THE REVERSE MATHEMATICS
OF THE TIETZE EXTENSION THEOREM

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(Communicated by Mirna Džamonja)

ABSTRACT. We prove that several versions of the Tietze extension theorem for functions with moduli of uniform continuity are equivalent to $\mathsf{WKL_0}$ over $\mathsf{RCA_0}$. This confirms a conjecture of Giusto and Simpson (2000) that was also phrased as a question in Montalbán's *Open questions in reverse mathematics* (2011).

1. INTRODUCTION

The *Tietze extension theorem* states that if X is a metric space, $C \subseteq X$ is closed, and $f\colon C \to \mathbb{R}$ is continuous, then there is a continuous $F\colon X \to \mathbb{R}$ extending f (meaning that $F(x) = f(x)$ for all $x \in C$). It is a fundamental theorem of real analysis and topology, and, as such, the question of its logical strength is natural and ripe for consideration. In this work, we analyze the logical strengths of formalized versions of the Tietze extension theorem in the setting of *reverse mathematics*, a foundational program designed by Friedman to classify mathematical theorems according to the strengths of the axioms required to prove them [2].

In reverse mathematics, we fix a weak base axiom system WeakSystem for second-order arithmetic and consider the implications that are provable in WeakSystem. If φ and ψ are two statements in second-order arithmetic, typically expressing two well-known theorems, and WeakSystem $\vdash \varphi \to \psi$, then we say that φ implies ψ over WeakSystem and think of the logical strength of φ as being at least that of ψ. We also like to appeal to the equivalence of WeakSystem $\vdash \varphi \to \psi$ and WeakSystem $+\varphi \vdash \psi$ in order to think of the strength of φ in terms of the additional statements ψ that become provable once φ is considered as a new axiom and added to the axioms of WeakSystem.

Often, as in this work, we wish to compare a theorem φ to an axiom system StrongSystem that is stronger than WeakSystem and proves φ. In this situation, if WeakSystem $+\varphi \vdash \psi$ for every axiom ψ of StrongSystem, then we say that φ is equivalent to StrongSystem over WeakSystem. The proof of (the axioms of) StrongSystem from WeakSystem $+\varphi$ is called a *reversal*, from which 'reverse mathematics' gets its name. It is a remarkable phenomenon that equivalences of this sort are the usual case: a theorem is typically either provable in the standard WeakSystem or equivalent to one of four well-known stronger systems. These five systems together are

known as the Big Five. There are, however, many fascinating examples of misfit theorems as well, and we refer the reader to [4] for the tip of that particular iceberg.

It is possible to formalize the Tietze extension theorem in second-order arithmetic in several different ways, and different formalizations may exhibit different logical strengths. The logical systems in play are the first three of the Big Five, which are

- the base system RCA_0 (for *recursive comprehension axiom*), which corresponds to computable mathematics and is the standard $\mathsf{WeakSystem}$;
- the stronger system WKL_0 (for *weak König's lemma*), which adds the ability to make compactness arguments; and
- the yet stronger system ACA_0 (for *arithmetical comprehension axiom*), which adds the ability to form sets defined by any number of first-order quantifiers (but no second-order quantifiers).

The differences among the formalizations of the Tietze extension theorem that we consider arise from two sources. The first source is the problem of coding closed subsets of complete separable metric spaces, which are most naturally thought of as third-order objects, as second-order objects. Closed sets can be coded by negative information (in which case they are simply called *closed*), positive information (in which case they are called *separably closed*), or both simultaneously (in which case they are called *closed and separably closed*). In a compact complete separable metric space, a set is closed if and only if it is separably closed, but both directions of this equivalence are themselves equivalent to ACA_0 over RCA_0 [1, Theorem 3.3]. These notions of closedness are thus distinct when working in RCA_0.

The second source of differences is the fact that the statement "every continuous function $f\colon X \to \mathbb{R}$ on a compact complete separable metric space X is uniformly continuous" is equivalent to WKL_0 over RCA_0 (see [7, Theorem IV.2.2 and Theorem IV.2.3]) and therefore has non-trivial logical strength. Here 'uniformly continuous' means having a *modulus of uniform continuity*, which is a function that, when given an $\epsilon > 0$, returns a $\delta > 0$ such that $(\forall x, y \in X)(d(x, y) < \delta \to d(f(x), f(y)) < \epsilon)$. Thus though the two statements

(1) for every compact complete separable metric space X, every closed $C \subseteq X$, and every continuous $f\colon C \to \mathbb{R}$, there is a continuous $F\colon X \to \mathbb{R}$ extending f,

(2) for every compact complete separable metric space X, every closed $C \subseteq X$, and every uniformly continuous $f\colon C \to \mathbb{R}$, there is a uniformly continuous $F\colon X \to \mathbb{R}$ extending f,

are obviously equivalent in ordinary mathematics, the situation over RCA_0 is more complicated. Following Giusto and Simpson's terminology from [3], we call statement (1) the *Tietze extension theorem* and statement (2) the *strong Tietze extension theorem*. The following list summarizes some of the known results:

- The Tietze extension theorem for closed sets (i.e., the negative information coding) is provable in RCA_0 (see [7, Theorem II.7.5]). In this case the assumption that X is compact may be dropped if f is assumed to be bounded.
- The Tietze extension theorem for separably closed sets is equivalent to ACA_0 over RCA_0 [3, Theorem 6.9].
- The strong Tietze extension theorem for separably closed sets is equivalent to WKL_0 over RCA_0 [3, Theorem 6.14].

- The strong Tietze extension theorem for closed sets is provable in WKL_0 because the Tietze extension theorem for closed sets is provable in RCA_0, and WKL_0 proves that continuous functions on compact complete separable metric spaces are uniformly continuous.
- The strong Tietze extension theorem for closed and separably closed sets is not provable in RCA_0. In fact, it implies the existence of diagonally non-recursive functions [3, Lemma 6.17].

Notice that the above list of results leaves open the precise logical strength of the strong Tietze extension theorem for closed sets and for closed and separably closed sets. Giusto and Simpson conjecture that these theorems are equivalent to WKL_0. Specifically, they make the following conjecture.

Conjecture 1.1 ([3, Conjecture 6.15]). *The following are equivalent over* RCA_0:

(1) WKL_0.
(2) *Let* X *be a compact complete separable metric space, let* C *be a closed subset of* X, *and let* $f \colon C \to \mathbb{R}$ *be a continuous function with a modulus of uniform continuity. Then there is a continuous function* $F \colon X \to \mathbb{R}$ *with a modulus of uniform continuity that extends* f.
(3) *Same as (2) with 'closed' replaced by 'closed and separably closed'.*
(4) *Special case of (2) with* $X = [0, 1]$.
(5) *Special case of (3) with* $X = [0, 1]$.

The question of whether or not this conjecture holds also appears as Question 16 in Montalbán's *Open questions in reverse mathematics* [6]. Let $\mathsf{sTET}_{[0,1]}$ denote statement (5) in Conjecture 1.1 (the notation is chosen to evoke *the strong Tietze extension theorem for* $[0, 1]$). We prove that Conjecture 1.1 is true by proving that $\mathsf{RCA}_0 + \mathsf{sTET}_{[0,1]} \vdash \mathsf{WKL}_0$.

Before continuing, we remark that Giusto and Simpson's *Located sets and reverse mathematics* [3], which contains Conjecture 1.1, is largely concerned with the notion of a located set, where a closed or separably closed subset C of a complete separable metric space X is called *located* if there is a continuous distance function $f \colon X \to \mathbb{R}$, where $f(x) = \inf\{d(x, y) : y \in C\}$ for every $x \in X$. With the assumption of locatedness, the equivalence between closed and separably closed becomes provable in RCA_0: for compact complete separable metric spaces, RCA_0 proves that every closed and located set is separably closed and that every separably closed and located set is closed. Furthermore, the strong Tietze extension theorem for closed and located sets (and thus for separably closed and located sets) is provable in RCA_0. These results and many others appear in [3]. However, located sets are not relevant to Conjecture 1.1, so we make no use of them here.

For ease of comparison, the table below displays the strengths of eight versions of the Tietze extension theorem, taking into account the confirmation of Conjecture 1.1 proven here. The row labeled 'TET' corresponds to versions of the theorem where f and its extension are not required to be uniformly continuous, and the row labeled 'sTET' corresponds to versions of the theorem where f and its extension are required to be uniformly continuous. The columns represent the different assumptions on the domain C of f. The column labeled 'located' means that C is assumed to be closed and located, which in RCA_0 is equivalent to assuming that C is separably closed and located.

	located	closed & separably closed	closed	separably closed
TET	RCA_0	RCA_0	RCA_0	ACA_0
sTET	RCA_0	WKL_0	WKL_0	WKL_0

2. Background

We introduce the systems RCA_0, WKL_0, and ACA_0 and then define in RCA_0 the analytic and topological notions relevant to Conjecture 1.1. The standard reference for reverse mathematics is Simpson's *Subsystems of Second Order Arithmetic* [7], and almost all of this section's material can be found in expert detail therein. Simpson's book also contains many, many examples of theorems that are provable in RCA_0, theorems that are equivalent to WKL_0 over RCA_0, and theorems that are equivalent to ACA_0 over RCA_0.

2.1. RCA_0, WKL_0, **and** ACA_0. The language of second-order arithmetic consists of the *first-order variables* (intended to range over natural numbers), the *second-order variables* (intended to range over sets of natural numbers), the *constant symbols* 0 and 1, the *function symbols* $+$ and $\times$, and the *relation symbols* $=$, $<$, and $\in$. The equality relation is restricted to first-order objects, and second-order equality is defined from $\in$. *Terms* are expressions built from the first-order variables, constants, and functions, such as $((1+1+1) \times n) + m + 1$. *Atomic formulas* are expressions built by combining relations with terms (and second-order variables, in the case of $\in$), such as $s = t$, $s < t$, and $t \in X$. *Formulas* are expressions built by combining atomic formulas with propositional connectives ($\wedge$, $\vee$, $\rightarrow$, etc.), first-order quantifiers $\exists n$ and $\forall n$, and second-order quantifiers $\exists X$ and $\forall X$. A formula is *arithmetical* if it contains no second-order quantifiers. Arithmetical formulas (in prenex normal form) are classified according to the following *arithmetical hierarchy* by counting alternations of unbounded quantifiers. The *bounded quantifiers* $\exists n < t$ and $\forall n < t$ are defined by letting $(\exists n < t)\varphi$ abbreviate $\exists n(n < t \wedge \varphi)$ and letting $(\forall n < t)\varphi$ abbreviate $\forall n(n < t \rightarrow \varphi)$, where n is a first-order variable, t is a term, and φ is a formula. The Σ_0^0 formulas and the Π_0^0 formulas are the arithmetical formulas in which all quantifiers are bounded. The definitions of the classes Σ_n^0 and Π_n^0 proceed by induction. A formula φ is Σ_{n+1}^0 if it is of the form $\exists m\psi$ for a formula ψ that is Π_n^0, and a formula φ is Π_{n+1}^0 if it is of the form $\forall m\psi$ for a formula ψ that is Σ_n^0. Finally, a predicate is Δ_n^0 if it can be expressed by both a Σ_n^0 formula and a Π_n^0 formula.

The axioms of RCA_0 are: a first-order sentence expressing that $\mathbb{N}$ is a discretely ordered commutative semi-ring with identity; the Σ_1^0 *induction scheme*, which consists of the universal closures (by both first- and second-order quantifiers) of all formulas of the form

$$[\varphi(0) \wedge \forall n(\varphi(n) \rightarrow \varphi(n+1))] \rightarrow \forall n \varphi(n),$$

where φ is Σ_1^0; and the Δ_1^0 *comprehension scheme*, which consists of the universal closures (by both first- and second-order quantifiers) of all formulas of the form

$$\forall n(\varphi(n) \leftrightarrow \psi(n)) \rightarrow \exists X \forall n(n \in X \leftrightarrow \varphi(n)),$$

where φ is Σ_1^0, ψ is Π_1^0, and X is not free in φ.

RCA_0 is the standard base system and captures what might be called *effective mathematics*. The name 'RCA_0', which stands for *recursive comprehension axiom*, refers to the Δ_1^0 comprehension scheme because a set X is Δ_1^0 in a set Y if and only

if X is recursive in Y. The subscript '0' refers to the fact that induction in RCA_0 is limited to Σ_1^0 formulas.

RCA_0 proves enough number-theoretic facts to implement the codings of finite sets and sequences that are ubiquitous in recursion theory. Therefore, in RCA_0 we can represent the set $\mathbb{N}^{<\mathbb{N}}$ of all finite sequences as well as its subset $2^{<\mathbb{N}}$ of all finite binary sequences, and we can give the usual definition of a tree as subset of $\mathbb{N}^{<\mathbb{N}}$ that is closed under initial segments. Thus, in RCA_0 we can formulate (but not prove) *weak König's lemma*, which is the statement "every infinite subtree of $2^{<\mathbb{N}}$ has an infinite path". WKL_0 is then the system RCA_0 + weak König's lemma. The fact that there is a recursive infinite subtree of $2^{<\mathbb{N}}$ with no recursive infinite path can be used to show that WKL_0 is strictly stronger than RCA_0. WKL_0 captures the mathematics of compactness. For example, WKL_0 is equivalent to the Heine-Borel compactness of $[0,1]$ (see [7, Theorem IV.1.2]), a fact that is crucial for our analysis of $\mathsf{sTET}_{[0,1]}$.

An important strategy for proving that a theorem implies WKL_0 over RCA_0 is to employ the following lemma, which states that WKL_0 is equivalent over RCA_0 to the statement that for every pair of injections with disjoint ranges, there is a set that separates the two ranges.

Lemma 2.1 (see [7, Lemma IV.4.4]). *The following are equivalent over RCA_0:*

  (i) WKL_0.
  (ii) *If $g_0, g_1\colon \mathbb{N} \to \mathbb{N}$ are injections such that $\forall m \forall n (g_0(m) \neq g_1(n))$, then there is a set X such that $\forall m (g_0(m) \in X \wedge g_1(m) \notin X)$.*

For comparison, ACA_0, introduced next, is equivalent over RCA_0 to the statement that for every injection there is a set consisting of exactly the elements in the injection's range.

The axioms of ACA_0 are those of RCA_0, plus the *arithmetical comprehension scheme*, which consists of the universal closures (by both first- and second-order quantifiers) of all formulas of the form

$$\exists X \forall n (n \in X \leftrightarrow \varphi(n)),$$

where φ is an arithmetical formula in which X is not free.

Jockusch and Soare's famous low basis theorem [5] can be used to prove that ACA_0 is strictly stronger than WKL_0. The strength of ACA_0 is great enough to provide a natural and extensive development of most classical mathematics. Though we do not make further use of ACA_0 here, it is relevant to the discussion in the introduction.

2.2. **Analytic and topological notions in RCA_0.** Following [7, Section II.4], we code integers as pairs of natural numbers and rational numbers as pairs of integers. A real number is then coded by a sequence of rational numbers $\langle q_k : k \in \mathbb{N} \rangle$ such that $\forall k \forall i (|q_k - q_{k+i}| \leq 2^{-k})$. The expression '$x \in \mathbb{R}$' abbreviates the predicate "x codes a real number". Two real numbers coded by $\langle q_k : k \in \mathbb{N} \rangle$ and $\langle q_k' : k \in \mathbb{N} \rangle$ are equal if $\forall k (|q_k - q_k'| \leq 2^{-k+1})$.

The definition of a complete separable metric space generalizes the coding of reals by rapidly converging Cauchy sequences.

Definition 2.2 (RCA_0; see [7, Definition II.5.1]). A *complete separable metric space* $\widehat{A}$ is coded by a non-empty set $A \subseteq \mathbb{N}$ and a distance function $d\colon A \times A \to \mathbb{R}$ such

that, for all $a, b, c \in A$, $d(a,a) = 0$, $d(a,b) = d(b,a) \geq 0$, and $d(a,b) + d(b,c) \geq d(a,c)$.

A *point* in $\widehat{A}$ is coded by a sequence $\langle a_k : k \in \mathbb{N} \rangle$ of elements of A such that $\forall k \forall i (d(a_k, a_{k+i}) \leq 2^{-k})$. The expression '$x \in \widehat{A}$' abbreviates the predicate "x codes a point in $\widehat{A}$".

If $x = \langle a_k : k \in \mathbb{N} \rangle$ and $y = \langle b_k : k \in \mathbb{N} \rangle$ code points in $\widehat{A}$, then $d(x,y)$ is defined to be $\lim_k d(x_k, y_k)$, and (the points coded by) x and y are defined to be equal if $d(x,y) = 0$.

Definition 2.3 (RCA$_0$; see [7, Definition III.2.3]). A complete separable metric space $\widehat{A}$ is *compact* if there is a sequence of finite sequences $\langle \langle x_{i,j} : j < n_i \rangle : i \in \mathbb{N} \rangle$ of points in $\widehat{A}$ such that

$$(\forall z \in \widehat{A})(\forall i \in \mathbb{N})(\exists j < n_i)(d(z, x_{i,j}) < 2^{-i}).$$

For example, the unit interval $[0,1]$ is the complete separable metric space coded by $\{q \in \mathbb{Q} : 0 \leq q \leq 1\}$ with the usual metric, and the sequence $\langle \langle j 2^{-i} : j \leq 2^i \rangle : i \in \mathbb{N} \rangle$ witnesses that $[0,1]$ is compact.

In complete separable metric spaces, open sets are coded by enumerations of open balls, and closed sets are complements of open sets. For the purposes of the remaining definitions, $\mathbb{Q}^+ = \{q \in \mathbb{Q} : q > 0\}$ denotes the set of positive rationals.

Definition 2.4 (RCA$_0$; see [7, Definition II.5.6 and Definition II.5.12]). An *open set* in a complete separable metric space $\widehat{A}$ is coded by a set $U \subseteq \mathbb{N} \times A \times \mathbb{Q}^+$. A point $x \in \widehat{A}$ belongs to U (abbreviated '$x \in U$') if

$$\exists n \exists a \exists r (d(x,a) < r \land \langle n, a, r \rangle \in U).$$

A *closed set* C in a complete separable metric space is also coded by a set $U \subseteq \mathbb{N} \times A \times \mathbb{Q}^+$, but now a point $x \in \widehat{A}$ belongs to C (abbreviated '$x \in C$') if $x \notin U$.

The idea here is that the pair $\langle a, q \rangle \in A \times \mathbb{Q}^+$ codes the open ball $B(a,q)$ of radius q centered at a and that a set $U \subseteq \mathbb{N} \times A \times \mathbb{Q}^+$ codes some sequence $\langle B(a_k, q_k) : k \in \mathbb{N} \rangle$ of open balls and hence codes the open set $\bigcup_{k \in \mathbb{N}} B(a_k, q_k)$. Thus in this scheme, open sets are coded by positive information (enumerations of open balls contained in the open set), and closed sets are coded by negative information (enumerations of open balls disjoint from the closed set). Alternatively, a closed set C can be coded by positive information by enumerating a sequence of points whose closure is C. Such a set is called *separably closed*.

Definition 2.5 (RCA$_0$; see [3, Definition 4.1]). A *separably closed set* in a complete separable metric space $\widehat{A}$ is coded by a sequence $C = \langle x_k : k \in \mathbb{N} \rangle$ of points in $\widehat{A}$. A point $x \in \widehat{A}$ belongs to C (abbreviated '$x \in C$') if

$$(\forall q \in \mathbb{Q}^+)(\exists n \in \mathbb{N})(d(x, x_n) < q).$$

We can now define a complete separable metric space to be *Heine-Borel compact* if for every sequence $\langle U_k : k \in \mathbb{N} \rangle$ of open sets such that $(\forall x \in \widehat{A})(\exists k \in \mathbb{N})(x \in U_k)$, there is an $N \in \mathbb{N}$ such that $(\forall x \in \widehat{A})(\exists k < N)(x \in U_k)$. Although RCA$_0$ proves that $[0,1]$ is a compact complete separable metric space in the sense of Definition 2.3, the Heine-Borel compactness of $[0,1]$ is equivalent to WKL$_0$ over RCA$_0$.

Theorem 2.6 (see [7, Theorem IV.1.2 and Theorem IV.1.5]). *The following are equivalent over* RCA_0:

(1) WKL_0.
(2) *Every compact complete separable metric space is Heine-Borel compact.*
(3) *The unit interval* $[0, 1]$ *is Heine-Borel compact.*
(4) *For every sequence* $\langle (a_k, b_k) : k \in \mathbb{N} \rangle$ *of intervals with rational endpoints such that* $(\forall x \in [0,1])(\exists k \in \mathbb{N})(x \in (a_k, b_k))$, *there is an* $N \in \mathbb{N}$ *such that* $(\forall x \in [0,1])(\exists k < N)(x \in (a_k, b_k))$.

Finally, we define continuous functions and moduli of uniform continuity.

Definition 2.7 (RCA_0; see [7, Definition II.6.1]). Let $\widehat{A}$ and $\widehat{B}$ be complete separable metric spaces. A continuous partial function from $\widehat{A}$ to $\widehat{B}$ is coded by a set $\Phi \subseteq \mathbb{N} \times A \times \mathbb{Q}^+ \times B \times \mathbb{Q}^+$ that satisfies the properties below. Let $\langle a, r \rangle \Phi \langle b, s \rangle$ denote $\exists n (\langle n, a, r, b, s \rangle \in \Phi)$. For $a, a' \in A$ and $r, r' \in \mathbb{Q}^+$, let $\langle a', r' \rangle \prec \langle a, r \rangle$ denote $d(a, a') + r' < r$ and similarly for $b, b' \in B$ and $s, s' \in \mathbb{Q}^+$. The properties that Φ must satisfy are that, for all $a, a' \in A$, all $b, b' \in B$, and all $r, r', s, s' \in \mathbb{Q}^+$,

- if $\langle a, r \rangle \Phi \langle b, s \rangle$ and $\langle a, r \rangle \Phi \langle b', s' \rangle$, then $d(b, b') \leq s + s'$;
- if $\langle a, r \rangle \Phi \langle b, s \rangle$ and $\langle a', r' \rangle \prec \langle a, r \rangle$, then $\langle a', r' \rangle \Phi \langle b, s \rangle$; and
- if $\langle a, r \rangle \Phi \langle b, s \rangle$ and $\langle b, s \rangle \prec \langle b', s' \rangle$, then $\langle a, r \rangle \Phi \langle b', s' \rangle$.

The domain of the function f coded by Φ is the set of all $x \in \widehat{A}$ such that

$$(\forall q \in \mathbb{Q}^+)(\exists \langle a, r \rangle \in A \times \mathbb{Q}^+)$$
$$(\exists \langle b, s \rangle \in B \times \mathbb{Q}^+)(\langle a, r \rangle \Phi \langle b, s \rangle \wedge d(x, a) < r \wedge s < q).$$

If $x \in \operatorname{dom} f$, then $f(x)$ is the unique $y \in \widehat{B}$ such that

$$(\forall \langle a, r \rangle \in A \times \mathbb{Q}^+)(\forall \langle b, s \rangle \in B \times \mathbb{Q}^+)((d(x, a) < r \wedge \langle a, r \rangle \Phi \langle b, s \rangle) \rightarrow d(y, b) \leq s).$$

The idea behind Definition 2.7 is that Φ lists pairs of open balls $\langle B(a, r), B(b, s) \rangle$ (i.e., the pairs of balls coded by the $\langle a, r \rangle$ and $\langle b, s \rangle$ such that $\langle a, r \rangle \Phi \langle b, s \rangle$) with the property that if f is the function being coded by Φ and x is in both $B(a, r)$ and $\operatorname{dom} f$, then $f(x)$ is in the closure of $B(b, s)$.

Definition 2.8 (RCA_0; see [7, Definition IV.2.1]). Let $\widehat{A}$ and $\widehat{B}$ be complete separable metric spaces, and let f be a partial continuous function from $\widehat{A}$ to $\widehat{B}$. A *modulus of uniform continuity* for f is a function $h \colon \mathbb{N} \to \mathbb{N}$ such that

$$(\forall x, y \in \operatorname{dom} f)(\forall n \in \mathbb{N})(d(x, y) < 2^{-h(n)} \rightarrow d(f(x), f(y)) < 2^{-n}).$$

3. Reversing the strong Tietze extension theorem to weak König's lemma

In their analysis of $\mathsf{sTET}_{[0,1]}$, Giusto and Simpson first show that $\mathsf{RCA}_0 \nvdash$ $\mathsf{sTET}_{[0,1]}$ by showing that $\mathsf{sTET}_{[0,1]}$ fails in REC, the model of RCA_0 whose first-order part is the standard natural numbers and whose second-order part is the recursive sets [3, Lemma 6.16]. To do this, they take advantage of Theorem 2.6, the fact that WKL_0 fails in REC, and the fact that RCA_0 proves that a continuous real-valued function on $[0, 1]$ has a modulus of uniform continuity if and only if it has a *Weierstraß approximation* (see [7, Theorem IV.2.4]). Here, a Weierstraß approximation of a continuous function $f \colon [0, 1] \to \mathbb{R}$ is a sequence of polynomials $\langle p_n : n \in \mathbb{N} \rangle$ from $\mathbb{Q}[x]$ such that $(\forall n \in \mathbb{N})(\forall x \in [0, 1])(|f(x) - p_n(x)| < 2^{-n})$.

The goal in proving that REC $\not\models$ sTET$_{[0,1]}$ is thus to produce a recursive code for a closed and separably closed $C \subseteq [0,1]$, a recursive code for a continuous $f \colon C \to \mathbb{R}$, and a recursive modulus of uniform continuity for f such that no continuous extension of f to $[0,1]$ has a recursive Weierstraß approximation. To this end, let $I_e = [2^{-(2e+1)}, 2^{-2e}]$ for each $e \in \mathbb{N}$, and let $D = \{0\} \cup \bigcup_{e \in \mathbb{N}} I_e$. We call D the *pre-domain* of f, as $C \subseteq D$ is obtained from D by enumerating additional open intervals into the complement of C. The plan is to define $f(0) = 0$; then for each $e \in \mathbb{N}$ to define C and f on I_e to diagonalize against Φ_e being a Weierstraß approximation to an extension of f. Thus on each I_e we implement the following strategy. First, by the fact that Theorem 2.6 item (4) fails in REC, fix a recursive enumeration $\langle (a_k, b_k) : k \in \mathbb{N} \rangle$ of open intervals with rational endpoints that covers (the recursive reals in) $[0,1]$ but has no finite subcover. Transfer this cover to a cover $\langle (a_k^e, b_k^e) : k \in \mathbb{N} \rangle$ of I_e that has no finite subcover by the linear transformation $x \mapsto (x+1)/2^{2e+1}$. Enumerate the intervals of $\langle (a_k^e, b_k^e) : k \in \mathbb{N} \rangle$ into the complement of C until a stage s is reached that witnesses $\Phi_{e,s}(2e+1){\downarrow} = p$, where p is (a code for) a polynomial in $\mathbb{Q}[x]$. If $\Phi_e(2e+1){\uparrow}$, then s is never found, and all the intervals in the sequence $\langle (a_k^e, b_k^e) : k \in \mathbb{N} \rangle$ are enumerated into the complement of C. In this case, I_e is erased from the domain of f, so we do not need to take any action to define f there. If s is found, then at stage s only the intervals of $\langle (a_k^e, b_k^e) : k < s \rangle$ have been enumerated into the complement of C. We then stop the enumeration, which makes $C \cap I_e = I_e \setminus \bigcup_{k<s}(a_k^e, b_k^e)$. As no finite set of intervals from $\langle (a_k^e, b_k^e) : k \in \mathbb{N} \rangle$ covers I_e, we can find a rational $q \in I_e \setminus \bigcup_{k<s}(a_k^e, b_k^e)$. Then we define f on $I_e \setminus \bigcup_{k<s}(a_k^e, b_k^e)$ by making it be constantly 2^{-2e} if $p(q) \le 0$ and making it be constantly -2^{-2e} otherwise. In both cases we ensure that $|f(q) - p(q)| \ge 2^{-2e}$, which successfully diagonalizes against Φ_e because if Φ_e were a Weierstraß approximation to an extension of f, then we would have $|f(q) - p(q)| < 2^{-(2e+1)}$. Furthermore, C is closed and separably closed by Lemma 3.1 below, and it is easy to write down a modulus of uniform continuity for f.

Our plan to prove that RCA$_0$ + sTET$_{[0,1]}$ $\vdash$ WKL$_0$ is to formalize and elaborate upon the preceding argument. Observe, however, that the above argument relies very heavily on the fact that $[0,1]$ is not Heine-Borel compact in REC. To replicate this style of argument, we appeal to Theorem 2.6 and work in RCA$_0$ + $\neg$WKL$_0$. The overall strategy is thus to produce the contradiction RCA$_0$ + $\neg$WKL$_0$ + sTET$_{[0,1]}$ $\vdash$ WKL$_0$.

Let $g_0, g_1 \colon \mathbb{N} \to \mathbb{N}$ be two injections with disjoint ranges. By Lemma 2.1, we wish to separate the ranges of g_0 and g_1 using sTET$_{[0,1]}$. A first idea would be to follow the proof that REC $\not\models$ sTET$_{[0,1]}$ and use I_e to code whether or not e should be in a separating set. Enumerate the intervals of $\langle (a_k^e, b_k^e) : k \in \mathbb{N} \rangle$ into the complement of C until a stage s is reached that witnesses either $g_0(s) = e$ or $g_1(s) = e$. If $g_0(s) = e$, then define f to be 2^{-2e} on the remaining portion of I_e; and if $g_1(s) = e$, then define f to be -2^{-2e} on the remaining portion of I_e. The idea would then be to decode a separating set from an extension F of f by checking whether or not F is ≥ 0 on I_e. The problem is of course that not every $F(q)$ for $q \in I_e$ correctly codes whether or not e should be in a separating set. We would need to find a $q \in I_e$ that is sufficiently close to a member of C, where the meaning of 'sufficiently close' is determined by F's modulus of uniform continuity.

We refine this idea by replacing each I_e with a sequence of disjoint closed intervals $\langle I_{e,m} : m \in \mathbb{N} \rangle$ where the length of each $I_{e,m}$ is at most 2^{-m}, and we choose a rational $q_{e,m} \in I_{e,m}$ for each $e, m \in \mathbb{N}$. The pre-domain for our f is $\{0\} \cup \bigcup_{e,m \in \mathbb{N}} I_{e,m}$. The refined strategy is to implement the above naïve coding plan for I_e on each interval $I_{e,m}$. In the end, if e is in the range of g_0 or g_1, then $C \cap I_{e,m}$ is non-empty for every $m \in \mathbb{N}$. So in this case, for every $m \in \mathbb{N}$, $q_{e,m}$ is a point in $I_{e,m}$ that is within 2^{-m} of a point in C. Thus we are able to decode whether or not e should be in a separating set from an extension of f and the extension's modulus of uniform continuity.

The first lemma says that the closed sets we consider are also separably closed. It is implicit in [3], but we make it explicit as a matter of convenience.

Lemma 3.1 (RCA$_0$). *If $\langle J_e : e \in \mathbb{N} \rangle$ is a sequence of pairwise disjoint closed subintervals of $[0,1]$ with rational endpoints such that $C = \{0\} \cup \bigcup_{e \in \mathbb{N}} J_e$ is closed, then C is also separably closed.*

Proof. Let $Q = \langle q_n : n \in \mathbb{N} \rangle$ be an enumeration of the rationals in $\{0\} \cup \bigcup_{e \in \mathbb{N}} J_e$. We show that the closure of Q is C. Clearly 0 is in the closure of Q, and if $x \in J_e$ it is easy to see that x is in the closure of the rationals in J_e. Conversely, suppose that $x \notin \{0\} \cup \bigcup_{e \in \mathbb{N}} J_e$. Then x is in some open interval (a, b) contained in the complement of C. By shrinking this interval, we can find an $m \in \mathbb{N} \setminus \{0\}$ such that $(x - 1/m, x + 1/m)$ is contained in the complement of C. Thus $\forall n(|x - q_n| \geq 1/m)$, so x is not in the closure of Q. $\square$

We remark that in Lemma 3.1, Q can even be taken to be a set of rationals, rather than a sequence of rationals. Let Q contain 0 and the set of rationals q such that there is an e less than q's code with $q \in J_e$.

The next lemma prepares f's pre-domain $\{0\} \cup \bigcup_{e,m \in \mathbb{N}} I_{e,m}$.

Lemma 3.2 (RCA$_0$ + ¬WKL$_0$). *For each $e \in \mathbb{N}$, let $I_e = [2^{-(2e+1)}, 2^{-2e}]$. Then there are pairwise disjoint closed intervals with rational endpoints $\langle I_{e,m} : e, m \in \mathbb{N} \rangle$, rationals $\langle q_{e,m} : e, m \in \mathbb{N} \rangle$, and open intervals with rational endpoints $\langle (a_k^{e,m}, b_k^{e,m}) : e, m, k \in \mathbb{N} \rangle$ such that*

- (i) $\{0\} \cup \bigcup_{e,m \in \mathbb{N}} I_{e,m}$ *is closed;*
- (ii) $q_{e,m} \in I_{e,m}$;
- (iii) $I_{e,m} \subseteq I_e$, *and the length of $I_{e,m}$ is less than 2^{-m};*
- (iv) $\langle (a_k^{e,m}, b_k^{e,m}) : k \in \mathbb{N} \rangle$ *is an open cover of $I_{e,m}$ with no finite subcover;*
- (v) *if $\langle e, m \rangle \neq \langle e', m' \rangle$, then $I_{e,m}$ and $(a_k^{e',m'}, b_k^{e',m'})$ are disjoint.*

Proof. By ¬WKL$_0$ in the form of the negation of Theorem 2.6 item (4), fix an open cover $\langle (a_k, b_k) : k \in \mathbb{N} \rangle$ of $[0,1]$ by open intervals with rational endpoints that has no finite subcover. By adjusting the endpoints of the intervals as necessary, assume that $(\forall k \in \mathbb{N})(-2^{-2} < a_k < b_k < 1 + 2^{-2})$. For each $e \in \mathbb{N}$, transfer $\langle (a_k, b_k) : k \in \mathbb{N} \rangle$ to I_e via the linear transformation $x \mapsto (x+1)/2^{2e+1}$, and denote the transferred sequence of intervals by $\langle (a_k^e, b_k^e) : k \in \mathbb{N} \rangle$. Notice that if $e \neq e'$, then (a_k^e, b_k^e) and $(a_{k'}^{e'}, b_{k'}^{e'})$ are disjoint for all k and k'.

The procedure described below is clearly uniform in e, so we think of fixing an $e \in \mathbb{N}$ and enumerating

- $\langle I_{e,m} : m \in \mathbb{N} \rangle$;
- $\langle q_{e,m} : m \in \mathbb{N} \rangle$;

- helper pairwise disjoint open intervals with rational endpoints $\langle (c^{e,m}, d^{e,m}) : m \in \mathbb{N} \rangle$ (used to later define $\langle (a_k^{e,m}, b_k^{e,m}) : k \in \mathbb{N} \rangle$ for each m); and
- an increasing sequence of indices $\langle k_{e,m} : m \in \mathbb{N} \rangle$ and a sequence of finite sets of open intervals with rational endpoints $\langle U_{e,m} : m \in \mathbb{N} \rangle$ such that, for all $m \in \mathbb{N}$, $\bigcup_{n \leq m} I_{e,n} \subseteq \bigcup_{k \leq k_{e,m}} (a_k^e, b_k^e)$ and $\bigcup_{k \leq k_{e,m}} (a_k^e, b_k^e) \setminus \bigcup_{n \leq m} I_{e,n} = \bigcup_{n \leq m} \bigcup_{O \in U_{e,n}} O$.

To start, let $k_{e,0}$ be the least k such that (a_k^e, b_k^e) intersects I_e, and choose an open interval $(c^{e,0}, d^{e,0}) \subseteq I_e \cap (a_{k_{e,0}}^e, b_{k_{e,0}}^e)$. Now choose a closed interval $I_{e,0} \subseteq (c^{e,0}, d^{e,0})$ of length at most 2^{-0} and a rational $q_{e,0} \in I_{e,0}$. Enumerate (the at most finitely many intervals coding) $\bigcup_{k \leq k_{e,0}} (a_k^e, b_k^e) \setminus I_{e,0}$ into $U_{e,0}$. The construction proceeds in this manner. Suppose at stage $m + 1$ we have $\langle I_{e,n} : n \leq m \rangle$, $\langle q_{e,n} : n \leq m \rangle$, $\langle (c^{e,n}, d^{e,n}) : n \leq m \rangle$, $\langle k_{e,n} : n \leq m \rangle$, and $\langle U_{e,n} : n \leq m \rangle$. The set $D_{e,m} = I_e \setminus \bigcup_{k \leq k_{e,m}} (a_k^e, b_k^e)$ must be a finite union of closed intervals where at least one of the intervals is non-degenerate (i.e., not a point) because no finite collection of intervals from $\langle (a_k^e, b_k^e) : k \in \mathbb{N} \rangle$ covers I_e. Let $k_{e,m+1}$ be the least k such that (a_k^e, b_k^e) intersects a non-degenerate component interval of $D_{e,m}$, and choose an open interval $(c^{e,m+1}, d^{e,m+1}) \subseteq D_{e,m} \cap (a_{k_{e,m+1}}^e, b_{k_{e,m+1}}^e)$, a closed interval $I_{e,m+1} \subseteq (c^{e,m+1}, d^{e,m+1})$ of length at most $2^{-(m+1)}$, and a rational $q_{e,m+1} \in I_{e,m+1}$. Enumerate (the at most finitely many intervals coding) $\bigcup_{k \leq k_{e,m+1}} (a_k^e, b_k^e) \setminus \bigcup_{n \leq m+1} I_{e,n}$ into $U_{e,m+1}$.

Immediately we see that (ii) and (iii) are satisfied. For (i), consider the closed set C that is described by the simultaneous enumeration of the open intervals $\langle (2^{-(2e+2)}, 2^{-(2e+1)}) : e \in \mathbb{N} \rangle$ and the open intervals in $\bigcup_{e,m \in \mathbb{N}} U_{e,m}$. Suppose that $x \in \{0\} \cup \bigcup_{e,m \in \mathbb{N}} I_{e,m}$. If $x = 0$, then it is clear that $x \in C$. If $x \in I_{e,m}$, then x is in no interval of the form $(2^{-(2e+2)}, 2^{-(2e+1)})$, and it is in no interval $O \in \bigcup_{n \in \mathbb{N}} U_{e',n}$ for an $e' \neq e$. Furthermore, x is in no interval $O \in \bigcup_{n \in \mathbb{N}} U_{e,n}$ either. This is because when $I_{e,m}$ is defined at stage m for e, $I_{e,m}$ is chosen disjoint from the intervals in $\bigcup_{n < m} U_{e,n}$, and at stages $n \geq m$ the intervals added to $U_{e,n}$ are chosen to be disjoint from $I_{e,m}$. Hence $x \in C$. Conversely, suppose that $x \notin \{0\} \cup \bigcup_{e,m \in \mathbb{N}} I_{e,m}$. If x is not in any I_e for $e \in \mathbb{N}$, then clearly $x \notin C$. So suppose that $x \in I_e$. Let k be such that $x \in (a_k^e, b_k^e)$, and let m be such that $k < k_{e,m}$. Then $x \in \bigcup_{k \leq k_{e,m}} (a_k^e, b_k^e) \setminus \bigcup_{n \leq m} I_{e,n}$, so $x \in \bigcup_{n \leq m} \bigcup_{O \in U_{e,n}} O$. Thus $C = \{0\} \cup \bigcup_{e,m \in \mathbb{N}} I_{e,m}$, establishing (i).

To establish (iv), for each $e, m \in \mathbb{N}$, transfer $\langle (a_k, b_k) : k \in \mathbb{N} \rangle$ to $I_{e,m}$ via the linear transformation that maps 0 to the left endpoint of $I_{e,m}$ and maps 1 to the right endpoint of $I_{e,m}$. Denote the transferred sequence of intervals by $\langle (a_k^{e,m}, b_k^{e,m}) : k \in \mathbb{N} \rangle$. To also ensure (v), intersect each interval $(a_k^{e,m}, b_k^{e,m})$ with $(c^{e,m}, d^{e,m})$, which suffices because $I_{e,m} \subseteq (c^{e,m}, d^{e,m})$, and the intervals of $\langle (c^{e,m}, d^{e,m}) : m \in \mathbb{N} \rangle$ are pairwise disjoint. $\qquad\square$

Theorem 3.3. $\mathsf{RCA}_0 + \mathsf{sTET}_{[0,1]} \vdash \mathsf{WKL}_0$.

Proof. We derive the contradiction $\mathsf{RCA}_0 + \neg\mathsf{WKL}_0 + \mathsf{sTET}_{[0,1]} \vdash \mathsf{WKL}_0$. Let $g_0, g_1 \colon \mathbb{N} \to \mathbb{N}$ be injections with disjoint ranges. Our goal is to separate the ranges of g_0 and g_1.

For each $e \in \mathbb{N}$, let $I_e = [2^{-(2e+1)}, 2^{-2e}]$. By $\neg\mathsf{WKL}_0$, let $\langle I_{e,m} : e, m \in \mathbb{N} \rangle$, $\langle q_{e,m} : e, m \in \mathbb{N} \rangle$, and $\langle (a_k^{e,m}, b_k^{e,m}) : e, m, k \in \mathbb{N} \rangle$ be as in Lemma 3.2, and let D denote the closed set $\{0\} \cup \bigcup_{e,m \in \mathbb{N}} I_{e,m}$. The plan is to define a continuous function

f with a modulus of uniform continuity on a closed and separably closed subset of D such that if F is a continuous extension of f to $[0, 1]$ with a modulus of uniform continuity, then, for each $e \in \mathbb{N}$, the value of $F(q_{e,m})$, for an m chosen according to e and F's modulus of uniform continuity, codes whether or not e should be in a separating set.

Let E denote the closed set whose complement is coded by

$$\{(a_k^{e,m}, b_k^{e,m}) : e, m \in \mathbb{N} \wedge (\forall s \leq k)(g_0(s) \neq e \wedge g_1(s) \neq e)\}.$$

Let C be the closed set $D \cap E$. Notice that for each $e, m \in \mathbb{N}$, either $I_{e,m}$ and E are disjoint (if $\forall s(g_0(s) \neq e \wedge g_1(s) \neq e)$) or $I_{e,m} \cap E$ is a finite union of closed intervals with rational endpoints (if $\exists s(g_0(s) = e \vee g_1(s) = e)$). Thus C is of the form $\{0\} \cup \bigcup_{e \in \mathbb{N}} J_e$ for $\langle J_e : e \in \mathbb{N} \rangle$ a sequence of pairwise disjoint closed intervals with rational endpoints. Therefore C is also separably closed by Lemma 3.1.

We now define the continuous function f with modulus of uniform continuity h to which we apply $\mathsf{sTET}_{[0,1]}$. Let

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ 2^{-2e} & \text{if } x \in I_e \cap C \wedge \exists s(g_0(s) = e), \\ -2^{-2e} & \text{if } x \in I_e \cap C \wedge \exists s(g_1(s) = e). \end{cases}$$

To do this, for each $e, m \in \mathbb{N}$, wait while the intervals from $\langle (a_k^{e,m}, b_k^{e,m}) : k \in \mathbb{N} \rangle$ covering $I_{e,m}$ are being enumerated into the complement of C. If this enumeration never stops, then $I_{e,m}$ is disjoint from C and f is not defined on $I_{e,m}$. If this enumeration stops at some stage s, then either $g_0(s) = e$ or $g_1(s) = e$, and $I_{e,m} \cap C$ is determined at this stage. Thus the appropriate pairs of intervals can start being enumerated into the code for f to define $f(x) = 2^{-2e}$ on $I_{e,m} \cap C$ if $g_0(s) = e$ and $f(x) = -2^{-2e}$ on $I_{e,m} \cap C$ if $g_1(s) = e$.

Let $h \colon \mathbb{N} \to \mathbb{N}$ be the function $h(n) = 2n + 2$. We show that h is a modulus of uniform continuity for f. Suppose that $x < y$ are in C and satisfy $|x - y| < 2^{-(2n+2)}$. If $y \in I_e$ for an $e \leq n$, then $|x - y| < 2^{-(2n+2)}$ implies that x must also be in I_e, which means that $|f(x) - f(y)| = 0 < 2^{-n}$. If $y \in I_e$ for an $e > n$, then $|f(y)| = 2^{-2e}$ and $|f(x)| \leq 2^{-2e}$, so $|f(x) - f(y)| \leq 2^{-2e+1} < 2^{-n}$. Thus h is a modulus of uniform continuity for f.

By $\mathsf{sTET}_{[0,1]}$, let F be a continuous extension of f to $[0, 1]$ with modulus of uniform continuity H. Define a set X as follows. Given $e \in \mathbb{N}$, let $m = H(2e + 2)$, and use F to approximate $F(q_{e,m})$ to within $2^{-(2e+2)}$ (i.e., find a rational q such that $|F(q_{e,m}) - q| < 2^{-(2e+2)}$). Define $e \in X$ if and only if this approximation is ≥ 0. This X separates the ranges of g_0 and g_1. Suppose $\exists s(g_0(s) = e)$. Then $I_{e,m} \cap C \neq \emptyset$ and $F(x) = f(x) = 2^{-2e}$ on $I_{e,m} \cap C$. As $q_{e,m} \in I_{e,m}$ and $I_{e,m}$ has length at most $2^{-m} = 2^{-H(2e+2)}$, it must be that $|F(q_{e,m}) - 2^{-2e}| \leq 2^{-(2e+2)}$. Thus if q approximates $F(q_{e,m})$ to within $2^{-(2e+2)}$, then $|q - 2^{-2e}| \leq 2^{-(2e+1)}$, which implies that q is positive and hence that $e \in X$. Similarly, if $\exists s(g_1(s) = e)$, then any approximation of $F(q_{e,m})$ to within $2^{-(2e+2)}$ must be within $2^{-(2e+1)}$ of -2^{-2e} and thus must be negative, which implies that $e \notin X$. $\qquad \square$

Corollary 3.4. *Conjecture* 1.1 *is true.*

Proof. First, in Conjecture 1.1, (1) implies (2), (3), (4), and (5) as explained in the introduction: RCA_0 proves the Tietze extension theorem for closed sets (i.e., the version without uniform continuity; see [7, Theorem II.7.5]), and WKL_0 proves that

continuous functions on compact complete separable metric spaces have moduli of uniform continuity (see [7, Theorem IV.2.2]). Next, each of (2), (3), and (4) implies (5) because (5) is a special case of each of (2), (3), and (4). Finally, (5) implies (1) by Theorem 3.3.   □

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