

## PURELY SINGULAR CONTINUOUS SPECTRUM FOR STURMIAN CMV MATRICES VIA STRENGTHENED GORDON LEMMAS

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ABSTRACT. The Gordon Lemma refers to a class of results in spectral theory which prove that strong local repetitions in the structure of an operator preclude the existence of eigenvalues for said operator. We expand on recent work of Ong and prove versions of the Gordon Lemma that are valid for CMV matrices and which do not restrict the parity of scales upon which repetitions occur. The key ingredient in our approach is a formula of Damanik-Fillman-Lukic-Yessen which relates two classes of transfer matrices for a given CMV operator. There are many examples to which our result can be applied. We apply our theorem to complete the classification of the spectral type of CMV matrices with Sturmian Verblunsky coefficients; we prove that such CMV matrices have purely singular continuous spectrum supported on a Cantor set of zero Lebesgue measure for all (irrational) frequencies and all phases. We also discuss applications to CMV matrices with Verblunsky coefficients generated by general codings of rotations.

### 1. INTRODUCTION

In this paper, we will discuss the spectral type of aperiodic extended CMV matrices. Extended CMV matrices arise as a natural playground for spectral theoretic techniques, as they furnish canonical unitary analogs of Schrödinger operators and Jacobi matrices. Moreover, they are interesting in their own right, since they naturally arise in connection with quantum walks in one dimension [3, 6–8], the classical Ising model [9, 16], and orthogonal polynomials on the unit circle [36, 37]. More specifically, given a sequence  $\alpha = (\alpha_n)_{n \in \mathbb{Z}}$  with  $\alpha_n \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ , the corresponding extended CMV matrix is given by  $\mathcal{E} = \mathcal{L}\mathcal{M}$ , where

$$\mathcal{L} = \bigoplus_{j \in \mathbb{Z}} \Theta(\alpha_{2j}), \quad \mathcal{M} = \bigoplus_{j \in \mathbb{Z}} \Theta(\alpha_{2j+1}), \quad \Theta(\alpha) = \begin{pmatrix} \bar{\alpha} & \rho \\ \rho & -\alpha \end{pmatrix}, \quad \rho = \sqrt{1 - |\alpha|^2},$$

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[24, 25, 31] are inspired references for the interested reader. Given an alphabet with two letters, say  $\mathcal{A} = \{0, 1\}$ , consider a sequence  $\omega \in \mathcal{A}^{\mathbb{Z}}$ . The *factor complexity function* of  $\omega$  is defined by

$$p_\omega(n) = \#F_\omega(n), \quad n \geq 0,$$

where  $F_\omega(n)$  denotes the set of all distinct subwords of  $\omega$  of length  $n$ . The celebrated theorem of Hedlund and Morse states that a recurrent sequence  $\omega \in \mathcal{A}^{\mathbb{Z}}$  is periodic if and only if  $p_\omega(n) \leq n$  for some  $n \in \mathbb{Z}_+$  [27]. We say that  $\omega$  is *Sturmian* if it is recurrent and satisfies  $p_\omega(n) = n + 1$  for every  $n \geq 0$ . Note that considering  $n = 1$  already forces the underlying alphabet to have precisely two symbols. In light of the aforementioned Hedlund-Morse Theorem, Sturmian sequences are precisely those aperiodic sequences that are “closest to periodic” in the sense of combinatorial complexity. Equivalently, it turns out that  $\omega$  is Sturmian if and only if it is a so-called mechanical sequence of irrational slope. More precisely,  $\omega$  is Sturmian if and only if there exist  $\theta \in (0, 1) \setminus \mathbb{Q}$  and  $\varphi \in [0, 1)$  such that  $\omega = s_{\theta, \varphi}$  or  $\omega = s'_{\theta, \varphi}$ , where

$$\begin{aligned} s_{\theta, \varphi}(n) &= \lfloor (n + 1)\theta + \varphi \rfloor - \lfloor n\theta + \varphi \rfloor, \\ s'_{\theta, \varphi}(n) &= \lceil (n + 1)\theta + \varphi \rceil - \lceil n\theta + \varphi \rceil \end{aligned}$$

for  $n \in \mathbb{Z}$ . We call  $\theta$  the *frequency* and  $\varphi$  the *phase* of  $\omega$ . The set of all Sturmian words with a given frequency  $\theta$  is called the associated *subshift*:

$$\Omega = \Omega_\theta = \{s_{\theta, \varphi} : \varphi \in [0, 1)\} \cup \{s'_{\theta, \varphi} : \varphi \in [0, 1)\} \subseteq \mathcal{A}^{\mathbb{Z}}.$$

It is straightforward to check that  $\Omega_\theta$  is a compact subset of  $\mathcal{A}^{\mathbb{Z}}$  equipped with the product topology and that  $\Omega$  is invariant under the action of the left shift  $S : \mathcal{A}^{\mathbb{Z}} \rightarrow \mathcal{A}^{\mathbb{Z}}$  defined by  $(S\omega)_n = \omega_{n+1}$ . Additionally,  $(\Omega_\theta, S)$  is a strictly ergodic dynamical system; that is, it is both minimal and uniquely ergodic. By a Sturmian CMV matrix, we shall then mean one whose Verblunsky coefficients assume two distinct values, modulated by a specific Sturmian sequence. More precisely, we choose  $\beta \neq \gamma$  in  $\mathbb{D}$ , an irrational frequency  $\theta \in (0, 1) \setminus \mathbb{Q}$ , and a Sturmian sequence  $\omega \in \Omega_\theta$ , and then define an extended CMV matrix  $\mathcal{E} = \mathcal{E}_\omega$  by

$$(1.2) \quad \alpha_\omega(n) = \beta + \omega_n(\gamma - \beta) = \begin{cases} \beta & \omega_n = 0, \\ \gamma & \omega_n = 1. \end{cases}$$

**Theorem 1.1.** *For all  $\beta \neq \gamma$  in  $\mathbb{D}$  and all Sturmian sequences  $\omega = s_{\theta, \varphi}$  or  $\omega = s'_{\theta, \varphi} \in \mathcal{A}^{\mathbb{Z}}$ ,  $\mathcal{E}_\omega$  has purely singular continuous spectrum supported on a Cantor set of zero Lebesgue measure.*

It is well known that the spectrum  $\Sigma = \sigma(\mathcal{E}_\omega)$  is independent of  $\omega \in \Omega_\theta$ , i.e.,  $\Sigma$  depends only on  $\beta$ ,  $\gamma$ , and  $\theta$ . Moreover,  $\Sigma$  is a Cantor set of zero Lebesgue measure for all  $\beta \neq \gamma$ , and Sturmian  $\omega$ , and hence, it cannot support absolutely continuous measures [14, Theorem 1.1]; see also [13]. Consequently, the new content of Theorem 1.1 is supplied by proving that  $\mathcal{E}_\omega$  does not have eigenvalues. Let us note that this extends the results of [30] in two directions. First, we do not need to restrict the continued fraction coefficients of the frequency  $\theta$ . Secondly, [30] makes a statement for Lebesgue-almost every phase  $\varphi \in [0, 1)$ , while this result describes the spectral type for arbitrary phase.

**1.2. Codings of rotations.** We may also apply our new Gordon Lemmas to CMV matrices whose Verblunsky coefficients are generated by more general codings of rotations. Given  $\theta \in (0, 1) \setminus \mathbb{Q}$ ,  $\varphi \in [0, 1)$ , and a non-degenerate interval  $I \subseteq \mathbb{T} \stackrel{\text{def}}{=} \mathbb{R}/\mathbb{Z}$ , the associated *rotation coding sequence*  $u \in \{0, 1\}^{\mathbb{Z}}$  is given by

$$(1.3) \quad u(n) = \chi_I(n\theta + \varphi).$$

Notice that Sturmian sequences are a special case of rotation codings, obtained by setting  $I = [1 - \theta, 1)$  or  $I = (1 - \theta, 1]$ . As with Sturmian sequences, one can use codings of rotations to generate CMV matrices; more specifically, given  $\beta \neq \gamma$  in  $\mathbb{D}$ , and  $\theta, \varphi$ , and  $I$  as above, we consider  $\mathcal{E} = \mathcal{E}_{\theta, \varphi}^I$  given by

$$\alpha_{\theta, \varphi}^I(n) = \begin{cases} \beta & u(n) = 0, \\ \gamma & u(n) = 1, \end{cases}$$

where  $u_n$  is defined by (1.3). Our theorem for these is precisely [30, Theorem 4] with the hypothesis that  $\theta$  have infinitely many even continued fraction denominators removed.

**Theorem 1.2.** *Let  $\theta \in (0, 1) \setminus \mathbb{Q}$  have continued fraction expansion*

$$\theta = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

with  $a_j \in \mathbb{Z}_+$  for all  $j$ , and fix a non-degenerate interval  $I \subseteq \mathbb{T}$ . If  $\limsup_{j \rightarrow \infty} a_j \geq 4$ , then  $\mathcal{E}_{\theta, \varphi}^I$  has purely continuous spectrum for Lebesgue almost-every  $\varphi \in [0, 1)$ .

**1.3. Gordon Lemmas for CMV matrices.** Let us now precisely describe the new CMV Gordon Lemmas. Given  $\alpha \in \mathbb{D}$  and  $z \in \partial\mathbb{D}$ , the corresponding *Szegő matrix* is defined by

$$S(\alpha, z) = \frac{1}{\rho} \begin{pmatrix} z & -\bar{\alpha} \\ -\alpha z & 1 \end{pmatrix}, \quad \rho = \rho_\alpha = \sqrt{1 - |\alpha|^2}.$$

The Szegő transfer matrices associated to  $\mathcal{E}_\alpha$  are then defined by

$$T(n, m; z) = \begin{cases} S(\alpha_{n-1}, z) \cdots S(\alpha_m, z) & n > m, \\ I & n = m, \\ T(m, n; z)^{-1} & n < m, \end{cases}$$

for  $n, m \in \mathbb{Z}$  and  $z \in \partial\mathbb{D}$ . We will also consider the Gesztesy–Zinchenko transfer matrices [18]. Define

$$(1.4) \quad P(\alpha, z) = \frac{1}{\rho} \begin{pmatrix} -\alpha & z^{-1} \\ z & -\bar{\alpha} \end{pmatrix}, \quad Q(\alpha, z) = \frac{1}{\rho} \begin{pmatrix} -\bar{\alpha} & 1 \\ 1 & -\alpha \end{pmatrix}, \quad \alpha \in \mathbb{D}, z \in \mathbb{C} \setminus \{0\},$$

where  $\rho = (1 - |\alpha|^2)^{1/2}$  as before. One may use  $P$  and  $Q$  to capture the recursion described by the difference equation  $\mathcal{E}u = zu$  in the following sense. If  $u \in \mathbb{C}^{\mathbb{Z}}$  is such that  $\mathcal{E}u = zu$ , then we define  $v = \mathcal{M}u$ , and since  $\Theta(\alpha)$  is real-symmetric for all  $\alpha \in \mathbb{D}$ , we have

$$\mathcal{E}^\top v = \mathcal{M}\mathcal{L}\mathcal{M}u = zv.$$

Now, with  $\Phi(n) = (u_n, v_n)^\top$ , Gesztesy and Zinchenko’s arguments show that

$$(1.5) \quad \Phi(n + 1) = \begin{cases} P(\alpha_n, z)\Phi(n) & n \text{ is even,} \\ Q(\alpha_n, z)\Phi(n) & n \text{ is odd,} \end{cases}$$

for all  $n \in \mathbb{Z}$ ; see also [26, Proposition 3] for a version of this formalism for more general finite-width unitary operators. It is relatively easy to see how these matrices arise from the definitions of  $u$  and  $v$ . In particular, (1.5) is exactly what one observes if one solves

$$\Theta(\alpha_{2j-1}) \begin{pmatrix} u_{2j-1} \\ u_{2j} \end{pmatrix} = \begin{pmatrix} v_{2j-1} \\ v_{2j} \end{pmatrix}, \quad \Theta(\alpha_{2j}) \begin{pmatrix} v_{2j} \\ v_{2j+1} \end{pmatrix} = z \begin{pmatrix} u_{2j} \\ u_{2j+1} \end{pmatrix},$$

for  $u_{n+1}$  and  $v_{n+1}$  in terms of  $v_n$  and  $u_n$ . Notice that we have shifted Gesztesy–Zinchenko’s definitions slightly so that our conventions for the definition of  $\mathcal{E}$  line up with those of [36, 37]. In light of these observations, we denote  $Y(n, z) = P(\alpha_n, z)$  when  $n$  is even and  $Y(n, z) = Q(\alpha_n, z)$  when  $n$  is odd, and we define the *Gesztesy–Zinchenko cocycle* by

$$(1.6) \quad Z(n, m; z) = \begin{cases} Y(n - 1, z) \cdots Y(m, z) & n > m, \\ I & n = m, \\ Z(m, n; z)^{-1} & n < m. \end{cases}$$

If  $u, v$ , and  $\Phi$  are as above, we have

$$(1.7) \quad \Phi(n) = Z(n, m; z)\Phi(m) \quad \text{for all } n, m \in \mathbb{Z}$$

by (1.5); see also [18, Lemma 2.2]. We will typically abbreviate and refer to  $Z(n, m; z)$  as a GZ matrix.

Each family of cocycles has distinct applications, advantages, and drawbacks. The Szegő-type matrices are easier to work with in general, since every one-step Szegő matrix has the same structure, whereas the one-step GZ matrices alternate. However, the GZ matrices propagate solution data for the eigenvalue equation, which is more useful from the point of view of spectral theory, since one can translate information about solutions into information about resolvents rather directly. Delightfully, there is a rather straightforward manner in which the two families of matrices are related, which was observed in [10]:

$$Q(\alpha, z)P(\beta, z) = z^{-1}S(\alpha, z)S(\beta, z)$$

for all  $\alpha, \beta \in \mathbb{D}$  and  $z \in \mathbb{C} \setminus \{0\}$ . In particular, one has

$$(1.8) \quad z^{-n}T(2n, 0; z) = Z(2n, 0; z)$$

for all  $z \in \mathbb{C} \setminus \{0\}$  and all  $n \geq 0$ . This relationship is the key that enables us to prove new versions of the two- and three-block Gordon Lemmas for CMV matrices with no restrictions on the parity of the scales of repetitions. In order to avoid losing information when we use the identity (1.8), we will assume throughout that  $\alpha$  is bounded away from  $\partial\mathbb{D}$ , that is,  $\|\alpha\|_\infty < 1$ .

**Theorem 1.3** (Two-block Gordon Lemma). *Let  $\alpha$  be bounded away from  $\partial\mathbb{D}$ . Suppose that there exist  $z_0 \in \partial\mathbb{D}$ , a constant  $c > 0$ , and a sequence of positive integers  $n_k \rightarrow \infty$  such that*

$$(1.9) \quad \alpha(j) = \alpha(j + n_k), \quad 0 \leq j \leq n_k - 1,$$

and  $|\text{tr}(T(n_k, 0; z_0))| \leq c$  for all  $k$ . Then  $z_0$  is not an eigenvalue of  $\mathcal{E}_\alpha$ . More precisely, there is a constant  $\Delta > 0$  that depends only on  $c$  and  $\|\alpha\|_\infty$  such that

$$(1.10) \quad \max (\|\Phi(\underline{n}_k)\|, \|\Phi(2n_k)\|) \geq \Delta \|\Phi(0)\|$$

for all  $k$ , where  $\Phi(n) = (u_n, v_n)^\top$ ,  $u$  solves  $\mathcal{E}u = z_0u$ ,  $v = \mathcal{M}u$ , and  $\underline{n}_k = 2\lfloor n_k/2 \rfloor$  denotes the greatest even integer which fails to exceed  $n_k$ .

If the Verblunsky coefficients exhibit an additional block of repetitions, one may abandon the hypothesis on the traces.

**Theorem 1.4** (Three-block Gordon Lemma). *Suppose  $\alpha$  is bounded away from  $\partial\mathbb{D}$  and that there exist positive integers  $n_k \rightarrow \infty$  such that  $\alpha(j - n_k) = \alpha(j) = \alpha(j + n_k)$  for all  $0 \leq j \leq n_k - 1$ . Then  $\mathcal{E}_\alpha$  has purely continuous spectrum.*

Moreover, one need not insist that repetitions be exact in Theorem 1.4. We refer to sequences of Verblunsky coefficients that obey an approximate version of the three-block repetition property as Gordon sequences. More precisely, let us say that  $\alpha \in \mathbb{D}^\mathbb{Z}$  is a *Gordon sequence* if it is bounded away from  $\partial\mathbb{D}$  and there exist positive integers  $n_k \rightarrow \infty$  such that

$$(1.11) \quad \lim_{k \rightarrow \infty} C^{n_k} \max_{0 \leq j \leq n_k - 1} |\alpha(j) - \alpha(j \pm n_k)| = 0$$

for all  $C > 0$ . The  $\pm$  in (1.11) is meant to signify that the equation is true for either choice of sign. These sequences also produce CMV operators with empty point spectrum.

**Theorem 1.5.** *If  $\alpha$  is a Gordon sequence, then  $\mathcal{E}_\alpha$  has purely continuous spectrum.*

*Remark 1.6.* For simplicity and clarity, we have stated the Gordon Lemmas with repetitions starting at/centered around the origin. However, it is easy to see that one can suitably modify the statements to apply if one has repetitions which are centered around some other fixed integer  $\ell_0 \in \mathbb{Z}$ .

The structure of the paper is as follows. In Section 2, we prove the new Gordon Lemmas for CMV matrices, i.e., Theorems 1.3, 1.4 and 1.5. Section 3 uses the new Gordon Lemmas to prove Theorems 1.1 and 1.2. Additionally, we provide a simple proof that the spectra of Sturmian CMV matrices may be characterized as the set of spectral parameters at which the associated trace map enjoys a bounded orbit. To the best of our knowledge, trace bounds in this setting have not been worked out in this level of generality. Moreover, we believe that our proof which establishes that traces are bounded on the spectrum (i.e., the proof of Lemma 3.5) is somewhat simpler than the standard arguments; we use growth estimates for escaping trace orbits [11] and the fact that the Lyapunov exponent vanishes on the spectrum [14].

## 2. PROOFS OF GORDON LEMMAS

We now prove the new Gordon Lemmas. To deal with odd repetition scales, we apply Cayley–Hamilton to the Szegő transfer matrices, use the relationship (1.8) over a slightly smaller block of even length, and then use that  $\alpha$  is bounded away from  $\partial\mathbb{D}$  to see that one still has effective lower bounds on the GZ evolution. The precise details follow presently.

*Proof of Theorem 1.3.* Let  $u \in \mathbb{C}^{\mathbb{Z}}$  be a solution to  $\mathcal{E}u = z_0u$ , and define  $v = \mathcal{M}u$ ,  $\Phi(n) = (u_n, v_n)^\top$ , and  $\Phi = \Phi(0)$ . Thus, one has  $\Phi(n) = Z(n, 0; z_0)\Phi$  for all  $n \in \mathbb{Z}$ . Without loss, assume  $|u_0|^2 + |v_0|^2 = 1$ . Since  $\Theta(\alpha)$  is unitary for every  $\alpha \in \mathbb{D}$ , it follows that

$$(2.1) \quad \lim_{n \rightarrow \infty} u_n = 0 \iff \lim_{n \rightarrow \infty} v_n = 0 \iff \lim_{n \rightarrow \infty} \Phi(n) = 0.$$

We claim that (1.10) holds with  $\Delta$  defined by

$$(2.2) \quad \eta = \frac{1}{2} \min(1, c^{-1}), \quad \Gamma = \sup_{n \in \mathbb{Z}} \|S(\alpha_n, z)\|, \quad \Delta = \frac{\eta}{\Gamma}.$$

Notice that  $\Gamma < \infty$  (and hence  $\Delta > 0$ ), since  $\alpha$  is bounded away from  $\partial\mathbb{D}$ . Moreover, we also have  $\Gamma \geq 1$  (and hence  $\Delta \leq \eta$ ) since  $\det(S(\alpha, z_0)) = z_0$  for all  $\alpha \in \mathbb{D}$ . If  $n_k$  is even, one may apply the arguments from [30] without modification to deduce that

$$\max(\|\Phi(2n_k)\|, \|\Phi(\underline{n}_k)\|) = \max(\|\Phi(2n_k)\|, \|\Phi(n_k)\|) \geq \eta \geq \Delta,$$

since (1.8) implies that  $|\operatorname{tr}(Z(j, 0; z))| = |\operatorname{tr}(T(j, 0; z))|$  whenever  $j$  is even and  $z \in \partial\mathbb{D}$ . Now, suppose  $n_k$  is odd. Denote  $\Psi(n) = T(n, 0; z_0)\Phi$  for  $n \in \mathbb{Z}$ . By Cayley–Hamilton and (1.9), we have

$$\Psi(2n_k) - \operatorname{tr}(T(n_k, 0; z_0))\Psi(n_k) + z_0^{n_k}\Phi = 0,$$

so, consequently, we deduce

$$\max(\|\Psi(2n_k)\|, \|\Psi(n_k)\|) \geq \eta.$$

If  $\|\Psi(2n_k)\| \geq \eta$ , we may apply the identity (1.8) to see that

$$\|\Phi(2n_k)\| = \|z_0^{-n_k}T(2n_k, 0; z_0)\Phi\| = \|\Psi(2n_k)\| \geq \eta$$

as well. Otherwise,  $\|\Psi(n_k)\| \geq \eta$ . Notice that  $\underline{n}_k = n_k - 1$ , which yields

$$S(\alpha_{n_k-1}, z_0)\Phi(\underline{n}_k) = z^{-\frac{n_k-1}{2}}T(n_k, 0; z)\Phi.$$

Thus,  $\|\Phi(\underline{n}_k)\| \geq \Gamma^{-1}\|\Psi(n_k)\| \geq \Delta$ , which concludes the proof of (1.10). By (2.1),  $u \notin \ell^2(\mathbb{Z})$ . Since no non-trivial solutions  $u$  of  $\mathcal{E}u = z_0u$  are square-summable, we conclude that  $z_0$  is not an eigenvalue of  $\mathcal{E}$ .  $\square$

*Proof of Theorem 1.4.* Let  $z \in \partial\mathbb{D}$  be given, and let  $u, v, \Phi, \Psi$ , and  $\Gamma$  be as in the previous theorem. If  $|\operatorname{tr}(T(n_k, 0; z))| \leq 1$ , then the proof of the previous theorem yields

$$\max(\|\Phi(\underline{n}_k)\|, \|\Phi(2n_k)\|) \geq \Delta,$$

where  $\Delta$  is defined by (2.2) with  $c = 1$ , i.e.,  $\Delta = \frac{1}{2\Gamma}$ . Otherwise, suppose  $|\operatorname{tr}(T(n_k, 0; z))| > 1$ . Arguing as in [30] and suitably modifying the argument as in the previous proof for odd  $n_k$ , we obtain

$$\max(\|\Phi(-\underline{n}_k)\|, \|\Phi(\underline{n}_k)\|) \geq \frac{1}{2\Gamma} > 0.$$

More specifically, if  $n_k$  is even, then the arguments of [30] give

$$\max(\|\Phi(-\underline{n}_k)\|, \|\Phi(\underline{n}_k)\|) \geq \frac{1}{2} \geq \Delta.$$

Notice that this uses (1.8) again to deduce  $|\operatorname{tr}(T(j, 0; z))| = |\operatorname{tr}(Z(j, 0; z))|$  for even  $j$ . Now, assume  $n_k$  is odd. By Cayley–Hamilton and our assumptions on  $\alpha$ , we have

$$\Psi(n_k) - \operatorname{tr}(T(n_k, 0; z))\Phi + z\Psi(-n_k) = 0,$$

which implies

$$\max (\|\Psi(-n_k)\|, \|\Psi(n_k)\|) > \frac{1}{2},$$

since  $|\text{tr}(T(n_k, 0; z))| > 1$ . Arguing as in the proof of Theorem 1.3, we use (1.8) to obtain

$$\max (\|\Phi(n_k)\|, \|\Phi(-n_k)\|) \geq \Gamma^{-1} \max (\|\Psi(n_k)\|, \|\Psi(-n_k)\|) \geq \frac{1}{2\Gamma}.$$

Thus, we have shown that

$$(2.3) \quad \max (\|\Phi(-n_k)\|, \|\Phi(n_k)\|, \|\Phi(2n_k)\|) \geq \Delta$$

for all  $k$ . □

For the next proof, let us note the following simple estimate.

**Lemma 2.1.** *For every  $r \in (0, 1)$ , there is a constant  $C = C(r)$  such that the following holds true: For any sequences  $\alpha, \tilde{\alpha} \in \mathbb{D}^{\mathbb{Z}}$  such that  $\|\alpha\|_{\infty}, \|\tilde{\alpha}\|_{\infty} \leq r$ , and  $|\alpha(j) - \tilde{\alpha}(j)| < \delta$  for all  $j$  between  $n$  and  $m$ , one has*

$$\|Z(n, m; z) - \tilde{Z}(n, m; z)\| \leq \delta C^{|n-m|}$$

for all  $n, m \in \mathbb{Z}$  and all  $z \in \partial\mathbb{D}$ .

*Proof.* Since  $|\det(Z(n, m; z))| = 1$  for all  $n, m \in \mathbb{Z}$  and all  $z \in \mathbb{C} \setminus \{0\}$ , it suffices to consider  $m = 0$  and  $n > 0$ . Notice that

$$Z(n, 0; z) - \tilde{Z}(n, 0; z) = \sum_{j=0}^{n-1} \left( \tilde{Z}(n, j+1; z)Z(j+1, 0; z) - \tilde{Z}(n, j; z)Z(j, 0; z) \right).$$

The desired estimate follows readily by writing the summands as

$$\tilde{Z}(n, j+1; z)(Y(j, z) - \tilde{Y}(j, z))Z(j, 0; z).$$

In particular, there exists  $A = A(r) \in (1, \infty)$  such that

$$\begin{aligned} \|P(\alpha, z)\| &\leq A, \\ \|Q(\alpha, z)\| &\leq A, \\ \|P(\alpha, z) - P(\beta, z)\| &\leq A|\alpha - \beta|, \\ \|Q(\alpha, z) - Q(\beta, z)\| &\leq A|\alpha - \beta|, \end{aligned}$$

for all  $z \in \partial\mathbb{D}$  and all  $\alpha, \beta \in \mathbb{D}$  with  $|\alpha|, |\beta| \leq r$ . Consequently,

$$\|Z(n, 0; z) - \tilde{Z}(n, 0; z)\| \leq n\delta A^n,$$

which yields the desired result with  $C = A\sqrt[3]{3}$ . □

*Proof of Theorem 1.5.* For each  $k$ , consider the  $n_k$ -periodic CMV matrix  $\mathcal{E}_k$  whose Verblunsky coefficients are given by repeating the string  $\alpha_0, \dots, \alpha_{n_k-1}$  periodically, and consider  $u$  and  $\Phi$  as in the previous arguments. By (1.11) and Lemma 2.1, we deduce that

$$\lim_{k \rightarrow \infty} \|Z_k(2n_k, 0; z) - Z(2n_k, 0; z)\| = \lim_{k \rightarrow \infty} \|Z_k(-n_k, 0; z) - Z(-n_k, 0; z)\| = 0,$$

where  $Z_k$  denotes the family of GZ transfer matrices associated with  $\mathcal{E}_k$ . Applying (2.3) to  $\mathcal{E}_k$ , we may choose  $\ell_k \in \{-n_k, n_k, 2n_k\}$  such that

$$\|Z_k(\ell_k, 0; z)\Phi\| \geq \Delta,$$

where  $\Delta = \frac{1}{2\Gamma}$  as in the proof of Theorem 1.4. Thus, we have

$$\|\Phi(\ell_k)\| \geq \|Z_k(\ell_k, 0; z)\Phi\| - \|Z(\ell_k, 0; z) - Z_k(\ell_k, 0; z)\| \geq \frac{\Delta}{2}$$

for all sufficiently large  $k$ . Consequently,  $\Phi$  is not in  $\ell^2(\mathbb{Z})$ , so neither is  $u$ . □

### 3. APPLICATION: SINGULAR CONTINUOUS SPECTRUM FOR ALL STURMIAN CMV MATRICES

In this section, we prove Theorem 1.1. The key ingredients are the new two-block Gordon Lemma, trace bounds on the spectrum, and a combinatorial analysis of the structure of elements of the subshift  $\Omega_\theta$ . We begin by recalling a few highlights from the theory of continued fractions without proofs. For a detailed exposition of the theory with proofs, we refer the reader to Khinchin’s beautiful text [22].

To any  $\theta \in (0, 1) \setminus \mathbb{Q}$ , there is a unique associated sequence  $(a_j)_{j=1}^\infty$  of positive integers with

$$(3.1) \quad \theta = [a_1, a_2, \dots] = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots}}}$$

Stopping the continued fraction expansion of  $\theta$  after  $n$  steps produces rational numbers  $r_n = [a_1, \dots, a_n] = p_n/q_n$ . If we assume that all of these fractions are in lowest terms (i.e.  $\gcd(p_n, q_n) = 1$ ), then one obtains the following recursive formulae for  $p_n$  and  $q_n$ :

$$(3.2) \quad p_0 = 0, \quad p_1 = 1, \quad p_n = a_n p_{n-1} + p_{n-2},$$

$$(3.3) \quad q_0 = 1, \quad q_1 = a_1, \quad q_n = a_n q_{n-1} + q_{n-2}.$$

Let us now observe how the continued fraction coefficients of  $\theta$  influence the combinatorial structure of elements of the hull  $\Omega_\theta$ . Fix  $\omega_0 = s_{\theta, \theta}$ , and define a sequence of words  $\{v_n\}_{n=-1}^\infty$  as follows:

$$(3.4) \quad v_{-1} = 1, \quad v_0 = 0, \quad v_1 = v_0^{a_1-1} v_{-1}, \quad v_n = v_{n-1}^{a_n} v_{n-2}, \quad n \geq 1.$$

Concretely, each  $v_n$  is an element of  $\{0, 1\}^*$ , the free monoid with two generators, so the multiplication in (3.4) is given by concatenation. By [2, Lemma 1(b)], one has

$$(3.5) \quad v_k = \omega_0|_{[0, q_k-1]} \quad \text{for all } k \geq 1.$$

Note the change in normalization: [2] considers  $s_{\theta, 0}$ , while we consider  $\omega_0 = s_{\theta, \theta}$ , which enables us to begin the analysis at the origin instead of at 1 (so that we can apply our version of the two-block Gordon Lemma as stated in Theorem 1.3).

The combinatorial structure of  $\omega_0$  is naturally manifested in the associated transfer matrices. More specifically, for  $\zeta \in \mathbb{C}$ , define

$$M_{-1}(\zeta) = S(\gamma, \zeta)S(\beta, \zeta)^{-1} = \frac{1}{\rho_\beta \rho_\gamma} \begin{pmatrix} 1 - \beta\bar{\gamma} & \bar{\beta} - \bar{\gamma} \\ \beta - \gamma & 1 - \bar{\beta}\gamma \end{pmatrix},$$

$$M_0(\zeta) = \zeta^{-1/2} S(\beta, \zeta) = \frac{\zeta^{-1/2}}{\rho_\beta} \begin{pmatrix} \zeta & -\bar{\beta} \\ -\beta\zeta & 1 \end{pmatrix},$$

and, for  $n \geq 1$ , define

$$(3.6) \quad M_n(\zeta) = \zeta^{-q_n/2} T_{\omega_0}(q_n, 0; \zeta), \quad n \geq 1, \zeta \in \mathbb{C},$$

where  $T_{\omega_0}$  is used to denote the family of Szegő transfer matrices associated to  $\alpha_{\omega_0}$ , given by (1.2). For each fixed  $\zeta$ , one chooses branches inductively to ensure that

$$(3.7) \quad \zeta^{-q_n/2} = \zeta^{-a_n q_{n-1}/2} \zeta^{-q_{n-2}/2} \quad \text{for all } n \geq 2.$$

Of course,  $M_n(\zeta)$  also depends on  $\beta, \gamma$ , and  $\theta$ , but we suppress this for the sake of readability. These matrices obey a recursive relationship which comes from (3.5).

**Lemma 3.1.** *Fix  $\zeta \in \partial\mathbb{D}$ . We have*

$$(3.8) \quad M_n(\zeta) = M_{n-2}(\zeta)M_{n-1}(\zeta)^{a_n} \quad \text{for all } n \geq 1.$$

*Proof.* This follows immediately from (3.4), (3.5), and (3.7). □

Now, since the transfer matrices  $M_n$  obey the same recursive relationship that one observes in the Schrödinger case, the traces may be realized by sampling orbits of the same dynamical system, simply with different initial conditions. More concretely, if we denote

$$x_n = \frac{1}{2} \text{tr}(M_{n-1}), \quad y_n = \frac{1}{2} \text{tr}(M_n), \quad z_n = \frac{1}{2} \text{tr}(M_n M_{n-1}),$$

then there are polynomial maps  $F_n : \mathbb{C}^3 \rightarrow \mathbb{C}^3$  such that

$$F_n(x_n(\zeta), y_n(\zeta), z_n(\zeta)) = (x_{n+1}(\zeta), y_{n+1}(\zeta), z_{n+1}(\zeta))$$

for all  $n \geq 0$  and all  $\zeta \in \mathbb{C}$ . Moreover, these polynomial maps enjoy a first integral given by the Fricke-Vogt invariant. More precisely,

$$I = I(\zeta) \stackrel{\text{def}}{=} x_n(\zeta)^2 + y_n(\zeta)^2 + z_n(\zeta)^2 - 2x_n(\zeta)y_n(\zeta)z_n(\zeta) - 1$$

is independent of  $n$  (though it will depend on  $\zeta$ ). This is explicitly computed in [15]. They find that

$$I(\zeta) = \frac{\text{Re}(\zeta)}{2\rho_\beta^2 \rho_\gamma^2} (\rho_\beta^2 + \rho_\gamma^2 - K) + \frac{1}{4\rho_\beta^2 \rho_\gamma^2} (K^2 - 2K + 2\rho_\beta^2 + 2\rho_\gamma^2) - 1,$$

where  $K = 2 - 2\text{Re}(\beta\bar{\gamma})$ . One can now characterize the uniform spectrum  $\Sigma \stackrel{\text{def}}{=} \sigma(\mathcal{E}_{\omega_0})$  as the dynamical spectrum, that is, the set of energies such that the corresponding trace map enjoys a bounded orbit.

**Theorem 3.2.** *Define*

$$B = \{\zeta \in \mathbb{C} : \text{tr}(M_n(\zeta)) \text{ is a bounded sequence in } \mathbb{C}\},$$

$$B' = \{\zeta \in \mathbb{C} : (x_n(\zeta), y_n(\zeta), z_n(\zeta)) \text{ is a bounded sequence in } \mathbb{C}^3\}.$$

Then  $B' = \Sigma = B$ .

*Remark 3.3.* By minimality and strong operator convergence, we know that  $\sigma(\mathcal{E}_\omega) = \Sigma$  for all  $\omega \in \Omega_\theta$ .

When  $\theta = \frac{\sqrt{5}-1}{2}$ , which corresponds to  $a_n \equiv 1$ , Theorem 3.2 is precisely [37, Theorem 12.8.3]. One can extend Simon’s arguments to the general case in a reasonable fashion à la [2], though the details become more technical. However, there is a simpler argument using hyperbolicity and the characterization of bounded

trace orbits from [11], which we shall describe presently for the reader’s convenience. First, let us recall the *Lyapunov exponent*, which is given by

$$(3.9) \quad L(\zeta) = \lim_{n \rightarrow \infty} \frac{1}{q_n} \log \|M_n(\zeta)\| = \lim_{m \rightarrow \infty} \frac{1}{m} \log \|T_{\omega_0}(m, 0; \zeta)\|, \quad \zeta \in \partial\mathbb{D}.$$

Strictly speaking, we should define  $L$  as an average over the associated hull with respect to the unique shift-invariant measure, that is, we should define

$$L(\zeta) = \lim_{n \rightarrow \infty} \frac{1}{n} \int_{\Omega_\theta} \log \|T_\omega(n, 0; \zeta)\| d\mu(\omega),$$

where  $\mu$  is the unique shift-invariant Borel probability measure on  $\Omega_\theta$  and  $T_\omega$  denotes a Szegő transfer matrix associated to  $\alpha_\omega$  defined as in (1.2). However, by [14, Proposition 3.1], this is equivalent to (3.9). We then consider  $\mathcal{Z}$ , the set on which the Lyapunov exponent vanishes, i.e.,

$$\mathcal{Z} = \{\zeta \in \partial\mathbb{D} : L(\zeta) = 0\}.$$

By [14, Theorem 1.1],  $\Sigma = \mathcal{Z}$ . Thus, to prove Theorem 3.2, it suffices to show that  $B' = \mathcal{Z} = B$ . We shall do this in three steps, namely, we will show  $B' \subseteq \mathcal{Z} \subseteq B \subseteq B'$ .

First, we shall show that there are no decaying solutions to the generalized eigenvalue equation in  $B'$ . Since this precludes uniform hyperbolicity of the Gesztesy-Zinchenko cocycle [10] and non-uniform hyperbolicity is absent in the present setting [14], one can use the DFLY formula to see that this yields  $B' \subseteq \mathcal{Z}$  as an immediate byproduct. Moreover, once we prove  $\Sigma = B'$ , we will also have established absence of eigenvalues on the spectrum.

**Lemma 3.4.** *Let  $\mathcal{E} = \mathcal{E}_\omega$  and  $\zeta \in B'$  be given. If  $u$  is a non-trivial solution to the difference equation  $\mathcal{E}u = \zeta u$ , then  $u_n$  does not converge to zero as  $|n| \rightarrow \infty$ . In particular,  $B' \subseteq \mathcal{Z}$ .*

*Proof.* Let  $\zeta \in B'$  be given. Since we have bounds on traces, it remains to be seen that one can produce suitable repetitions in  $\omega$  in order to apply the two-block Gordon Lemma. The relevant combinatorial analysis is furnished by the proof of [12, Lemma 4.1]. More specifically, they prove that, for every  $k \geq 3$ , there exists a word  $s = s_k \in \{v_{k-1}, v_k, v_{k+1}, v_{k+1}v_k\}$  of length  $n_k$  such that  $\omega(0), \dots, \omega(2n_k - 1)$  is a cyclic permutation of  $ss$ . Thus, since  $\zeta \in B'$ , we know that  $\text{tr}(T_\omega(n_k, 0; \zeta))$  is uniformly bounded as  $k \rightarrow \infty$  by cyclicity of the trace. In particular, Theorem 1.3 implies that there are no non-trivial solutions  $u$  of  $\mathcal{E}u = \zeta u$  such that  $u_n \rightarrow 0$  as  $|n| \rightarrow \infty$ .

To prove the second statement, notice that the Gesztesy-Zinchenko cocycle cannot exhibit non-uniform hyperbolicity by [14, Proposition 3.1]. Additionally, by [10], it cannot exhibit uniform hyperbolicity (as uniform hyperbolicity would imply the presence of decaying solutions to the difference equation). Thus,  $L(\zeta) = 0$  by (1.8).  $\square$

Next, we prove  $\mathcal{Z} \subseteq B$ . Essentially, if  $\zeta \notin B$ , then [11, Lemma 4.4] establishes lower bounds on the growth of  $\text{tr}(M_n(\zeta))$ , which yields an obvious lower bound on the growth of  $\|M_n(\zeta)\|$ , and these lower bounds suffice to prove  $L(\zeta) > 0$ .

There is one small subtlety to address regarding the invocation of [11, Lemma 4.4] in the following proof. Since [11] works in the Schrödinger setting, they have (and use) the a priori estimate  $|\text{tr}(M_{-1})| \leq 2$ , which is false in the present setting for

some values of  $\beta$  and  $\gamma$ . However, if one replaces the condition  $\delta \geq 0$  with the condition  $\delta \geq \max(0, |\text{tr}(M_{-1})| - 2)$  in the statement of [11, Lemma 4.4], then the proof of the same carries through verbatim to the present setting.

**Lemma 3.5.** *Let  $\beta, \gamma,$  and  $\theta$  be given. Then  $\mathcal{Z} \subseteq B$ .*

*Proof.* Suppose  $\zeta \in \partial\mathbb{D} \setminus B$ . Then, by [11, Lemma 4.4], there exists  $\delta > 0$  and  $k_0 \in \mathbb{Z}_+$  such that

$$(3.10) \quad |y_{k_0+k}(\zeta)| \geq \frac{1}{2}(1 + \delta)^{G_k^{(k_0)}} \quad \text{for all } k \geq k_0,$$

where

$$(3.11) \quad G_0^{(k_0)} = 1, \quad G_1^{(k_0)} = a_{k_0+1}, \quad G_{k+1}^{(k_0)} = a_{k_0+k+1}G_k^{(k_0)} + G_{k-1}^{(k_0)}, \quad k \geq 1.$$

Notice that [11] works in the Schrödinger setting, but this only amounts to a change in initial conditions. More specifically, our transfer matrices obey (3.8), which is precisely the same recursion as the Schrödinger matrices, and this is all that is used in the proof of [11, Lemma 4.4], so their characterization of escaping trace orbits applies to the present setting. Comparing (3.11) to (3.3), we see that there is a constant  $C > 0$  such that

$$(3.12) \quad \frac{G_k^{(k_0)}}{q_{k+k_0}} \geq C \quad \text{for every } k \geq 0.$$

More specifically, define

$$C = \min(q_{k_0}^{-1}, a_{k_0+1}/q_{k_0+1})$$

so that (3.12) holds for  $k = 0$  and  $k = 1$ . One then obtains (3.12) for all  $k$  by induction. Consequently, we have

$$L(\zeta) \geq C \log(1 + \delta) > 0$$

by (3.10) and (3.12). □

Finally, we prove  $B \subseteq B'$ . This is almost immediate from the conservation of the Fricke-Vogt character:

$$(3.13) \quad x_n(\zeta)^2 + y_n(\zeta)^2 + z_n(\zeta)^2 - 2x_n(\zeta)y_n(\zeta)z_n(\zeta) - 1 = I(\zeta) \quad \text{for all } n \geq 0.$$

**Lemma 3.6.** *With setup as above,  $B \subseteq B'$ .*

*Proof.* Let  $\zeta \in B$  be given. To show  $\zeta \in B'$ , it suffices to show that  $z_n(\zeta)$  is a bounded sequence. However, we can solve (3.13) for  $z_n$  in terms of  $(x_n, y_n)$  and then use boundedness of the latter to deduce boundedness of the former. □

*Proof of Theorem 3.2.* This is immediate from Lemmas 3.4, 3.5, and 3.6. □

*Proof of Theorem 1.1.* By Lemma 3.4,  $\mathcal{E}_\omega$  has no eigenvalues in  $B'$ . Since  $\Sigma = B'$  by Theorem 3.2, we are done. □

We now turn towards codings of rotations. Unlike the Sturmian setting, the techniques of [30] apply without any modification.

*Proof of Theorem 1.2.* This follows immediately from the arguments that prove [30, Theorem 4]. Simply replace [30, Proposition 2] by Theorem 1.4. In particular, it follows from the arguments of [21] that the set of phases for which the associated Verblunsky coefficients satisfy the assumptions of three-block Gordon Lemma (Theorem 1.4) has Lebesgue measure at least

$$\limsup_{n \rightarrow \infty} \left( 1 - \frac{4q_n}{q_{n+1}} \right),$$

and [21, Lemma 4] implies that

$$\limsup_{n \rightarrow \infty} \frac{q_{n+1}}{q_n} \geq \frac{1}{2} \left[ \limsup_{n \rightarrow \infty} a_n + \sqrt{\left( \limsup_{n \rightarrow \infty} a_n \right)^2 + 4} \right].$$

This yields the desired statement by almost-sure invariance of the pure point spectrum [37, Theorem 10.16.1].  $\square$

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