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THE SYZYGIES OF SOME THICKENINGS OF DETERMINANTAL VARIETIES

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Abstract. The vector space of $m \times n$ complex matrices ($m \geq n$) admits a natural action of the group $\mathrm{GL} = \mathrm{GL}_m \times \mathrm{GL}_n$ via row and column operations. For positive integers a, b, we consider the ideal $I_{a\times b}$ defined as the smallest GL-equivariant ideal containing the b-th powers of the $a \times a$ minors of the generic $m \times n$ matrix. We compute the syzygies of the ideals $I_{a\times b}$ for all a, b, together with their GL-equivariant structure, generalizing earlier results of Lascoux for the ideals of minors ($b = 1$), and of Akin–Buchsbaum–Weyman for the powers of the ideals of maximal minors ($a = n$). Our methods rely on a nice connection between commutative algebra and the representation theory of the superalgebra $\mathfrak{gl}(m|n)$, as well as on our previous calculation of Ext modules done in the context of describing local cohomology with determinantal support. Our results constitute an important ingredient in the proof by Nagpal–Sam–Snowden of the first non-trivial Noetherianity results for twisted commutative algebras which are not generated in degree one.

1. Introduction

For positive integers $m \geq n$, we consider the ring $S = \mathrm{Sym}(\mathbb{C}^m \otimes \mathbb{C}^n)(= \mathbb{C}[z_{ij}])$ of polynomial functions on the vector space of $m \times n$ matrices with entries in the complex numbers. The ring S admits an action of the group $\mathrm{GL} = \mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$, and it decomposes into irreducible GL-representations according to Cauchy's formula:

$$S = \bigoplus_{\lambda=(\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_n\geq 0)} S_\lambda\mathbb{C}^m \otimes S_\lambda\mathbb{C}^n,$$

where S_λ denotes the Schur functor associated to a partition λ. For each λ, we let I_λ denote the ideal in S generated by the irreducible representation $S_\lambda\mathbb{C}^m \otimes S_\lambda\mathbb{C}^n$. Every ideal $I \subset S$ which is preserved by the GL-action is a sum of ideals I_λ: such ideals I have been classified and their geometry has been studied by De Concini, Eisenbud and Procesi in the 1980s [dCEP80]. Nevertheless, their syzygies are still mysterious, and in particular the following problem remains unsolved:

Problem 1.1. Describe the syzygies of the ideals I_λ, together with their GL-equivariant structure.

The goal of our paper is to give a quick solution to this problem in the case when λ is a rectangular partition, which means that there exist positive integers

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a, b such that $\lambda_1 = \cdots = \lambda_a = b$ and $\lambda_i = 0$ for $i > a$ (alternatively, the Young diagram associated to λ is the $a \times b$ rectangle). In this case we write $\lambda = a \times b$ and $I_\lambda = I_{a\times b}$. One can think of $I_{a\times b}$ as the smallest GL-equivariant ideal which contains the b-th powers of the $a \times a$ minors of the generic matrix of indeterminates $Z = (z_{ij})$. What distinguishes the ideals $I_{a\times b}$ among all the I_λ's is that they define a scheme without embedded components, so from a geometric point of view they form the simplest class of GL-equivariant ideals after the reduced (and prime) ideals of minors. Examples of ideals $I_{a\times b}$ include:

- $I_{a\times 1} = I_a$, the ideal generated by the $a \times a$ minors of Z.
- $I_{n\times b} = I_n^b$, the b-th power of the ideal I_n of maximal minors of Z.
- $I_{1\times b}$, the ideal of $b \times b$ permanents of Z: here by a $b \times b$ permanent of Z we mean the permanent of a $b \times b$ matrix obtained by selecting b rows and b columns of Z, not necessarily distinct; for instance, when $m = n = 2$ we have

$$I_{1\times 2} = (z_{11}^2, z_{12}^2, z_{21}^2, z_{22}^2, z_{11}z_{12}, z_{11}z_{21}, z_{12}z_{22}, z_{21}z_{22}, z_{11}z_{22} + z_{12}z_{21}). \tag{1.1}$$

To state our main result, we need to introduce some notation. We write Rep_{GL} for the representation ring of the group GL, and for a given GL-representation M, we let $[M] \in Rep_{\mathrm{GL}}$ denote its class in the representation ring. We let

$$B_{i,j}(I_{a\times b}) = \mathrm{Tor}_i^S(I_{a\times b}, \mathbb{C})_j \tag{1.2}$$

denote the vector space of i-syzygies of degree j of $I_{a\times b}$. We encode the syzygies of $I_{a\times b}$ into the equivariant Betti polynomial

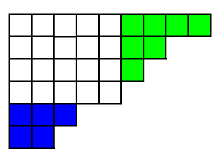
$$B_{a\times b}(z, w) = \sum_{i,j\in\mathbb{Z}} [B_{i,j}(I_{a\times b})] \cdot w^i \cdot z^j \in Rep_{\mathrm{GL}}[z, w], \tag{1.3}$$

so the variable z keeps track of the internal degree, while w keeps track of the homological degree.

If r, s are positive integers, α is a partition with at most r parts ($\alpha_i = 0$ for $i > r$) and β is a partition with parts of size at most s ($\beta_1 \leq s$), we construct the partition

$$\lambda(r, s; \alpha, \beta) = (s + \alpha_1, \cdots, s + \alpha_r, \beta_1, \beta_2, \cdots). \tag{1.4}$$

This is easiest to visualize in terms of Young diagrams: one starts with an $r \times s$ rectangle and attaches α to the right and β to the bottom of the rectangle. If $r = 4$, $s = 5$, $\alpha = (4, 2, 1)$, $\beta = (3, 2)$, then

(1.5) $\lambda(r, s; \alpha, \beta) =$ 

We write μ' for the conjugate partition to μ (obtained by transposing the Young diagram of μ) and consider the polynomials $h_{r\times s} \in Rep_{\mathrm{GL}}[z, w]$ given by

$$h_{r\times s}(z, w) = \sum_{\alpha,\beta} [S_{\lambda(r,s;\alpha,\beta)}\mathbb{C}^m \otimes S_{\lambda(r,s;\beta',\alpha')}\mathbb{C}^n] \cdot z^{r\cdot s+|\alpha|+|\beta|} \cdot w^{|\alpha|+|\beta|}, \tag{1.6}$$

where the sum is taken over partitions α, β such that α is contained in the $\min(r, s) \times (n - r)$ rectangle ($\alpha_1 \leq n - r$, $\alpha_1' \leq \min(r, s)$) and β is contained in the $(m-r) \times \min(r,s)$ rectangle ($\beta_1 \leq \min(r,s)$ and $\beta'_1 \leq m-r$). We also need to introduce the Gauss polynomial $\binom{r+s}{r}_w \in \mathbb{Z}[w]$,

$$\binom{r+s}{r}_w = \sum_{s \geq t_1 \geq \cdots \geq t_r \geq 0} w^{t_1+\cdots+t_r}, \tag{1.7}$$

which is the generating function for partitions contained inside the $r \times s$ rectangle. Note that $\binom{r+s}{r}_{w^2}$ is the Poincaré polynomial of the Grassmannian of r-dimensional subspaces of an $(r+s)$-dimensional vector space, and also that $\binom{r+s}{r}_1 = \binom{r+s}{r}$ is the usual binomial coefficient. Our main result is:

Theorem on Syzygies of Rectangular Ideals (Theorem 3.1)**.** *The equivariant Betti polynomial of the ideal $I_{a\times b}$ is*

$$B_{a\times b}(z,w) = \sum_{q=0}^{n-a} h_{(a+q)\times(b+q)} \cdot w^{q^2+2q} \cdot \binom{q+\min(a,b)-1}{q}_{w^2}.$$

When $b=1$, this recovers the result of Lascoux on syzygies of determinantal varieties [Las78]. When $a=n$, we obtain the syzygies of the powers of the ideals of maximal minors, as originally computed by Akin–Buchsbaum–Weyman [ABW81].

Example 1.2. When $m=n=2$, the ideal $I_{1\times 2}$ from (1.1) has the equivariant Betti polynomial

$$B_{1\times 2}(z,w) = h_{1\times 2} + h_{2\times 3} \cdot w^3,$$

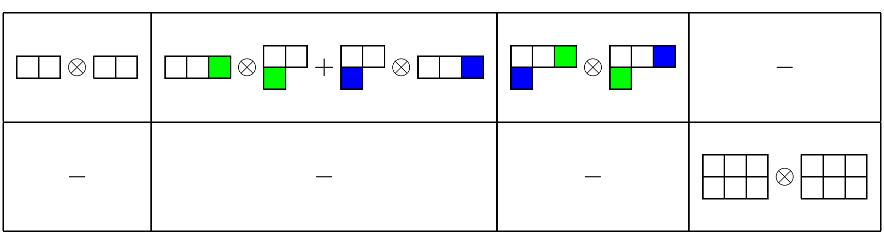
where

$$\begin{aligned} h_{1\times 2} = {} & [\mathrm{Sym}^2 \mathbb{C}^2 \otimes \mathrm{Sym}^2 \mathbb{C}^2] \cdot z^2 + ([\mathrm{Sym}^3 \mathbb{C}^2 \otimes S_{2,1}\mathbb{C}^2] \\ & + [S_{2,1}\mathbb{C}^2 \otimes \mathrm{Sym}^3 \mathbb{C}^2]) \cdot z^3 \cdot w + [S_{3,1}\mathbb{C}^2 \otimes S_{3,1}\mathbb{C}^2] \cdot z^4 \cdot w^2 \end{aligned}$$

and

$$h_{2\times 3} = [S_{3,3}\mathbb{C}^2 \otimes S_{3,3}\mathbb{C}^2] \cdot z^6.$$

The equivariant Betti table (where the (i,j)-entry is $[B_{i,i+j}(I_{1\times 2})] \in \mathit{Rep}_{\mathrm{GL}}$, represented pictorially in terms of Young diagrams; as in (1.5) we use empty boxes for the $r \times s$ rectangle inside $\lambda(r,s;\alpha,\beta)$ and $\lambda(r,s;\beta',\alpha')$, green boxes for the partitions α, α' and blue boxes for the partition β, β') then looks like



Taking dimensions of representations ($\dim(\mathrm{Sym}^r \mathbb{C}^2) = r+1$, $\dim(S_{r,1}\mathbb{C}^2) = r$, $\dim(S_{r,r}\mathbb{C}^2) = 1$), we get the usual Betti table, which can be verified for instance using Macaulay2 [GS]:

9	16	9	–
–	–	–	1

An immediate corollary of Theorem 3.1 is a uniform boundedness result for the syzygies of the ideals $I_{a\times b}$: it is easy to see that for fixed i, the coefficient of $w^i z^j$ in $B_{a\times b}(z,w)$ is zero when j is large enough, independent of the size $m\times n$ of our matrices. This is translated in [NSS16] into the fact that over the bivariate twisted commutative algebra $\mathcal{S}=\mathrm{Sym}(\mathbb{C}^\infty\otimes\mathbb{C}^\infty)$, the syzygy modules $\mathrm{Tor}_i^{\mathcal{S}}(I_{a\times b},\mathbb{C})$ have finite length, which is then used to derive a similar conclusion for the syzygy modules of any finitely generated $\mathcal{S}$-module. In the slightly different setup of Δ-modules, a similar boundedness result for the syzygies of Segre embeddings [Sno13, Prop. 5.1] has been used by Snowden to prove more refined finiteness properties for the said syzygies.

We expect that a complete solution to Problem 1.1 will be intimately related to the representation theory of the general linear Lie superalgebra $\mathfrak{gl}(m|n)$ [Kac77]: we briefly explain this connection for the experts, but then proceed with a more elementary approach; knowledge of the representation theory of superalgebras is therefore not required in order to understand the statements and proofs in our paper. The connection is explained as follows: the universal enveloping algebra of $\mathfrak{gl}(m|n)$ contains as a subalgebra the exterior algebra $\Lambda=\bigwedge(\mathbb{C}^m\otimes\mathbb{C}^n)$. Every $\mathfrak{gl}(m|n)$ module P can then be thought of as a Λ-module, which by the BGG correspondence [Eis05, Chapter 7] gives rise to a linear complex over the polynomial ring S. When P is a Kac module, the corresponding complex is just a Koszul complex, and it is exact in positive homological degree. The composition factors of Kac modules however give rise to complexes which are typically far from being exact, and in many cases their homology groups are closely related to the ideals I_λ. There is significant literature related to the character theory of $\mathfrak{gl}(m|n)$-modules [Ser96, Bru03], and in particular to Kac modules [HKVdJ92, SHK00, Su06], and it is our hope that this paper will provide sufficient motivation for a more systematic study of the corresponding complexes from a commutative algebra perspective.

The proof of Theorem 3.1 is based on the following two ingredients:

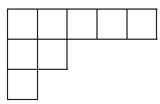
- Work of the second author with Akin [AW97, AW07]: they study the family of linear complexes $X_\bullet^{r\times s}$, arising via the BGG correspondence from the irreducible $\mathfrak{gl}(m|n)$-modules of lowest weight $(s^r,0^{m-r}|s^r,0^{n-r})$. The homology of these linear complexes consists entirely of direct sums of ideals $I_{(r+q)\times(s+q)}$. The polynomials $h_{r\times s}(z,w)$ introduced in (1.6) precisely encode the terms of these linear complexes, or equivalently they encode the characters of the corresponding simple $\mathfrak{gl}(m|n)$-modules.
- Work of the authors on computing local cohomology with support in determinantal ideals: in [RW14] we compute all the modules $\mathrm{Ext}_S^\bullet(I_{a\times b},S)$, together with their GL-equivariant structure.

Based on these two ingredients, our strategy is as follows. We obtain a nonminimal resolution of $I_{a\times b}$ via an iterated mapping cone construction involving the linear complexes $X_\bullet^{(a+q)\times(b+q)}$, $q\geq 0$. We then use the GL-equivariance to conclude that whenever cancellations occur for some of the terms of an $X_\bullet^{r\times s}$, they must in fact occur for all the terms of $X_\bullet^{r\times s}$. This implies that the minimal resolution of $I_{a\times b}$ is also built out of copies of $X_\bullet^{(a+q)\times(b+q)}$, and it remains to determine the number of such copies, as well as their homological shifts. This is done by dualizing the minimal resolution and using the GL-equivariant description of $\mathrm{Ext}_S^\bullet(I_{a\times b},S)$. We elaborate on this argument in Section 3, after we establish some notational conventions in Section 2, and collect some preliminary results on functoriality of syzygies, on the complexes $X_\bullet^{r\times s}$, and on the computation of Ext modules.

2. Preliminaries

2.1. **Representation theory** [FH91], [Wey03, Ch. 2]. If W is a complex vector space of dimension $\dim(W) = n$, a choice of basis determines an isomorphism between $\mathrm{GL}(W)$ and the group $\mathrm{GL}_n(\mathbb{C})$ of $n\times n$ invertible matrices. We will refer to n-tuples $\lambda = (\lambda_1,\cdots,\lambda_n)\in\mathbb{Z}^n$ as weights of the corresponding maximal torus of diagonal matrices. We say that λ is a dominant weight if $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_n$. Irreducible representations of $\mathrm{GL}(W)$ are in one-to-one correspondence with dominant weights λ. We denote by $S_\lambda W$ the irreducible representation associated to λ. We write $|\lambda|$ for the total size $\lambda_1+\cdots+\lambda_n$ of λ.

When λ is a dominant weight with $\lambda_n\geq 0$, we say that λ is a partition of $r = |\lambda|$. We will often represent a partition via its associated Young diagram, which consists of left–justified rows of boxes, with λ_i boxes in the i-th row: for example, the Young diagram associated to $\lambda = (5,2,1)$ is



Note that when we're dealing with partitions we often omit the trailing zeros. We define the length of a partition λ to be the number of its non-zero parts and denote it by $l(\lambda)$. If $l(\lambda)>\dim(W)$, then $S_\lambda W = 0$. The transpose λ' of a partition λ is obtained by transposing the corresponding Young diagram. For the example above, $\lambda' = (3,2,1,1,1)$, $l(\lambda) = 3$ and $l(\lambda') = 5$. If μ is another partition, we write $\mu\subset\lambda$ to indicate that $\mu_i\leq\lambda_i$ for all i and say that μ is contained in λ.

For a pair of finite dimensional vector spaces F, G, we write $\mathrm{GL}(F,G)$ (or simply GL when F, G are understood) for the group $\mathrm{GL}(F)\times\mathrm{GL}(G)$. If M is a $\mathrm{GL}(F,G)$-representation, we write

$$\langle S_\lambda F\otimes S_\mu G, M\rangle$$

for the multiplicity of the irreducible GL-representation $S_\lambda F\otimes S_\mu G$ inside M. If $M^\bullet$ is a cohomologically graded module, then we record the occurrences of $S_\lambda F\otimes S_\mu G$ inside the graded components of $M^\bullet$ by

$$\langle S_\lambda F\otimes S_\mu G, M^\bullet\rangle = \sum_{i\in\mathbb{Z}}\left\langle S_\lambda F\otimes S_\mu G, M^i\right\rangle\cdot w^i, \tag{2.1}$$

where the variable w encodes the cohomological degree (note a slight difference from (1.3), where w was used for homological degree).

2.2. **Functoriality of syzygies.** It will be useful to think of the polynomial ring $S = \mathrm{Sym}(\mathbb{C}^m\otimes\mathbb{C}^n)$ as a functor S which assigns to a pair (F,G) of finite dimensional vector spaces the polynomial ring $S(F,G) = \mathrm{Sym}(F\otimes G)$. For each a,b we obtain functors $I_{a\times b}$ which assign to (F,G) the corresponding ideal $I_{a\times b}(F,G)\subset S(F,G)$. The syzygy modules in (1.2) become functors $B_{i,j}^{a\times b}(-,-)$, defined by

$$B_{i,j}^{a\times b}(F,G) = \mathrm{Tor}_i^{S(F,G)}(I_{a\times b}(F,G),\mathbb{C})_j.$$

In fact, each $B_{i,j}^{a\times b}$ is a polynomial functor in the sense of [Mac95, Ch. I, Appendix A], since the terms in the Koszul complex resolving $\mathbb{C}$, as well as the functors

$I_{a\times b}(-,-)$ and $S(-,-)$, are polynomial functors. As such, we get a decomposition into a (usually infinite) direct sum indexed by pairs of partitions

$$B^{a\times b}_{i,j}(-,-)=\bigoplus_{|\lambda|=|\mu|=j}(S_{\lambda}(-)\otimes S_{\mu}(-))^{\bigoplus m_{\lambda,\mu}}. \tag{2.2}$$

When evaluating $B^{a\times b}_{i,j}$ on a pair of vector spaces (F,G), the only terms on the right hand side of (2.2) that survive are the ones for which $l(\lambda)\leq\dim(F)$ and $l(\mu)\leq\dim(G)$. The multiplicities $m_{\lambda,\mu}$ for such pairs (λ,μ) are then determined by the $\mathrm{GL}(F,G)$-equivariant structure of $B^{a\times b}_{i,j}(F,G)$. In particular, knowing the GL-equivariant structure for the syzygies of $I_{a\times b}(\mathbb{C}^m,\mathbb{C}^n)$ determines the syzygies of $I_{a\times b}(F,G)$ for all pairs of vector spaces (F,G) with $\dim(F)\leq m$, $\dim(G)\leq n$.

2.3. **The linear complexes $X^{r\times s}_{\bullet}$ of Akin and Weyman.** In [AW97, AW07], Akin and the second author construct linear complexes $X^{r\times s}_{\bullet}=X^{r\times s}_{\bullet}(F,G)$ which depend functorially on a pair of finite dimensional vector spaces (F,G). The terms in the complex are given (using notation (1.4)) by

$$X^{r\times s}_{i}(F,G)=\left(\bigoplus_{\substack{|\alpha|+|\beta|=i\\ \alpha'_1,\beta_1\leq\min(r,s)}}S_{\lambda(r,s;\alpha,\beta)}F\otimes S_{\lambda(r,s;\beta',\alpha')}G\right)\otimes S(F,G). \tag{2.3}$$

Note that since $S_{\lambda}W=0$ when $l(\lambda)>\dim(W)$, only finitely many of the terms $X^{r\times s}_{i}(F,G)$ in (2.3) are non-zero for a given pair (F,G). More precisely, we must have $\alpha_1\leq\dim(G)-r$, $\beta'_1\leq\dim(F)-r$, so $|\alpha|\leq\min(r,s)\cdot(\dim(G)-r)$, $|\beta|\leq\min(r,s)\cdot(\dim(F)-r)$, $i\leq\min(r,s)\cdot(\dim(F)+\dim(G)-2r)$. We can rewrite (1.6) as

$$h_{r\times s}(z,w)=\sum_{i=0}^{\min(r,s)\cdot(m+n-2\cdot r)}[X^{r\times s}_{i}(\mathbb{C}^m,\mathbb{C}^n)_{r\cdot s+i}]\cdot z^{r\cdot s+i}\cdot w^{i},$$

where $X^{r\times s}_{i}(\mathbb{C}^m,\mathbb{C}^n)_{r\cdot s+i}$ is the vector space of minimal generators of the free module $X^{r\times s}_{i}(\mathbb{C}^m,\mathbb{C}^n)$. As explained in [AW97, Section 2], the complex

$$X^{(a+q)\times(1+q)}_{\bullet}(\mathbb{C}^m,\mathbb{C}^n)$$

can be identified with the q-th linear strand of the Lascoux resolution of the ideal $I_{a\times 1}$ of $a\times a$ minors of the generic $m\times n$ matrix. In this paper we'll see that more generally, the complexes $X^{(a+q)\times(b+q)}_{\bullet}$, $q\geq 0$, form the building blocks of the minimal resolutions of the ideals $I_{a\times b}$.

In [AW07] the homology of the complexes $X^{r\times s}_{\bullet}$ is shown to consist of direct sums of the rectangular ideals $I_{(r+q)\times(s+q)}$. To state this more precisely, we need to introduce some notation. We denote by $P(r,s;i)$ the number of partitions of i contained in the $r\times s$ rectangle. The Gauss polynomial defined in (1.7) is then

$$\binom{r+s}{r}_{w}=\sum_{i=0}^{r\cdot s}P(r,s;i)w^{i}.$$

Theorem 2.1 ([AW07, Thm. 2]). *With the above notation, the homology groups of $X_\bullet^{r\times s}$ are*

(1) $H_{2j+1}(X_\bullet^{r\times s}) = 0$;

(2) $H_{2j}(X_\bullet^{r\times s}) = \bigoplus_{q=0}^{j} I_{(r+q)\times(s+q)}^{\oplus P(q,\min(r,s)-1;j-q)}$.

In [AW07] the projective dimension of the ideals $I_{a\times b}$ is calculated. The calculation of Ext modules in [RW14, Thm. 4.3] in fact allows one to compute the projective dimension and regularity for all the ideals I_λ, i.e. the shape of their minimal resolution. More work is however necessary in order to completely determine the syzygies.

2.4. **The** Ext **modules** $\mathrm{Ext}^\bullet_S(I_{a\times b}, S)$. In [RW14, Theorem 4.3] we determined the decomposition into irreducible GL-representations for all the modules $\mathrm{Ext}^\bullet_S(I_\mu, S)$. In the case when $\mu = a\times b$ is a rectangular partition, we obtain the following consequence which will be useful for our calculation of syzygies.

Theorem 2.2. *Assume that $m = n$ and write $q = n-a$, $S = S(\mathbb{C}^n, \mathbb{C}^n)$, $I_{a\times b} = I_{a\times b}(\mathbb{C}^n, \mathbb{C}^n)$, $\mathrm{GL} = \mathrm{GL}(\mathbb{C}^n, \mathbb{C}^n)$. The occurrences of the irreducible GL-representation $S_{(-b-q)^n}\mathbb{C}^n \otimes S_{(-b-q)^n}\mathbb{C}^n$ inside $\mathrm{Ext}^\bullet_S(I_{a\times b}, S)$ (see (2.1)) are encoded as*

$$\left\langle S_{(-b-q)^n}\mathbb{C}^n \otimes S_{(-b-q)^n}\mathbb{C}^n, \mathrm{Ext}^\bullet_S(I_{a\times b}, S)\right\rangle = w^{q^2+2q}\cdot \binom{q+\min(a,b)-1}{q}_{w^2}.$$

Proof. With the notation as in [RW14, Theorem 4.3], let $\underline{x} = (b^a) = a\times b$ be the partition defined by $x_1 = \cdots = x_a = b$, $x_{a+1} = \cdots = x_n = 0$. As explained in [RW14, Remark 4.4], since the inequality $x_p > x_{p+1}$ holds only for $p = a$, it suffices to consider only the terms with $p = a$ in [RW14, (4-3)]. We take $\lambda = (-b-q)^n$ to be the weight with all $\lambda_i = -b-q = a-b-n$ and investigate the values of the parameters $0 \le s \le t_1 \le \cdots \le t_{n-a} \le a-1$ for which λ is a member of $W'(\underline{x}, a; \underline{t}, s)$.

Condition [RW14, (4-4a)] becomes $\lambda_n = a-b-n \ge p - x_p - n = a-b-n$, which is trivially satisfied. Condition [RW14, (4-4b)] becomes $a-b-n \le t_j - n$, which is equivalent to $t_j \ge a-b$: since all t_j are non-negative, we can rewrite this as $t_j \ge \max(a-b, 0)$. Condition [RW14, (4-4c)] holds for precisely one value of s: $s = a-b$ if $a \ge b$, respectively $s = 0$ if $a < b$ (note that $\lambda_0 = \infty$ by convention): we can rewrite this for short as $s = \max(a-b, 0)$. Noting that [RW14, Theorem 4.3] describes $\mathrm{Ext}^\bullet_S(S/I_{a\times b}, S)$, which is the same as $\mathrm{Ext}^{\bullet-1}_S(I_{a\times b}, S)$ for $\bullet > 0$, we can then conclude from [RW14, (4-3)] that

$$\begin{aligned}(2.4)\qquad &\left\langle S_{(-b-q)^n}\mathbb{C}^n \otimes S_{(-b-q)^n}\mathbb{C}^n, \mathrm{Ext}^\bullet_S(I_{a\times b}, S)\right\rangle \\ &= \sum_{\max(a-b,0)\le t_1\le\cdots\le t_{n-a}\le a-1} w^{n^2-a^2-2\cdot\left(\sum_{j=1}^{n-a} t_j\right)}\end{aligned}$$

Making the change of variable $t_j' = a-1-t_j$ we get

$$n^2-a^2-2\cdot\sum_{j=1}^{n-a} t_j = n^2-a^2-2\cdot\sum_{j=1}^{n-a}(a-1-t_j') = n^2-a^2-2\cdot(a-1)\cdot(n-a)+2\cdot\sum_{j=1}^{n-a} t_j'.$$

Using the fact that $n-a = q$, we get

$$n^2-a^2-2\cdot(a-1)\cdot(n-a) = (n-a)\cdot(n+a-2a+2) = q^2+2q.$$

Using the fact that $a-1-\max(a-b,0)=\min(a,b)-1$, we can then rewrite the right hand side of (2.4) as

$$\sum_{\min(a,b)-1\geq t'_1\geq\cdots\geq t'_q\geq 0} w^{q^2+2q}\cdot(w^2)^{t'_1+\cdots+t'_q} \overset{(1.7)}{=} w^{q^2+2q}\cdot\binom{q+\min(a,b)-1}{q}_{w^2},$$

which is the desired conclusion. $\square$

3. The syzygies of the ideals $I_{a\times b}$

We now proceed to state and prove the main result of our paper:

Theorem 3.1. *The equivariant Betti polynomial of $I_{a\times b}\subset\mathrm{Sym}(\mathbb{C}^m\otimes\mathbb{C}^n)$, $m\geq n$, is*

$$B_{a\times b}(z,w)=\sum_{q=0}^{n-a} h_{(a+q)\times(b+q)}\cdot w^{q^2+2q}\cdot\binom{q+\min(a,b)-1}{q}_{w^2},$$

where $h_{r\times s}=h_{r\times s}(z,w)$ is as defined in (1.6).

We prove Theorem 3.1 in a few stages. We first note that by functoriality (Section 2.2) it is enough to prove the theorem in the case $m=n$, which we assume for the remainder of this section. For brevity, we will say that a complex $Y_\bullet$ is filtered by the linear complexes $X^i_\bullet$ if $Y_\bullet$ admits a filtration by subcomplexes in such a way that the subquotients are isomorphic to $X^i_\bullet$. Equivalently, $Y_\bullet$ can be built out of $X^i_\bullet$ via an iterated mapping cone construction. We have

Proposition 3.2. *The ideal $I_{a\times b}$ has a (not necessarily minimal) free* GL*-equivariant resolution over S, denoted $Y^{a\times b}_\bullet$, which is filtered by the complexes $X^{(a+q)\times(b+q)}_\bullet$.*

Proof. We prove by descending induction on q that $I_{(a+q)\times(b+q)}$ admits a (not necessarily minimal) resolution $Y^{(a+q)\times(b+q)}_\bullet$ which is filtered by complexes $X^{(a+q')\times(b+q')}_\bullet$ with $q'\geq q$. If $q=n-a$, then $I_{(a+q)\times(b+q)}=I_{n\times(b+n-a)}$ coincides with $X^{n\times(b+n-a)}_\bullet$: both are isomorphic to a free module of rank one, generated by the $(b+n-a)$-th power of the determinant of the generic $n\times n$ matrix. Assuming now that the result is true for the ideals $I_{(a+q)\times(b+q)}$ with $q>q_0$, we'll prove it for $q=q_0$ to finish the inductive argument. By Theorem 2.1 the higher homology of the linear complex $X^{(a+q_0)\times(b+q_0)}_\bullet$ consists of direct sums of ideals $I_{(a+q)\times(b+q)}$, $q>q_0$, and $H_0(X^{(a+q_0)\times(b+q_0)}_\bullet)=I_{(a+q_0)\times(b+q_0)}$. We can therefore construct a resolution $Y^{(a+q_0)\times(b+q_0)}_\bullet$ of $I_{(a+q_0)\times(b+q_0)}$ as a mapping cone of the maps from the complexes $Y^{(a+q)\times(b+q)}_\bullet$, $q>q_0$, to the complex $X^{(a+q_0)\times(b+q_0)}_\bullet$ that cancel its higher homology. $\square$

Let $Y^{a\times b}_\bullet$ be a non-minimal GL-equivariant resolution of the ideal $I_{a\times b}$ as in Proposition 3.2. We can minimize $Y^{a\times b}_\bullet$ by making appropriate cancellations. Notice that since the generators of the free modules appearing in $X^{(a+q)\times(b+q)}_\bullet$ and $X^{(a+q')\times(b+q')}_\bullet$ don't share isomorphic irreducible GL-subrepresentations for $q\neq q'$, the only cancellations that can occur are between the terms in various copies of the same $X^{(a+q)\times(b+q)}_\bullet$.

Lemma 3.3. *Any* $\mathrm{GL}(F,G)$*-equivariant endomorphism of $X^{r\times s}_\bullet(F,G)$ is a multiple of the identity.*

Proof. Let ψ denote a GL-equivariant endomorphism of $X_\bullet^{r\times s}$, and write ψ_i for its component in homological degree i. By GL-equivariance and using the decomposition (2.3), we have $\psi_i = \bigoplus_{\alpha,\beta} \psi_{\alpha,\beta}$, where $\psi_{\alpha,\beta}$ is the restriction of ψ_i to the free submodule $X_{\alpha,\beta}^{r\times s}$ generated by the irreducible representation $S_{\lambda(r,s;\alpha,\beta)}F \otimes S_{\lambda(r,s;\beta',\alpha')}G$. By Schur's lemma, such an endomorphism is necessarily a multiple of the identity. Writing $\psi_{\alpha,\beta} = \cdot c_{\alpha,\beta}$, multiplication by $c_{\alpha,\beta}$, it suffices to show that all $c_{\alpha,\beta}$ are the same. We prove this by induction on $i = |\alpha| + |\beta|$. When $i = 0$, there is only one pair (α, β) with $|\alpha| + |\beta| = 0$, namely $\alpha = \beta = 0$. The induction will show that all $c_{\alpha,\beta}$ are equal to $c_{0,0}$.

Consider (α, β) with $i = |\alpha| + |\beta| > 0$, and consider a pair $(\overline{\alpha}, \overline{\beta})$ with $|\overline{\alpha}| + |\overline{\beta}| = i - 1$, such that the restriction of the differential $\partial_i : X_i^{r\times s} \to X_{i-1}^{r\times s}$ to

$$X_{\alpha,\beta}^{r\times s} \xrightarrow{\partial_i} X_{\overline{\alpha},\overline{\beta}}^{r\times s}$$

is non-zero: such a pair exists since otherwise $S_{\lambda(r,s;\alpha,\beta)}F \otimes S_{\lambda(r,s;\beta',\alpha')}G$ would contribute to the homology of $X_\bullet^{r\times s}$ (note that this representation is not a coboundary, since the complex $X_\bullet^{r\times s}$ is minimal), which would contradict Theorem 2.1. Since ψ commutes with the differentials, we have a commutative diagram

$$\begin{tikzcd}
X_{\alpha,\beta}^{r\times s} \arrow[r, "\partial_i"] \arrow[d, "\cdot c_{\alpha,\beta}"'] & X_{\overline{\alpha},\overline{\beta}}^{r\times s} \arrow[d, "\cdot c_{\overline{\alpha},\overline{\beta}}"] \\
X_{\alpha,\beta}^{r\times s} \arrow[r, "\partial_i"] & X_{\overline{\alpha},\overline{\beta}}^{r\times s}
\end{tikzcd}$$

Since $\partial_i \neq 0$, it follows that $c_{\alpha,\beta} = c_{\overline{\alpha},\overline{\beta}}$, and we conclude by induction. □

The preceding discussion implies the following.

Corollary 3.4. *The minimal resolution of $I_{a\times b}$ is filtered by the complexes $X_\bullet^{(a+q)\times(b+q)}$, $q \geq 0$. In particular, there exist polynomials $M_{a\times b}^q(w)$ which account for the multiplicities of the complexes $X_\bullet^{(a+q)\times(b+q)}$ in the minimal resolution of $I_{a\times b}$, as well as for their homological shifts, i.e.*

$$B_{a\times b}(z, w) = \sum_{q=0}^{n-a} h_{(a+q)\times(b+q)} \cdot M_{a\times b}^q(w).$$

Proof. By Proposition 3.2, $I_{a\times b}$ admits a non-minimal resolution $Y_\bullet^{a\times b}$, filtered by the complexes $X_\bullet^{(a+q)\times(b+q)}$. To get to the minimal resolution of $I_{a\times b}$, one must perform appropriate cancellations, which can only occur between copies of the same $X_\bullet^{(a+q)\times(b+q)}$. By Lemma 3.3, such cancellations occur either for all the terms in $X_\bullet^{(a+q)\times(b+q)}$ or for none. This shows that the minimal resolution of $I_{a\times b}$ is filtered by (possibly fewer) copies of $X_\bullet^{(a+q)\times(b+q)}$, from which the remaining part of the conclusion follows formally. □

We are now ready to prove the main result of the paper:

Proof of Theorem 3.1. It remains to calculate the polynomials $M^q_{a\times b}(w)$. We fix q and shrink n if necessary to assume that $n = a+q$ (see Section 2.2), so $X_\bullet^{(a+q)\times(b+q)} = X_\bullet^{n\times(b+q)}$ consists of a single free module, generated by the irreducible GL-representation $S_{(b+q)^n}\mathbb{C}^n \otimes S_{(b+q)^n}\mathbb{C}^n$. Dualizing the minimal resolution Y of $I_{a\times b}$ and computing the cohomology $\mathrm{Ext}^\bullet_S(I_{a\times b}, S)$ of the resulting complex $Y_\bullet^\vee$, we get

(a) each occurrence of $X_\bullet^{n\times(b+q)}$ in $Y_\bullet$ yields a copy of $S_{(-b-q)^n}\mathbb{C}^n \otimes S_{(-b-q)^n}\mathbb{C}^n$ in $\mathrm{Ext}^\bullet_S(I_{a\times b}, S)$;
(b) the only occurrences of $S_{(-b-q)^n}\mathbb{C}^n \otimes S_{(-b-q)^n}\mathbb{C}^n$ inside $\mathrm{Ext}^\bullet_S(I_{a\times b}, S)$ arise in this way.

To prove (a), note that there are no non-zero maps going into the free module $X_\bullet^{n\times(b+q)}$: this follows from the fact that $(b+q)^n$ is the highest weight appearing among the minimal generators of the free modules in $Y_\bullet$, which is a consequence of Corollary 3.4 and the definition (1.6) of the polynomials $h_{(a+q)\times(b+q)}$. Since there are no maps into $X_\bullet^{n\times(b+q)}$, the dual free module $\mathrm{Hom}_S(X_\bullet^{n\times(b+q)}, S)$ consists entirely of cocycles in $Y_\bullet^\vee$. Since $Y_\bullet^\vee$ is minimal, the space $S_{(-b-q)^n}\mathbb{C}^n \otimes S_{(-b-q)^n}\mathbb{C}^n$ of minimal generators of $(X_\bullet^{n\times(b+q)})^\vee = \mathrm{Hom}_S(X_\bullet^{n\times(b+q)}, S)$ contains no coboundaries, so (a) follows. If (b) failed, one could find a free submodule $M^* \otimes S$ in $Y_\bullet^\vee$ containing $S_{(-b-q)^n}\mathbb{C}^n \otimes S_{(-b-q)^n}\mathbb{C}^n$, where M is an irreducible GL-representation appearing as a subspace of minimal generators in some complex $X_\bullet^{(a+q')\times(b+q')}$, $q' < q$. The condition $S_{(-b-q)^n}\mathbb{C}^n \otimes S_{(-b-q)^n}\mathbb{C}^n \subset M^* \otimes S$ implies that M appears as a subrepresentation of $S_{(b+q)^n}\mathbb{C}^n \otimes S_{(b+q)^n}\mathbb{C}^n \otimes S$. This can only happen if $M = S_\lambda\mathbb{C}^n \otimes S_\mu\mathbb{C}^n$, where λ, μ are partitions containing the $n \times (b+q)$ rectangle. By (1.4), M can only occur inside $X_\bullet^{n\times(b+q)}$.

It follows from (a) and (b) that there is a one-to-one correspondence between occurrences of $X_\bullet^{n\times(b+q)}$ inside Y and those of $S_{(-b-q)^n}\mathbb{C}^n \otimes S_{(-b-q)^n}\mathbb{C}^n$ inside $\mathrm{Ext}^\bullet_S(I_{a\times b}, S)$, and moreover this correspondence replaces homological shifts with cohomological shifts. We get (see (2.1) and the remark following it) that

$$\begin{aligned} M^q_{a\times b}(w) &= \left\langle S_{(-b-q)^n}\mathbb{C}^n \otimes S_{(-b-q)^n}\mathbb{C}^n, \mathrm{Ext}^\bullet_S(I_{a\times b}, S)\right\rangle \\ &\overset{\text{Thm. 2.2}}{=} w^{q^2+2q} \cdot \binom{q+\min(a,b)-1}{q}_{w^2}. \end{aligned}$$

This concludes the proof of Theorem 3.1. □

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