ for any $i \in \mathbb{N}$ and $\bar{\mathtt{d}}_0 v = \gamma v$ for some $\gamma \in \mathbb{C}$. We denote this $\mathcal{B}_r$-module by M_γ. Clearly

$$F(M_\gamma, A(\alpha, \beta)) \cong A(\alpha, \beta + \gamma).$$

The following theorem was given by Liu-Lu-Zhao; see Theorem 4 and Theorem 5 in [15].

Theorem 2.6. *The following statements hold:*

(a) *Let M be an irreducible $\mathcal{B}_r$-module. Then the Virasoro module $F(M, A(\alpha, \beta))$ is reducible if and only if $M \cong M_\gamma$ and $\alpha \in \mathbb{Z}, \beta + \gamma = 0$ or 1.*

(b) *Let M, M' be an infinite dimensional irreducible module over $\mathcal{B}_r$, and α, α', $\beta, \beta' \in \mathbb{C}$. Then $F(M, A(\alpha, \beta)) \cong F(M', A(\alpha', \beta'))$ if and only if $M \cong M'$, $\alpha - \alpha' \in \mathbb{Z}$ and $\beta = \beta'$.*

3. Irreducibility and the isomorphism classes of $F(M, \Omega(\lambda, \beta))$

In this section, we will determine the irreducibility and the isomorphism classes of the Virasoro modules $F(M, \Omega(\lambda, \beta))$ for any irreducible $\mathcal{B}_r$-module M.

3.1. **Irreducibility of** $F(M, \Omega(\lambda, \beta))$**.** From (2.6) and (2.8), the $\mathcal{V}$-module structure on $F(M, \Omega(\lambda, \beta))$ is given by

$$\mathtt{d}_m(v \otimes f(t)) = v \otimes \lambda^m (t - m\beta) f(t - m) + \sum_{i=0}^{r} \frac{m^{i+1}\bar{\mathtt{d}}_i v}{(i+1)!} \otimes \lambda^m f(t - m). \tag{3.1}$$

Lemma 3.1. *Let M be an irreducible module over $\mathcal{B}_r$. Then either $\bar{\mathtt{d}}_r M = 0$ or the action of $\bar{\mathtt{d}}_r$ on M is bijective.*

Proof. It is straightforward to check that $\bar{\mathtt{d}}_r M$ and $\mathrm{ann}_M(\bar{\mathtt{d}}_r) = \{v \in M | \bar{\mathtt{d}}_r v = 0\}$ are submodules of V. Then the lemma follows from the simplicity of M. $\square$

For any $n \in \mathbb{Z}_+, m \in \mathbb{Z}$, let $h_m^0 = 1$ and

$$h_m^n = \prod_{j=m+1}^{m+n} (t - j), \forall\ n > 0.$$

It is clear that $\{h_m^n \mid n \in \mathbb{Z}_+\}$ forms a basis of $\Omega(\lambda, \beta)$ for any $m \in \mathbb{Z}$. Moreover

$$h_m^n - h_{m+1}^n = n h_{m+1}^{n-1}. \tag{3.2}$$

From the definition of $\Omega(\lambda, \beta)$, we have that

$$\mathtt{d}_m h_k^n = \lambda^m (t - m\beta) h_{k+m}^n, \quad x^m h_k^n = \lambda^m h_{k+m}^n, \tag{3.3}$$

for any $m, k \in \mathbb{Z}, n \in \mathbb{Z}_+$.

Theorem 3.2. *Let M be an irreducible module over $\mathcal{B}_r$, $\lambda \in \mathbb{C} \setminus \{0\}$. Then the Virasoro module $F(M, \Omega(\lambda, \beta))$ is reducible if and only if $M \cong M_\beta$.*

Proof. If M is finite dimensional, then $M \cong M_\gamma$ for some $\gamma \in \mathbb{C}$. In this case, we have that

$$\mathtt{d}_m(v \otimes f(t)) = v \otimes (t - m(\beta - \gamma))f(t - m),$$

for any $f(t) \in \Omega(\lambda, \beta)$. Hence

$$F(M_\gamma, \Omega(\lambda, \beta)) \cong \Omega(\lambda, \beta - \gamma).$$

Thus $F(M_\gamma, \Omega(\lambda, \beta))$ is reducible if and only if $\beta - \gamma = 0$, i.e., $\beta = \gamma$.

Next we assume that M is infinite dimensional. By Lemma 3.1, there exists an $r_1 \in \mathbb{N}$ with $r_1 \le r$ such that the action of $\bar{\mathtt{d}}_{r_1}$ on M is injective and $\bar{\mathtt{d}}_i M = 0$ for $i > r_1$. We may simply assume that $r_1 = r$.

Let N be a non-zero submodule of $F(M, \Omega(\lambda, \beta))$. Since

$$F(M, \Omega(\lambda, \beta)) \cong F(M, \Omega(1, \beta)),$$

see Theorem 3.6 below, we may further assume that $\lambda = 1$.

Claim 1. If $u = \sum_{n=0}^{l} v_n \otimes h_0^n$ is a non-zero element in N with $v_l \ne 0$, then $\sum_{n=0}^{l} (\bar{\mathtt{d}}_r^2 v_n) \otimes h_m^n \in N$ for any $m \in \mathbb{Z}$.

By (3.1) and (3.3),

$$\mathtt{d}_m(v_n \otimes h_0^n) = v_n \otimes (t - m\beta)h_m^n + (\mathtt{g}(m)v_n) \otimes h_m^n.$$

Consequently

$$\begin{aligned} &\mathtt{d}_k \mathtt{d}_{m-k}(v_n \otimes h_0^n) \\ &= \mathtt{d}_k \left(v_n \otimes (t - (m-k)\beta)h_{m-k}^n + \mathtt{g}(m-k)v_n \otimes h_{m-k}^n \right) \\ &= v_n \otimes (t - k\beta)(t - (m-k)\beta - k)h_m^n + \mathtt{g}(m-k)v_n \otimes (t - k\beta)h_m^n \\ &\quad + \mathtt{g}(k)v_n \otimes (t - (m-k)\beta - k)h_m^n + \mathtt{g}(k)\mathtt{g}(m-k)v_n \otimes h_m^n. \end{aligned}$$

Since k is an arbitrary integer, considering the coefficient of k^{2r+2} in $\mathtt{d}_k \mathtt{d}_{m-k} u$, we obtain that $\sum_{n=0}^{l} (\bar{\mathtt{d}}_r^2 v_n) \otimes h_m^n \in N$ for $m \in \mathbb{Z}$. Then Claim 1 follows.

Claim 2. If $u = \sum_{n=0}^{l} v_n \otimes h_0^n$ is a non-zero element in N, then $(\bar{\mathtt{d}}_r^2 v_l) \otimes 1 \in N$. Consequently $(\bar{\mathtt{d}}_r^2 v_l) \otimes \Omega(\lambda, \beta) \subset N$.

Note that $h_m^k \in \mathbb{C}[t]$ can be expressed as a polynomial in m with degree k, and the highest term is a constant (i.e., a degree 0 polynomial in t). From Claim 1,

$$u(m) := \sum_{n=0}^{l} (\bar{\mathtt{d}}_r^2 v_n) \otimes h_m^n \in N,$$

for any $m \in \mathbb{Z}$. Considering the coefficient of m^l in $u(m)$, we know that $(\bar{\mathtt{d}}_r^2 v_l) \otimes 1 \in N$. From $\mathtt{d}_0((\bar{\mathtt{d}}_r^2 v_l) \otimes 1) = (\bar{\mathtt{d}}_r^2 v_l) \otimes t$, we have that $(\bar{\mathtt{d}}_r^2 v_l) \otimes \Omega(\lambda, \beta) \subset N$. Then Claim 2 follows.

Since the action of $\bar{\mathtt{d}}_r$ on M is injective, $(\bar{\mathtt{d}}_r^2 v_l) \ne 0$. From Claim 2, there exists a non-zero element $v \in M$ such that $v \otimes \Omega(\lambda, \beta) \subset N$.

Claim 3. If $v \otimes \Omega(\lambda) \subset N$, then $(\bar{\mathtt{d}}_i v) \otimes \Omega(\lambda) \subset N$ for any $i \in \mathbb{Z}_+$.

From $\mathtt{d}_m(v \otimes h_{k-m}^n) = v \otimes (t - m\beta)h_k^n + \left(\sum_{i=0}^{r} \frac{m^{i+1} \bar{\mathtt{d}}_i v}{(i+1)!} \right) \otimes h_k^n$, we see that $\left(\sum_{i=0}^{r} \frac{m^{i+1} \bar{\mathtt{d}}_i v}{(i+1)!} \right) \otimes h_k^n \in N$, for any $k, m \in \mathbb{Z}, n \in \mathbb{Z}_+$. Hence $(\bar{\mathtt{d}}_i v) \otimes h_k^n \in N$ for any $i = 0, 1, \ldots, r$.

Since M is an irreducible $\mathcal{B}_r$-module, from Claim 3, we obtain that $M \otimes \Omega(\lambda) = N$. Therefore $F(M, \Omega(\lambda, \beta))$ is irreducible when M is infinite dimensional. $\square$

3.2. **Isomorphism of $F(V, \Omega(\lambda, \beta))$.** We will first recall the weighting functor introduced in [24].

For $a \in \mathbb{C}$, let I_a be the maximal ideal of $\mathbb{C}[\mathtt{d}_0]$ generated by $\mathtt{d}_0 - a$. For a $\mathcal{V}$-module M and $n \in \mathbb{Z}$, let

$$M_n := M/I_n M, \quad \mathfrak{W}(M) := \oplus_{n\in\mathbb{Z}}(M_n \otimes x^n).$$

By Proposition 8 in [24], we have the following construction.

Proposition 3.3. *The vector space $\mathfrak{W}(M)$ becomes a weight $\mathcal{V}$-module under the following action:*

$$\mathtt{d}_m \cdot ((v + I_n M) \otimes x^n) := (\mathtt{d}_m v + I_{n+m} M) \otimes x^{n+m}. \tag{3.4}$$

We first establish the following useful lemma.

Lemma 3.4. *We have $\mathfrak{W}(\Omega(\lambda, \beta)) \cong A(0, 1 - \beta)$.*

Proof. It is easy to see that $\dim(\Omega(\lambda, \beta)/I_n(\Omega(\lambda, \beta))) = 1$ for any $n \in \mathbb{Z}$. Let $v_n = 1 + I_n(\Omega(\lambda, \beta)) \in \Omega(\lambda, \beta)/I_n(\Omega(\lambda, \beta))$. We see that

$$\begin{aligned} \mathtt{d}_m v_n =& \lambda^m (t - m\beta) + I_{m+n}(\Omega(\lambda, \beta)) \\ =& \lambda^m (m + n - m\beta) + I_{m+n}(\Omega(\lambda, \beta)) \\ =& \lambda^m (n + m(1 - \beta)) v_{m+n}. \end{aligned}$$

Set $w_n = \lambda^n v_n$. Then $\mathtt{d}_m w_n = (n + m(1 - \beta)) w_{m+n}$. Thus the lemma follows. $\square$

We know that $\Omega(\lambda, 1)$ is irreducible, and $A(0, 0)$ is reducible as $\mathcal{V}$-modules. Thus the weighting functor $\mathfrak{W}$ does not map irreducible modules to irreducible modules.

Proposition 3.5. *As Virasoro modules, we have the isomorphism*

$$\mathfrak{W}(F(M, \Omega(\lambda, \beta)) \cong F(M, A(0, 1 - \beta)).$$

Proof. Note that $F(M, \Omega(\lambda, \beta)) = M \otimes \Omega(\lambda, \beta)$. For any $n \in \mathbb{Z}$, using (3.1), we have

$$I_n(F(M, \Omega(\lambda, \beta))) = I_n(M \otimes \Omega(\lambda, \beta)) = M \otimes I_n(\Omega(\lambda, \beta)).$$

We can easily deduce that

$$\mathfrak{W}(F(M, \Omega(\lambda, \beta)) \cong F(M, A(0, 1 - \beta)).$$

$\square$

Combining Proposition 3.5 with Theorem 2.6, we obtain the following isomorphism criterion.

Theorem 3.6. *Let M, M' be two infinite dimensional irreducible $\mathcal{B}_r$-modules, $\lambda, \lambda' \in \mathbb{C} \setminus \{0\}$, $\beta, \beta' \in \mathbb{C}$. Then $F(M, \Omega(\lambda, \beta) \cong F(M', \Omega(\lambda', \beta')$ if and only if $M \cong M'$, $\beta = \beta'$.*

Remark 3.7. From Theorem 3.2, we can construct irreducible non-weight Virasoro modules from irreducible $\mathcal{B}_r$-modules. All irreducible modules over $\mathcal{B}_1$ were classified in [1], while all irreducible modules over $\mathcal{B}_2$ were classified in [22]. The classification of irreducible modules over $\mathcal{B}_r$ remains open, for any $r > 2$.

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References

[1] Richard E. Block, *The irreducible representations of the Lie algebra* $\mathfrak{sl}(2)$ *and of the Weyl algebra*, Adv. in Math. **39** (1981), no. 1, 69–110, DOI 10.1016/0001-8708(81)90058-X. MR605353

[2] Yuly Billig and Kaiming Zhao, *Weight modules over exp-polynomial Lie algebras*, J. Pure Appl. Algebra **191** (2004), no. 1-2, 23–42, DOI 10.1016/j.jpaa.2003.12.004. MR2048305

[3] N. Chair, V. K. Dobrev, and H. Kanno, $\mathrm{SO}(2, \mathbf{C})$ *invariant ring structure of BRST cohomology and singular vectors in* $2D$ *gravity with* $c < 1$ *matter*, Phys. Lett. B **283** (1992), no. 3-4, 194–202, DOI 10.1016/0370-2693(92)90007-Q. MR1170971

[4] Hongjia Chen, Xiangqian Guo, and Kaiming Zhao, *Tensor product weight modules over the Virasoro algebra*, J. Lond. Math. Soc. (2) **88** (2013), no. 3, 829–844, DOI 10.1112/jlms/jdt046. MR3145133

[5] Charles H. Conley and Christiane Martin, *A family of irreducible representations of the Witt Lie algebra with infinite-dimensional weight spaces*, Compositio Math. **128** (2001), no. 2, 153–175, DOI 10.1023/A:1017566220585. MR1850181

[6] Askar Dzhumadil′daev, *Virasoro type Lie algebras and deformations*, Z. Phys. C **72** (1996), no. 3, 509–517, DOI 10.1007/s002880050272. MR1415385

[7] B. L. Feĭgin and D. B. Fuks, *Verma modules over a Virasoro algebra*, Funktsional. Anal. i Prilozhen. **17** (1983), no. 3, 91–92. MR714236

[8] Ewa Felińska, Zbigniew Jaskólski, and Michał Kosztołowicz, *Whittaker pairs for the Virasoro algebra and the Gaiotto-Bonelli-Maruyoshi-Tanzini states*, J. Math. Phys. **53** (2012), no. 3, 033504, 16, DOI 10.1063/1.3692188. MR2798231

[9] Philippe Di Francesco, Pierre Mathieu, and David Sénéchal, *Conformal field theory*, Graduate Texts in Contemporary Physics, Springer-Verlag, New York, 1997. MR1424041

[10] Peter Goddard and David Olive, *Kac-Moody and Virasoro algebras in relation to quantum physics*, Internat. J. Modern Phys. A **1** (1986), no. 2, 303–414, DOI 10.1142/S0217751X86000149. MR864165

[11] Xiangqian Guo, Rencai Lu, and Kaiming Zhao, *Fraction representations and highest-weight-like representations of the Virasoro algebra*, J. Algebra **387** (2013), 68–86, DOI 10.1016/j.jalgebra.2013.04.012. MR3056686

[12] Kenji Iohara and Yoshiyuki Koga, *Representation theory of the Virasoro algebra*, Springer Monographs in Mathematics, Springer-Verlag London, Ltd., London, 2011. MR2744610

[13] Victor G. Kac, *Infinite-dimensional Lie algebras*, 3rd ed., Cambridge University Press, Cambridge, 1990. MR1104219

[14] V. G. Kac and A. K. Raina, *Bombay lectures on highest weight representations of infinite-dimensional Lie algebras*, Advanced Series in Mathematical Physics, vol. 2, World Scientific Publishing Co., Inc., Teaneck, NJ, 1987. MR1021978

[15] Genqiang Liu, Rencai Lu, and Kaiming Zhao, *A class of simple weight Virasoro modules*, J. Algebra **424** (2015), 506–521, DOI 10.1016/j.jalgebra.2014.08.054. MR3293231

[16] Rencai Lü, Xiangqian Guo, and Kaiming Zhao, *Irreducible modules over the Virasoro algebra*, Doc. Math. **16** (2011), 709–721. MR2861395

[17] R. Lü and K. Zhao, *A family of simple weight modules over the Virasoro algebra*, arXiv:1303.0702.

[18] Rencai Lu and Kaiming Zhao, *Irreducible Virasoro modules from irreducible Weyl modules*, J. Algebra **414** (2014), 271–287, DOI 10.1016/j.jalgebra.2014.04.029. MR3223399

[19] Olivier Mathieu, *Classification of Harish-Chandra modules over the Virasoro Lie algebra*, Invent. Math. **107** (1992), no. 2, 225–234, DOI 10.1007/BF01231888. MR1144422

[20] Volodymyr Mazorchuk and Emilie Wiesner, *Simple Virasoro modules induced from codimension one subalgebras of the positive part*, Proc. Amer. Math. Soc. **142** (2014), no. 11, 3695–3703, DOI 10.1090/S0002-9939-2014-12098-3. MR3251711

[21] Volodymyr Mazorchuk and Kaiming Zhao, *Classification of simple weight Virasoro modules with a finite-dimensional weight space*, J. Algebra **307** (2007), no. 1, 209–214, DOI 10.1016/j.jalgebra.2006.05.007. MR2278050

[22] Volodymyr Mazorchuk and Kaiming Zhao, *Simple Virasoro modules which are locally finite over a positive part*, Selecta Math. (N.S.) **20** (2014), no. 3, 839–854, DOI 10.1007/s00029-013-0140-8. MR3217463

[23] Jonathan Nilsson, *Simple $\mathfrak{sl}_{n+1}$-module structures on $\mathcal{U}(\mathfrak{h})$*, J. Algebra **424** (2015), 294–329, DOI 10.1016/j.jalgebra.2014.09.036. MR3293222

[24] Jonathan Nilsson, *$\mathcal{U}(\mathfrak{h})$-free modules and coherent families*, J. Pure Appl. Algebra **220** (2016), no. 4, 1475–1488, DOI 10.1016/j.jpaa.2015.09.013. MR3423459

[25] Matthew Ondrus and Emilie Wiesner, *Whittaker modules for the Virasoro algebra*, J. Algebra Appl. **8** (2009), no. 3, 363–377, DOI 10.1142/S0219498809003370. MR2535995

[26] Matthew Ondrus and Emilie Wiesner, *Whittaker categories for the Virasoro algebra*, Comm. Algebra **41** (2013), no. 10, 3910–3930, DOI 10.1080/00927872.2012.693557. MR3169498

[27] Haijun Tan and Kaiming Zhao, *Irreducible Virasoro modules from tensor products (II)*, J. Algebra **394** (2013), 357–373, DOI 10.1016/j.jalgebra.2013.07.023. MR3092725

[28] Haijun Tan and Kaiming Zhao, *$\mathcal{W}_n^+$- and $\mathcal{W}_n$-module structures on $U(\mathfrak{h}_n)$*, J. Algebra **424** (2015), 357–375, DOI 10.1016/j.jalgebra.2014.09.031. MR3293224

[29] Shintarou Yanagida, *Whittaker vectors of the Virasoro algebra in terms of Jack symmetric polynomial*, J. Algebra **333** (2011), 273–294, DOI 10.1016/j.jalgebra.2011.02.039. MR2785949

[30] Hechun Zhang, *A class of representations over the Virasoro algebra*, J. Algebra **190** (1997), no. 1, 1–10, DOI 10.1006/jabr.1996.6565. MR1442144

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