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EXISTENCE AND GLOBAL ANALYTIC BIFURCATION FOR SINGULAR BIHARMONIC EQUATION WITH NAVIER BOUNDARY CONDITION

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Abstract. Let Ω be a bounded smooth domain in $\mathbb{R}^N$, $N \geq 2$, and let ρ denote the distance to the boundary function. Let K be a positive function such that $K = O(\rho^{-\beta})$ near $\partial\Omega$ for some $\beta \geq 0$. We consider the following fourth order singular elliptic problem (for $\alpha > 0$):

$$(P) \qquad \begin{cases} \Delta^2 u = K(x)u^{-\alpha} & \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \ u|_{\partial\Omega} = 0, \ \Delta u|_{\partial\Omega} = 0. \end{cases}$$

We show the existence of a unique $C^2(\overline{\Omega})$ solution to the above problem when $0 < \alpha + \beta < 2$. We also show the nonexistence of such C^2 solutions when $K \sim \rho^{-\beta}$ near $\partial\Omega$ with $\alpha + \beta \geq 2$.

We then consider an associated bifurcation problem involving a nonlinearity of the type $Ku^{-\alpha} + \lambda f(u)$ where f is taken to be super-linear at infinity. We show the existence of a global (in λ) path-connected analytic branch of solutions to this bifurcation problem when again $\alpha + \beta < 2$.

1. Introduction

Let Ω be a bounded smooth domain in $\mathbb{R}^N$, $N \geq 2$ and denote $\rho(x) := d(x, \partial\Omega)$, $x \in \Omega$. Denote by ϕ_1 and λ_1 respectively the first (positive and normalized) eigenfunction and the first eigenvalue of $-\Delta$ in the space $H_0^1(\Omega)$.

We make the following assumption on K for most parts of this work:

$$(K_1) \qquad \begin{cases} K \in C^{\nu}_{loc}(\Omega) \ \text{ for some } \ \nu \in (0,1), \inf_\Omega K > 0 \ \text{ and} \\ K = O(\rho^{-\beta}) \ \text{ near } \ \partial\Omega, \ \text{ for some } \ \beta \geq 0. \end{cases}$$

Given $\alpha > 0$, we consider the following fourth order singular elliptic problem:

$$(P) \qquad \begin{cases} \Delta^2 u = K(x)u^{-\alpha} & \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \ u|_{\partial\Omega} = 0, \ \Delta u|_{\partial\Omega} = 0. \end{cases}$$

There is a large literature concerning such singular problems (as well as the corresponding systems) for second order elliptic operators wherein questions of existence, uniqueness and multiplicity can be found; for instance, [1], [3], [10], [11], [15], [21], [25], [27], [28]. Similar results for the quasilinear case have been obtained in [22] and [23]. We refer the reader to the two excellent surveys [19] and [26] for more details.

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There are very few results available which concern fourth order singular problems similar to (P). In [20], the author studies the problem $\Delta^2 u = u^{-\alpha}$, $\alpha < 1$, but with Dirichlet boundary condition. Furthermore, the author assumes that the domain is a perturbation of the ball to ensure positivity of the associated Green's function. Using the Schauder fixed point theorem to a suitable integral formulation of the problem in an appropriate cone of positive continuous functions, the existence and the uniqueness of a solution in $C^2(\Omega)\cap C_0^1(\overline{\Omega})$ that behaves like ρ^2 near the boundary is shown in this work. Since such a boundary behaviour is expected, the restriction $\alpha < 1$ is necessary.

In contrast with [20], we consider the problem (P) for a general smooth bounded domain Ω with Navier boundary conditions. We first clarify the notion of a solution to (P):

Definition 1.1. A function $u \in C^2(\overline{\Omega})$ is a solution to (P) if $u > 0$ in Ω, $u = \Delta u = 0$ on $\partial\Omega$ and satisfies the following integral identity for any $\psi \in C^2(\overline{\Omega}) \cap C_0(\overline{\Omega})$:

$$\int_\Omega \Delta u \Delta \psi \mathrm{d}x = \int_\Omega K(x) u^{-\alpha} \psi \mathrm{d}x. \tag{1.1}$$

Remark 1.2.
(1) We require $C^2(\overline{\Omega})$ regularity to be able to define $\Delta u = 0$ on $\partial\Omega$.
(2) A consequence of the above definition is that a solution u to (P) necessarily satisfies
$$\int_\Omega K(x) u^{-\alpha}(x) \rho(x) \mathrm{d}x < \infty, \quad \text{i.e.}, Ku^{-\alpha} \in L^1(\rho\, dx).$$
To see this, plug in the test function $\psi = \phi_1$ in (1.1).
(3) The class of test functions used in the above definition is reminiscent of the ones usually used when dealing with very weak solutions of $-\Delta u = f$ where u, f are assumed only to be in $L^1(\Omega)$ (see for example [6]).

It is easy to see that the equation in (P) is equivalent to the following second order elliptic system:

$$(PS) \qquad \begin{cases} -\Delta u = v \quad \text{in } \Omega,\ u > 0 \quad \text{in } \Omega, \\ -\Delta v = K(x) u^{-\alpha} \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0,\ v|_{\partial\Omega} = 0. \end{cases}$$

Nevertheless, (PS) is not a cooperative system and hence monotone methods cannot be used to prove the existence of solutions to (PS), as is done in [11] for the single equation. Furthermore, for $\alpha \in (0,1)$, the problem (P) has a variational structure and the energy functional J associated to (P) is defined as follows:

$$J(w) \stackrel{\text{def}}{=} \frac{1}{2}\int_\Omega (\Delta w)^2 \mathrm{d}x - \frac{1}{1-\alpha}\int_\Omega K(x) w^{1-\alpha} \mathrm{d}x \quad \text{for} \quad w \in X \stackrel{\text{def}}{=} H^2(\Omega) \cap H_0^1(\Omega). \tag{1.2}$$

Clearly, J is well defined in the cone of nonnegative functions in X provided K has moderate singularity near $\partial\Omega$. But the main difficulty is that truncation techniques (which work in the case of second order elliptic equations) cannot be used directly since we are in the H^2-framework. This makes it difficult to employ variational methods for studying (P). Another difficulty is that the Schauder fixed point theorem (used in [20]) works only in the case $\alpha < 1$ where the invariance of the solution operator with respect to a cone of positive solutions can be ensured.

For these reasons, our approach in this paper is slightly different. We first approximate the singular problem (P) by a family of problems (P_ϵ) with regular terms as given below and use a priori estimates to show the existence of the solution. We now state the results that we prove:

Theorem 1.3. *Assume that $\alpha+\beta<2$ and (K_1) holds. Then there exists a unique solution u to (P). Furthermore, there exist $c_1, c_2>0$ such that*

$$c_1\rho(x) \le u(x) \le c_2\rho(x). \tag{1.3}$$

The idea behind the proof (see Section 3) is to approximate the problem in the following way:

$$(P_\epsilon) \qquad \begin{cases} \Delta^2 u = K_\epsilon(x)(u+\epsilon)^{-\alpha} & \text{in } \Omega, \ \ K_\epsilon := \min(\frac{1}{\epsilon}, K), \\ u>0 \quad \text{in } \Omega, \ \ u|_{\partial\Omega}=0, \ \Delta u|_{\partial\Omega}=0. \end{cases}$$

The existence of the solution u_ϵ to (P_ϵ) can be obtained by the Schauder fixed point theorem. We then prove a priori estimates on $\{u_\epsilon\}_{\epsilon>0}$ using crucially the restrictions on α, β and pass to the limit as $\epsilon \to 0^+$.

We now have the following nonexistence result:

Theorem 1.4. *Assume that*

$$(K_2) \qquad \inf_\Omega K\rho^\beta > 0 \ \ \textit{for some} \ \ \beta \ge 0.$$

If $\alpha+\beta \ge 2$, there is no solution to (P).

The following corollary which follows at once from Theorems 1.3 and 1.4 shows that the restriction $\alpha+\beta<2$ is sharp:

Corollary 1.5. *Let K satisfy both (K_1) and (K_2) for some $\beta \ge 0$, i.e., K is a locally Hölder continuous positive function on Ω such that $K \sim \rho^{-\beta}$ near $\partial\Omega$. Then (P) has a solution if and only if $\alpha+\beta<2$.*

Next, we use Theorem 1.3 to obtain the existence of a path-connected branch of solutions to the following bifurcation problem:

$$(P_\lambda) \qquad \begin{cases} \Delta^2 u = K(x)u^{-\alpha} + \lambda f(u) & \text{in } \Omega, \\ u>0 \quad \text{in } \Omega, \ \ u|_{\partial\Omega}=0, \ \Delta u|_{\partial\Omega}=0, \end{cases} \tag{1.4}$$

where λ is the bifurcation parameter and f a function satisfying the following assumptions:

- (f_0) $f:[0,\infty)\to[0,\infty)$ is a twice continuously differentiable map with $f(0)=0$.
- (f_1) $f(t)$ is a finite product of functions of the form $g(t^p), p>0$, where g is a real entire function on $\mathbb{R}$.
- (f_2) $f' \ge 0$ and $\liminf_{t\to\infty} \dfrac{f(t)}{t} > 0$.

Given a positive continuous function ϕ on Ω, denote by

$$\mathcal{C}_\phi(\Omega) := \Big\{ u \in C(\overline{\Omega}) : \sup_\Omega \Big|\frac{u}{\phi}\Big| < +\infty \Big\},$$

with the norm

$$\|u\|_{\mathcal{C}_\phi(\Omega)} := \sup_\Omega \Big|\frac{u}{\phi}\Big|$$

and the "positive cone"

$$\mathcal{C}_\phi^+(\Omega) := \Big\{ u \in \mathcal{C}_\phi(\Omega) : \inf_\Omega \frac{u}{\phi} > 0 \Big\}.$$

We define the inverse of the biharmonic operator denoted as $(\Delta^2)^{-1}$ as follows:

$$(\Delta^2)^{-1}h = u$$

where for h in an appropriate space, u solves the inhomogeneous problem:

$$\begin{cases} \Delta^2 u = h \quad \text{in } \Omega, \\ u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0. \end{cases} \tag{1.5}$$

The bifurcation analysis is done in the space $\mathbb{R} \times \mathcal{C}_{\phi_1}(\Omega)$. Therefore, we consider the following set of all solutions (in the sense of Definition 1.1):

$$\mathcal{S} = \{u \in C^4(\Omega) \cap C^2(\overline{\Omega}), u > 0 \text{ solves } (P_\lambda)\} \subset \mathbb{R} \times \mathcal{C}_{\phi_1}(\Omega). \tag{1.6}$$

Consider the following solution operator associated to (P_λ):

$$F(\lambda, u) = u - (\Delta^2)^{-1}(K(x)u^{-\alpha} + \lambda f(u)),\ (\lambda, u) \in \mathbb{R} \times \mathcal{C}^+_{\phi_1}(\Omega),\ 0 < \alpha + \beta < 2. \tag{1.7}$$

Using the framework of analytic bifurcation theory as developed in the works [8] and [9] (see also [4] and [7]), we obtain an analytic global unbounded path of solutions to (P_λ):

Theorem 1.6. *Let f satisfy conditions $(f_0) - (f_2)$ and $\alpha + \beta < 2$. Then, $F : \mathbb{R} \times \mathcal{C}^+_{\phi_1}(\Omega) \to \mathcal{C}_{\phi_1}(\Omega)$ is an analytic map (see Definition 5.1). Furthermore, there exist $\Lambda \in (0, \infty)$ and an unbounded set $\mathcal{A} \subset (-\infty, \Lambda] \times \mathcal{C}^+_{\phi_1}(\Omega) \subset \mathcal{S}$ of solutions to (P_λ) which is globally parametrised by a continuous map:*

$$(-\infty, \infty) \ni s \to (\lambda(s), u(s)) \in \mathcal{A}.$$

Moreover, the following properties hold along this path $\mathcal{A}$:

(i) *$(\lambda(s), u(s)) \to (0, u_0)$ in $\mathbb{R} \times \mathcal{C}_{\phi_1}(\Omega)$ as $s \to 0$, where u_0 is the unique solution to (P).*

(ii) *$\|u(s)\|_{\mathcal{C}_{\phi_1}(\Omega)} \to \infty$ as $s \to \infty$.*

(iii) *$\mathcal{A}$ has at least one asymptotic bifurcation point $\Lambda_a \in [0, \Lambda]$. That is, there exist sequences $\{s_n\}_{n\in\mathbb{N}} \subset (0, \infty)$, $\{(\lambda(s_n), u(s_n))\} \subset \mathcal{A}$ such that $s_n \to \infty$, $\lambda(s_n) \to \Lambda_a$ and $\|u(s_n)\|_{\mathcal{C}_{\phi_1}(\Omega)} \to \infty$.*

(iv) *$\{s \geq 0 : \partial_u F(\lambda(s), u(s))$ is not invertible$\}$ is a discrete set.*

(v) *($\mathcal{A}$ is an "analytic" path) At each of its points $\mathcal{A}$ has a local analytic re-parameterisation in the following sense: For each $s^* \in \mathbb{R}$ there exists a continuous, injective map $\rho^* : (-1, 1) \to \mathbb{R}$ such that $\rho^*(0) = s^*$ and the re-parameterisation*

$$(-1, 1) \ni t \to (\lambda(\rho^*(t)), u(\rho^*(t))) \in \mathcal{A} \text{ is analytic.}$$

Furthermore, the map $s \mapsto \lambda(s)$ is injective in a neighborhood of $s = 0$ and for each $s^* > 0$ there exists $\epsilon^* > 0$ such that λ is injective on $[s^*, s^* + \epsilon^*]$ and on $[s^* - \epsilon^*, s^*]$.

(vi) *For any $\lambda \leq 0$, there exists atmost one solution to (P_λ) and $\mathcal{A} \cap (-\infty, 0) \times \mathcal{C}_{\phi_1}(\Omega)$ is a single analytic curve which is a graph from the λ axis consisting of nondegenerate solutions u_λ. In particular, we can take $\lambda(s) = s$ for $s < 0$.*

The paper is organized as follows: In Section 3, we prove Theorem 1.3 using a version of the Hopf principle recalled in Proposition 2.1. In Section 4, we study the equivalence between the two definitions of a solution and prove Theorem 1.4. Finally, in Section 5 we prove Theorem 1.6.

2. Some preliminary results for Theorem 1.3

We first prove a version of the Hopf principle.

Proposition 2.1. *Let $h \in L^\infty(\Omega)$ be a nonnegative function. Let u be the classical solution to* (1.5)*. Then there exists a constant $C > 0$ (independent of h) such that the following inequality holds:*

$$u(x) \geq C\rho(x) \int_\Omega h(y)\rho(y)\mathrm{d}y. \tag{2.1}$$

Proof. Since $h \in L^\infty(\Omega)$, u solves the following system:

$$\begin{cases} -\Delta u = v & \text{in } \Omega, \\ -\Delta v = h & \text{in } \Omega, \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0. \end{cases} \tag{2.2}$$

Recall from lemma 3.2 in [5] that for any nonnegative function $h \in L^\infty(\Omega)$, the unique solution w to the problem

$$\begin{cases} -\Delta w = h & \text{in } \Omega, \\ w|_{\partial\Omega} = 0, \end{cases}$$

satisfies the estimate:

$$w(x) \geq C\rho(x) \int_\Omega h(y)\rho(y)\mathrm{d}y \quad x \in \Omega,$$

where the constant C does not depend on h.

We apply the previous inequality to u and v to get

$$u(x) \geq C\rho(x) \int_\Omega v(y)\rho(y)\mathrm{d}y \geq C^2\rho(x) \int_\Omega \rho(y)^2\mathrm{d}y \int_\Omega h(z)\rho(z)\mathrm{d}z$$

which completes the proof. □

By a simple approximation argument and the maximum principle, we have the

Corollary 2.2. *Let $h\rho \in L^1(\Omega)$ and be nonnegative. Then any u solving* (1.5) *(in the sense of Definition* 1.1*) satisfies the inequality* (2.1)*.*

We next have the following regularity and uniform estimate result:

Lemma 2.3. *Let $h \in C^\nu_{loc}(\Omega)$ be a nonnegative function such that $h\rho^\delta \in L^\infty(\Omega)$ for some $0 < \delta < 2$. Let $u \in C^2(\overline{\Omega})$ be the solution (in the sense of Definition* 1.1*) to* (1.5)*. Then there exist constants $C > 0$ (dependent on $\|h\rho^\delta\|_{L^\infty(\Omega)}$, ν and δ) and $0 < \theta < 1$ (depending on ν and δ) such that the following inequality holds:*

$$\|u\|_{C^{2,\theta}(\overline{\Omega})} \leq C. \tag{2.3}$$

Proof. Since $h\rho \in L^1(\Omega)$, from the above corollary we obtain that $u \geq c\rho$ for some $c > 0$. Since $u \in C^2(\overline{\Omega}) \cap C_0(\overline{\Omega})$, we obtain that $u \sim \rho$ near $\partial\Omega$. We note that $v := -\Delta u \in C^{2,\nu}_{loc}(\Omega)$ by elliptic regularity and is a nonnegative function by the maximum principle. Consider the equivalent system for u, v as in (2.2). Then we have

$$|\Delta v| \leq C_0\rho^{-\delta}, \quad \text{where} \quad C_0 := \|h\rho^\delta\|_{L^\infty(\Omega)}. \tag{2.4}$$

Let $w := w(\delta)$ denote the unique positive solution to

$$\begin{cases} -\Delta w = w^{-\delta} \text{ in } \Omega, \\ w = 0 \text{ on } \partial\Omega. \end{cases}$$

From [11], there exist positive constants $c_1 < c_2$ such that the following estimates hold:

$$\begin{aligned} c_1\rho \leq \quad & w \quad \leq c_2\rho \quad \text{if} \quad 0<\delta<1, \\ c_1\rho\ln\left(\frac{D}{\rho}\right) \leq \quad & w \quad \leq c_2\rho\ln\left(\frac{D}{\rho}\right) \quad \text{if} \quad \delta=1 \ \ (D:=diam(\Omega)) \quad \text{and} \\ c_1\rho^{\frac{2}{\delta+1}} \leq \quad & w \quad \leq c_2\rho^{\frac{2}{\delta+1}} \quad \text{if} \quad 1<\delta<2. \end{aligned}$$

Choosing the constant $M>0$ large enough (depending on C_0, c_1 and δ) and using the weak comparison principle, we can conclude $v - Mw \leq 0$. Thus, we have

$$0 \leq v \leq Mw \leq Mc_2\rho^{\mu} \quad \text{for some} \ \ \mu>0. \tag{2.5}$$

By noting (2.4) and (2.5), appealing to proposition 3.4 in [24] we obtain that $v \in C^{0,\theta}(\overline{\Omega})$ for some $\theta := \theta(\delta,\nu) \in (0,1)$ and $\|v\|_{C^{0,\theta}(\overline{\Omega})} \leq C = C(C_0,\delta)$. We then apply the classical elliptic theory to get $u \in C^{2,\theta}(\overline{\Omega})$ and $\|u\|_{C^{2,\theta}(\overline{\Omega})} \leq \tilde{C} = \tilde{C}(C_0,\delta,\nu)$. □

We can now show the following result on the existence of the $C^2(\overline{\Omega})$ solution (as in Definition 1.1) by means of a simple approximation argument:

Proposition 2.4. *Let h be a nonnegative function such that $h\rho^{\delta} \in L^{\infty}(\Omega)$ for some $0<\delta<2$. Then there exists a unique solution $u \in C^2(\overline{\Omega})$ solving* (1.5).

Proof. Define $h_n := \min\{h,n\}$. Let $u_n \in C^2(\overline{\Omega})$ be the unique solution to (1.5) with $h = h_n$. We note that given $\psi \in C^2(\overline{\Omega}) \cap C_0(\overline{\Omega})$ there exists $p>1$ such that

$$\Big\{h_n\psi\Big\} \quad \text{is a bounded sequence in} \quad L^p(\Omega).$$

Then, by Vitali's convergence theorem

$$\int_{\Omega} h_n\psi \to \int_{\Omega} h\psi \quad \text{as} \quad n\to\infty.$$

By appealing to Lemma 2.3 we obtain as well that for some $\theta \in (0,1)$,

$$\Big\{u_n\Big\} \quad \text{is a bounded sequence in} \quad C^{2,\theta}(\overline{\Omega}).$$

Thus, up to a subsequence, $u_n \to u$ in $C^2(\overline{\Omega})$. It is then easy to see that u solves (1.5). □

We recall the Hardy Inequality for H^s spaces:

Lemma 2.5 (see Lemma 3.2.6.1 in [29])**.** *Let $s \in [0,2]$ such that $s \neq \frac{1}{2}$ and $s \neq \frac{3}{2}$. Then the following generalisation of Hardy's Inequality holds:*

$$\|\rho^{-s}g\|_{L^2(\Omega)} \leq C\|g\|_{H^s(\Omega)} \quad \textit{for all} \quad g \in H_0^s(\Omega). \tag{2.6}$$

Finally, we state the following regularity result.

Lemma 2.6. *Assume that h is a nonnegative function such that $h\rho^{\delta} \in L^{\infty}(\Omega)$ for some $0<\delta<2$. Let $u \in C^2(\overline{\Omega}) \cap C_0(\overline{\Omega})$ be the solution of* (1.5)*. Then $u \in W^{4,p}_{loc}(\Omega) \cap H^{4-s}(\Omega)$ for any $1 \leq p < \infty$ and $s \in (\delta - \frac{1}{2}, 2] \setminus \{\frac{1}{2}, \frac{3}{2}\}$.*

Proof. That $u \in W^{4,p}_{loc}(\Omega)$ for $1 \le p < \infty$ follows from a standard elliptic regularity result. Fix any $s \in (\delta - \frac{1}{2}, 2] \setminus \{\frac{1}{2}, \frac{3}{2}\}$. We claim that $h \in H^{-s}(\Omega)$. Indeed, using Lemma 2.5, for any $\xi \in H^s_0(\Omega)$,

$$\begin{aligned} \Big|\int_\Omega h\xi\Big| &= \Big|\int_\Omega (h\rho^s)(\xi\rho^{-s})\Big| \\ &\le C\int_\Omega (\rho^{s-\delta})(\rho^{-s}|\xi|) \\ &\le C\|\rho^{s-\delta}\|_{L^2(\Omega)}\|\rho^{-s}\xi\|_{L^2(\Omega)} \\ &\le C\|\xi\|_{H^s_0(\Omega)}. \end{aligned}$$

Hence by elliptic regularity used successively to v and u we obtain that $u \in H^{4-s}(\Omega)$. □

3. Proof of Theorem 1.3

We first show that the solution is unique. Let u_1 and u_2 be two solutions to (P). Then,

$$\int_\Omega (\Delta(u_1 - u_2))^2 \mathrm{d}x = \int_\Omega K(x)(u_1^{-\alpha} - u_2^{-\alpha})(u_1 - u_2)\mathrm{d}x \le 0. \tag{3.1}$$

Therefore, since $u_1, u_2 \in H^1_0(\Omega)$, we obtain $u_1 \equiv u_2$.

Fix $\epsilon > 0$. We next prove the existence of a unique solution to (P_ϵ). Let $\mathcal{W}$ be the positive cone of $C_0(\overline{\Omega})$, i.e.,

$$\mathcal{W} \stackrel{\text{def}}{=} \big\{u \in C_0(\overline{\Omega}) \,|\, u \ge 0 \text{ in } \Omega\big\}.$$

We define the functional $\Phi : \mathcal{W} \to \mathcal{W}$ as the solution to the following problem:

$$\begin{cases} \Delta^2\Phi(u) = K_\epsilon(x)(u+\epsilon)^{-\alpha} & \text{in } \Omega, \\ \Phi(u)|_{\partial\Omega} = 0,\ \Delta\Phi(u)|_{\partial\Omega} = 0. \end{cases}$$

By the elliptic regularity theory, Φ is a compact linear operator on $C_0(\overline{\Omega})$, and by the weak comparison principle, leaves the closed convex set $\mathcal{W}$ invariant. Hence, by the Schauder fixed point theorem, there exists the $u_\epsilon \in \mathcal{W}$ solution to (P_ϵ). Using a similar argument as in (3.1), u_ϵ is the unique solution to (P_ϵ). By elliptic regularity, we also have $u_\epsilon \in C^2(\overline{\Omega})$.

Multiplying the equation satisfied by u_ϵ by the first (positive) eigenfunction ϕ_1 of $-\Delta$ in the space $H^1_0(\Omega)$, with the corresponding eigenvalue denoted by λ_1, we obtain that

$$\lambda_1^2 \int_\Omega u_\epsilon\phi_1 \mathrm{d}x = \int_\Omega \Delta^2 u_\epsilon \phi_1 \mathrm{d}x = \int_\Omega K_\epsilon(x)\phi_1(u_\epsilon(x)+\epsilon)^{-\alpha}\mathrm{d}x. \tag{3.2}$$

First we show a uniform lower bound:

Proposition 3.1. *There exists a constant $C > 0$ independent of ϵ such that $u_\epsilon \ge C\rho$ in Ω.*

Proof. We first show the following fact:

$$\inf_{\epsilon>0} \int_\Omega K_\epsilon\phi_1(u_\epsilon+\epsilon)^{-\alpha}\mathrm{d}x > 0. \tag{3.3}$$

We argue by contradiction. Suppose, up to a subsequence,

$$\int_\Omega K_\epsilon \phi_1 (u_\epsilon + \epsilon)^{-\alpha} \mathrm{d}x \to 0 \quad \text{as} \quad \epsilon \to 0^+.$$

Using (3.2) this implies that $\int_\Omega u_\epsilon \phi_1 \mathrm{d}x \to 0$ and hence $u_\epsilon \to 0$ in $L^1_{loc}(\Omega)$ as $\epsilon \to 0^+$.

Again up to a subsequence, we deduce that

$$u_\epsilon \to 0 \quad \text{and} \quad K_\epsilon (u_\epsilon + \epsilon)^{-\alpha} \to 0 \ \text{ a.e. in } \Omega \text{ as } \epsilon \to 0^+ \tag{3.4}$$

which is a contradiction. This proves (3.3) above.

By elliptic regularity theory, $u_\epsilon \in C^{2,\gamma}(\overline{\Omega})$ for any $\gamma \in (0,1)$ and from Proposition 2.1 the estimate

$$u_\epsilon(x) \geq C\rho(x) \int_\Omega K_\epsilon (u_\epsilon + \epsilon)^{-\alpha} \rho \, \mathrm{d}y \tag{3.5}$$

holds. The conclusion follows from (3.3). □

Proposition 3.2. *There exists $\theta \in (0,1)$ independent of $\epsilon > 0$ such that*

$$\sup_{\epsilon > 0} \|u_\epsilon\|_{C^{2,\theta}(\Omega)} < +\infty.$$

Proof. From the last proposition, it follows that

$$K_\epsilon (u_\epsilon + \epsilon)^{-\alpha} \rho^{\alpha+\beta} \in L^\infty(\Omega). \tag{3.6}$$

Noting that $0 < \alpha + \beta < 2$ and invoking Lemma 2.3, the conclusion follows. □

Let $u_\epsilon \to u$ in $C^2(\overline{\Omega})$ as $\epsilon \to 0$. From (3.6), we note that given $\psi \in C^2(\overline{\Omega}) \cap C_0(\overline{\Omega})$ there exists $p > 1$ such that

$$\left\{ K_\epsilon (u_\epsilon + \epsilon)^{-\alpha} \psi \right\}_{\epsilon > 0} \quad \text{is a bounded family in } L^p(\Omega).$$

We can now use Vitali's convergence theorem to directly pass to the limit as $\epsilon \to 0$ in (P_ϵ) to conclude that u solves (P).

4. Proof of Theorem 1.4

Let $u \in C^2(\overline{\Omega}) \cap C_0(\overline{\Omega})$ be a solution to (P). From Remark 1.2 (2), we obtain that $Ku^{-\alpha}\rho \in L^1(\Omega)$. Hence by Corollary 2.2 we obtain that $u \geq c\rho$ in Ω for some $c = c(K,u) > 0$. Noting that $u \in C^1(\overline{\Omega})$ vanishes on the boundary, we therefore obtain that $u \sim \rho$ near $\partial\Omega$. Since K satisfies assumption (K_2), we obtain that

$$C \int_\Omega \rho^{1-\alpha-\beta} \leq \int_\Omega K u^{-\alpha} \rho < +\infty \quad \text{for some } \ C > 0.$$

This necessarily implies $\alpha + \beta < 2$. □

5. Bifurcation results

In this section we prove Theorem 1.6. We consider the following bifurcation framework (see Chapter 9 in [7] or Theorem 1.13 in [4] for more details):

Let $\mathcal{X}$, $\mathcal{Y}$ be real Banach spaces, $\mathcal{U} \subset \mathbb{R}^+ \times \mathcal{X}$ an open set. Let $\Psi : \mathcal{U} \to \mathcal{Y}$ be a map.

Definition 5.1. Ψ is said to be real analytic on $\mathcal{U}$ if for each $x \in \mathcal{U}$ there is an $\varepsilon > 0$ and continuous k-homogeneous polynomials $P_k : \mathcal{U} \to \mathcal{Y}$ such that $\Psi(x+h) = \sum_{k=0}^\infty P_k(h)$ if $\|h\| < \varepsilon$.

Define the solution set

$$\mathcal{S} = \{(\lambda, x) \in \mathcal{U} \,:\, \Psi(\lambda, x) = 0\}$$

and the nonsingular solution set

$$\mathcal{N} = \{(\lambda, x) \in \mathcal{S} \,:\, Ker(\partial_x \Psi(\lambda, x)) = \{0\}\}\,.$$

Definition 5.2. A distinguished arc is a maximal connected subset of $\mathcal{N}$.

Suppose that

(G1) Bounded closed subsets of $\mathcal{S}$ are compact in $\mathbb{R} \times \mathcal{X}$.
(G2) $\partial_x \Psi(\lambda, x)$ is a Fredholm operator of index zero for all $(\lambda, x) \in \mathcal{S}$.
(G3) There exists an analytic function $(\lambda, u) : (-\epsilon, \epsilon) \to \mathcal{S}$ such that $\partial_x \Psi(\lambda(s), u(s))$ is invertible for all $s \in (-\epsilon, \epsilon)$ and $\lim_{s \to 0^+} (\lambda(s), u(s)) = (0, u_0)$ where $u_0 \in \mathcal{X}$ is the unique solution to $\Psi(0, u_0) = 0$.

Let

$$\mathcal{A}_0 = \{(\lambda(s), u(s)) \,:\, s \in (-\epsilon, \epsilon)\}\,.$$

Obviously, $\mathcal{A}_0 \subset \mathcal{S}$. The following result gives a global extension of the function (λ, u) from $(-\epsilon, \epsilon)$ to $(-\infty, \infty)$ in the real analytic case.

Theorem 5.3. *Suppose (G1)-(G3) hold. Then, (λ, u) can be extended as a continuous map (still called) $(\lambda, u) : (-\infty, \infty) \to \mathcal{S}$ with the following properties:*

(a) *Let $\mathcal{A} \stackrel{\text{def}}{=} \{(\lambda(s), u(s)) : s \in \mathbb{R}\}$. Then, $\mathcal{A} \cap \mathcal{N}$ is an at most countable union of distinct distinguished arcs $\bigcup_{i=1}^{n} \mathcal{A}_i$, $n \leq \infty$.*
(b) *$\mathcal{A}_0 \subset \mathcal{A}_1$.*
(c) *$\{s \in \mathbb{R} \,:\, ker(\partial_x \Psi(\lambda(s), u(s))) \neq \{0\}\}$ is a discrete set.*
(d) *At each of its points $\mathcal{A}$ has a local analytic re-parameterisation in the following sense: For each $s^* \in \mathbb{R}$ there exists a continuous, injective map $\rho^* : (-1, 1) \to \mathbb{R}$ such that $\rho^*(0) = s^*$ and the re-parameterisation*

$$(-1, 1) \ni t \to (\lambda(\rho^*(t)), u(\rho^*(t))) \in \mathcal{A} \text{ is analytic.}$$

Furthermore, the map $s \mapsto \lambda(s)$ is injective in a neighborhood of $s = 0$ and for each $s^* \neq 0$ there exists $\epsilon^* > 0$ such that λ is injective on $[s^*, s^* + \epsilon^*]$ and on $[s^* - \epsilon^*, s^*]$.
(e) *Only one of the following alternatives occurs:*
  (i) *$\|(\lambda(s), u(s))\|_{\mathbb{R} \times \mathcal{X}} \to \infty$ as $s \to +\infty$ (resp. $s \to -\infty$).*
  (ii) *A subsequence $\{(\lambda(s_n), u(s_n))\}$ approaches the boundary of $\mathcal{U}$ as $s_n \to +\infty$ (resp. $s_n \to -\infty$).*
  (iii) *$\mathcal{A}$ is the closed loop:*

$$\mathcal{A} = \{(\lambda(s), u(s)) \,:\, -T \leq s \leq T, (\lambda(T), u(T)) = (\lambda(-T), u(-T)) \text{ for some } T > 0\}\,.$$

In this case, choosing the smallest such $T > 0$ we have

$$(\lambda(s + 2T), u(s + 2T)) = (\lambda(s), u(s)) \quad \text{for all} \quad s \in \mathbb{R}.$$

(f) *Suppose $\partial_x \Psi(\lambda(s_1), u(s_1))$ is invertible for some $s_1 \in \mathbb{R}$. If for some $s_2 \neq s_1$, we have $(\lambda(s_1), u(s_1)) = (\lambda(s_2), u(s_2))$, then (e)(iii) occurs and $|s_1 - s_2|$ is an integer multiple of $2T$. In particular, the map $s \mapsto (\lambda(s), u(s))$ is injective on $[-T, T)$.*

Remark 5.4. We remark that theorem 9.1.1 in [7] deals with "bifurcation from the first eigenvalue" type of situation whereas theorem 1.13 in [4] concerns the bifurcation from origin. The conditions $(G1)-(G3)$ assumed there are required only to ensure that the starting analytic path corresponding to $\mathcal{A}_0$ is available for global extension. In our case, we make this as an assumption $(G3)$ above. Hence the proof given in [7] and in [4] holds good in our case as well.

We recall the following result from [4] (proposition 2.1).

Proposition 5.5. *Let $g:\mathbb{R}\to\mathbb{R}$ be an entire function with $g(0)=0$. Define $M_k(a)=\max_{[-a,a]} g^{(k)}$, $k=1,2,3,\ldots$. Assume that for any $a\geq 0$, there exists $\mu>0$ such that the series $\sum_{k=0}^{\infty}\frac{M_k(a)}{k!}\mu^k$ converges. Then, for any $\phi\in C_0(\overline{\Omega}), \phi>0$ in Ω, we have $\mathcal{C}_\phi(\Omega)\ni u\mapsto g(u)\in\mathcal{C}_\phi(\Omega)$ is an analytic map. Furthermore, if $\inf_{[0,\infty)} g'>0$, then g maps $\mathcal{C}^+_\phi(\Omega)$ into itself.*

Consider now the solution operator F associated to (P_λ) defined in (1.7).

Proposition 5.6. *The map F takes $\mathbb{R}\times\mathcal{C}^+_{\phi_1}(\Omega)$ into $\mathcal{C}_{\phi_1}(\Omega)$ and is analytic. Furthermore, if $\lambda\geq 0$, then $F(\lambda,\cdot)$ maps $\mathcal{C}^+_{\phi_1}(\Omega)$ into $\mathcal{C}^+_{\phi_1}(\Omega)$.*

Proof.

Step 1: *The map $\mathcal{C}^+_{\phi_1}(\Omega)\ni u\longmapsto K(x)u^{-\alpha}+\lambda f(u)\in\mathcal{C}_{\phi_1^{-\alpha-\beta}}(\Omega)$ is analytic.* Given $u\in\mathcal{C}^+_{\phi_1}(\Omega)$, it follows that $K(x)u^{-\alpha}\in\mathcal{C}_{\phi_1^{-\alpha-\beta}}(\Omega)$. Then following the arguments in Step 1 of prop.2.3 in [4], we obtain the analyticity of the map.

Step 2: *The map $\mathcal{C}_{\phi_1^{-\alpha-\beta}}(\Omega)\ni u\longmapsto(\Delta^2)^{-1}u\in\mathcal{C}_{\phi_1}(\Omega)$ is a linear continuous map (and hence analytic).* Furthermore, this map takes $\mathcal{C}^+_{\phi_1^{-\alpha-\beta}}(\Omega)$ into $\mathcal{C}^+_{\phi_1}(\Omega)$.

We observe that $(\Delta^2)^{-1}$ is well defined on $\mathcal{C}_{\phi_1^{-\alpha-\beta}}(\Omega)$. Indeed, since $\alpha+\beta<2$, from Lemma 2.3, there exists a unique solution $w\in C^{2,\theta}(\overline{\Omega}), 0<\theta<1$, solving

$$\begin{cases}\Delta^2 w=u \ \text{ in } \ \Omega, \quad u\in\mathcal{C}_{\phi_1^{-\alpha-\beta}}(\Omega),\\ w=\Delta w=0 \text{ on } \partial\Omega.\end{cases}\tag{5.1}$$

Clearly, $w:=(\Delta^2)^{-1}u\in\mathcal{C}_{\phi_1}(\Omega)$. If additionally $u\in\mathcal{C}^+_{\phi_1}(\Omega)$, from the Hopf principle in Corollary 2.2, we also have that $w\in\mathcal{C}^+_{\phi_1}(\Omega)$. The proof of the proposition follows by combining Steps 1 and 2. □

We now prove the existence of $\mathcal{A}_0$:

Proposition 5.7. *Let $0<\alpha+\beta<2$. There exists a $\lambda_0>0$ such that for all $\lambda\in(-\lambda_0,\lambda_0)$, there exists a nondegenerate solution $u_\lambda\in\mathcal{C}^+_{\phi_1}(\Omega)$ to (P_λ). Furthermore, the map*

$$(-\lambda_0,\lambda_0)\ni\lambda\to u_\lambda\in\mathcal{C}^+_{\phi_1}(\Omega)$$

is analytic and $\|u_\lambda-u_0\|_{\mathcal{C}_{\phi_1}(\Omega)}\to 0$ as $\lambda\to 0$ where u_0 is the unique solution to (P).

Proof. We would like to apply the analytic version of the implicit function theorem. Given $u\in\mathcal{C}^+_{\phi_1}$, we can check that the linearised operator $\partial_u F(\lambda,u):\mathcal{C}_{\phi_1}(\Omega)\to\mathcal{C}_{\phi_1}(\Omega)$ is given by

$$\partial_u F(\lambda,u)\phi=\phi+(\Delta^2)^{-1}\Big([\alpha K(x)u^{-\alpha-1}-\lambda f'(u)]\phi\Big).\tag{5.2}$$

Note that $K(x)u^{-\alpha-1}\phi \in \mathcal{C}_{\phi_1^{-\alpha-\beta}}(\Omega)$. Indeed, for some $C > 0$,

$$\|K(x)u^{-\alpha-1}\phi\|_{\mathcal{C}_{\phi_1^{-\alpha-\beta}}(\Omega)} \leq C\|\phi\|_{\mathcal{C}_{\phi_1}(\Omega)}.$$

Therefore from Lemma 2.3, we obtain a constant $\theta \in (0,1)$ (depending only on ν, α and β) such that

$$\left\{K(x)u^{-\alpha-1}\phi\right\} \text{ bounded in } \mathcal{C}_{\phi_1^{-\alpha-\beta}}(\Omega)$$
$$\Longrightarrow \left\{(\Delta^2)^{-1}(K(x)u^{-\alpha-1}\phi)\right\} \text{ bounded in } C^{2,\theta}(\overline{\Omega}).$$

We infer then that $\partial_u F(\lambda, u)$ is a compact perturbation of the identity and hence is a Fredholm operator of index zero. Next, we show that $\partial_u F(\lambda, u)$ is invertible for $\lambda \leq 0$. If ϕ belongs to the kernel of $\partial_u F(\lambda, u)$, denoted by $N(\partial_u F(\lambda, u))$, we will have

$$\int_\Omega (\Delta\phi)^2 \mathrm{d}x + \alpha \int_\Omega K(x)u^{-\alpha-1}\phi^2 \mathrm{d}x - \lambda \int_\Omega f'(u)\phi^2 \mathrm{d}x = 0.$$

Using (f_2) and nonpositivity of λ we get $\phi \equiv 0$ from the above identity. Therefore, if $\lambda \leq 0$, we have $N(\partial_u F(\lambda, u)) = \{0\}$ which implies that $\partial_u F(\lambda, u)$ is invertible.

Appealing to the real analytic version of implicit function theorem (see [7]), we obtain a $\lambda_0 > 0$ and an analytic branch of solutions, $\lambda \to (\lambda, u_\lambda)$ to $F(\lambda, u) = 0$ for $\lambda \in (-\lambda_0, \lambda_0)$. By taking λ_0 smaller if required, from the smoothness of the map F we obtain that $\partial_u F(\lambda, u_\lambda)$ is invertible for all $-\lambda_0 < \lambda < \lambda_0$. That is, the solution u_λ is nondegenerate for all such λ. □

Proposition 5.8. *There exists $\Lambda > 0$ such that (P_λ) admits no solution for $\lambda > \Lambda$.*

Proof. Using the assumption on K and (f_2) we note that for some positive constants c_1, c_2,

$$K(x)t^{-\alpha} + \lambda f(t) \geq c_1 + c_2\lambda t \quad \text{for all} \quad x \in \Omega, t > 0.$$

Let $\lambda > 0$. We multiply the equation in (P_λ) by ϕ_1 and use the above inequality to get for any solution u to (P_λ):

$$\lambda_1^2 \int_\Omega u\phi_1 \mathrm{d}x = \int_\Omega K(x)\phi_1 u^{-\alpha} \mathrm{d}x + \lambda \int_\Omega f(u)\phi_1 \mathrm{d}x \geq c_1 \int_\Omega \phi_1 \mathrm{d}x + c_2\lambda \int_\Omega u\phi_1 \mathrm{d}x.$$

Hence, necessarily, $\lambda \leq \frac{\lambda_1^2}{c_2}$. □

We finally prove Theorem 1.6.

Proof. Let $\mathcal{U} \stackrel{\text{def}}{=} \mathbb{R} \times \mathcal{C}^+_{\phi_1}(\Omega)$. We first check that F satisfies the conditions $(G1) - (G3)$ in order to apply Theorem 5.3. From the regularity estimate in Lemma 2.3, we deduce that any bounded subset of $\mathcal{S}$ is relatively compact in $\mathbb{R} \times \mathcal{C}_{\phi_1}$, i.e., $(G1)$ holds. $(G2) - (G3)$ follow from the above proposition. Hence Theorem 5.3 asserts the existence of $\mathcal{A} \subset \mathcal{S}$ satisfying $(a) - (e)$.

(i) follows from Proposition 5.7.

(ii) We first prove that the assertion in the alternative $(e)(i)$ of Theorem 5.3 occurs. We do this by ruling out the possibilities $(e)(ii)$ and $(e)(iii)$.

The case $(e)(ii)$ can be ruled out as follows. Suppose there exists a sequence $\{(\lambda(s_n), u(s_n))\} \subset \mathcal{A}$ such that $(\lambda(s_n), u(s_n)) \to (\tilde{\lambda}, \tilde{u}) \in \partial\mathcal{U}$ in $\mathbb{R} \times \mathcal{C}_{\phi_1}(\Omega)$ as

$s_n \to \infty$. In particular, $\tilde{u} \notin \mathcal{C}^+_{\phi_1}(\Omega)$. Applying Corollary 2.2, we get for some $C > 0$ independent of n,

$$u(s_n)(x) \geq C\rho(x) \int_\Omega K(y)u(s_n)^{-\alpha} + \lambda(s_n)f(u(s_n))\rho(y)\mathrm{d}y \ \ \forall x \in \Omega. \tag{5.3}$$

Passing to the limit as $n \to \infty$ and using Fatou's lemma, we get

$$\tilde{u}(x) \geq C\rho(x) \int_\Omega (K(y)\tilde{u}^{-\alpha} + \tilde{\lambda} f(\tilde{u}))\rho(y)\mathrm{d}y \tag{5.4}$$

which contradicts the assumption $\tilde{u} \notin \mathcal{C}^+_{\phi_1}(\Omega)$ if $\tilde{\lambda} \geq 0$.

Next we rule out alternative $(e)(iii)$. For that, we observe that u_0 is the unique solution to (P_0) and from the implicit function theorem, $\mathcal{A}_0$ is the unique branch of solutions emanating from $(0, u_0)$. Therefore, $\mathcal{A}$ cannot bend back to join the point $(0, u_0)$.

Hence alternative $(e)(i)$ of Theorem 5.3 holds. From Proposition 5.8, the conclusion (ii) of the theorem follows.

(iii) follows in view of (ii) and the fact that there is no solution for all large λ (Proposition 5.8).

(iv) and (v) of Theorem 1.6 follow directly from (c) and (d) of Theorem 5.3.

(vi) We also note that (see Proposition 5.7) since $\partial_u F(\lambda, u_\lambda)$ is an invertible operator for $\lambda < 0$, the negative portion of $\mathcal{A}$, i.e., $\mathcal{A} \cap (-\infty, 0) \times \mathcal{C}_{\phi_1}(\Omega)$, is a single analytic curve (indeed a graph from the λ axis) consisting of nondegenerate solutions u_λ. In particular, this curve does not undergo any bifurcations.

This completes the proof of the theorem. □

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