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DEGREE-INVARIANT, ANALYTIC EQUIVALENCE RELATIONS WITHOUT PERFECTLY MANY CLASSES

ANTONIO MONTALBÁN

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Abstract. We show that there is only one natural Turing-degree invariant, analytic equivalence relation with $\aleph_1$ many equivalence classes: the equivalence $X \equiv_{\omega_1} Y \iff \omega_1^X = \omega_1^Y$. More precisely, under $PD + \neg CH$, we show that every Turing-degree invariant, analytic equivalence relation with $\aleph_1$ many equivalence classes is equal to $\equiv_{\omega_1}$ on a Turing cone.

1. Introduction

A function $f\colon 2^\omega \to 2^\omega$ is said to be *Turing-degree invariant* if $X \equiv_T Y \Rightarrow f(X) \equiv_T f(Y)$ for all $X, Y \in 2^\omega$. There are not very many natural degree-invariant functions. The easy examples are the identity function, the constant functions, the Turing jump, and iterates of the Turing jump. Martin's famous conjecture states precisely that: Under AD, every degree-invariant $f\colon 2^\omega \to 2^\omega$ is Turing equivalent to either a constant function, the identity function or a transfinite iterate of the Turing jump almost everywhere with respect to Martin's measure. Martin's measure is the one that assigns a set $C \subseteq 2^\omega$ measure 1 if it contains a *Turing cone* (i.e., a set of the form $\{X \in 2^\omega : X \geq_T Y\}$ for some Y), and measure 0 if it is disjoint from a Turing cone. (Martin's Turing determinacy theorem states that, under AD, every degree-invariant set either contains or is disjoint from a cone.) Martin's conjecture was proved for uniformly degree-invariant functions and for order-preserving functions by Slaman and Steel [Ste82, SS88], but is still a major open question for non-uniformly degree-invariant functions (see [MSS] for a current survey).

In this paper, we consider equivalence relations instead of functions and, in particular, equivalence relations with $\aleph_1$ many classes. An equivalence relation $\sim$ on 2^ω is said to be *degree-invariant* if $X \equiv_T Y \Rightarrow X \sim Y$ for all $X, Y \in 2^\omega$. If $\aleph_1 < 2^{\aleph_0}$, there are not very many natural degree-invariant equivalence relations with $\aleph_1$ many classes. An example is the equivalence relation

$$X \equiv_{\omega_1} Y \iff \omega_1^X = \omega_1^Y,$$

where ω_1^X is the least non-X-computable ordinal. Our theorem states that this is the only natural such equivalence relation.

Theorem 1 (ZF+$\mathbf{\Sigma}^1_2$-DET)**.** *If $\sim$ is a degree-invariant, analytic equivalence relation on 2^ω without perfectly many classes, then, on a cone, $\sim$ is equal to either the trivial equivalence relation or to $\equiv_{\omega_1}$.*

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By the *trivial equivalence relation*, we mean the relation where all reals are equivalent. When we say that an equivalence relation E has *perfectly many classes*, we mean that there exists a perfect subset of 2^ω, all of whose elements are non-E-equivalent. Burgess's theorem [Bur79] says that an analytic equivalence relation without perfectly many classes can have at most $\aleph_1$ many classes. Note that if $\aleph_1 < 2^{\aleph_0}$, Burgess's theorem is an equivalence.

Since almost all proofs in computability theory relativize, if E is a "natural" equivalence relation and one can prove that it is either equal to or different from $\equiv_{\omega_1}$, then one would expect that proof to relativize and hold in any cone. One would also expect that proof not to depend on whether $\mathbf{\Sigma}^1_2$-DET holds or not. This is why we read Theorem 1 as saying that $\sim_{\omega_1}$ is the only *natural*, degree-invariant, analytic equivalence relation with uncountably many, but not perfectly many, classes.

As a side note, let us remark that there are other natural, analytic, non-degree-invariant equivalence relations with $\aleph_1$ many classes. The known examples are isomorphisms of well-orderings (letting non-well-orders be equivalent to each other) [Spe55], bi-embeddability of linear orderings [Mon07], bi-embeddability of torsion abelian groups [GM08], and isomorphism on the models of a counterexample to Vaught's conjecture (if it exists) [Mon13]. See [Mon] for more on these.

2. The proof

A key lemma that we will use a couple of times is the following:

Lemma 2 (Martin)**.** *(ZF+Σ^1_2-DET) Let $f\colon 2^\omega \to \omega_1$ be a function invariant under Turing equivalence which has a Σ^1_2 presentation; i.e., there is a Σ^1_2 function $g\colon 2^\omega \to 2^\omega$ such that, for every X, $g(X)$ is a well-ordering of ω isomorphic to $f(X)$. If $f(X) < \omega_1^X$ for every X, then f is constant on a cone.*

See [Mon13, Lemma 2.5] for a proof.

Proof of Theorem 1. Assume $\sim$ is not trivial on any cone, and let us prove that it must be equal to $\equiv_{\omega_1}$ on some cone.

By a result of Burgess [Bur79, Corollary 1], there is a nested, decreasing sequence of Borel equivalence relations, $\sim_\alpha$ for $\alpha \in \omega_1$, whose intersection is $\sim$, or in other words, such that $X \sim Y \iff (\forall \alpha < \omega_1)\ X \sim_\alpha Y$. Furthermore, if $\sim$ is lightface Σ^1_1, we can also require that, for all $X, Y \in 2^\omega$,

$$X \sim Y \iff X \sim_{\omega_1^{X\oplus Y}} Y$$

(see [Mon, Lemma 2.1 and Remark 2.2]). Note that by relativizing to an oracle if necessary, we can assume $\sim$ is lightface Σ^1_1. Moreover, from the proof of [Mon, Lemma 2.1], we also get that $\sim_\alpha$ is $\Sigma^0_{\alpha+1}$ uniformly in α and that the sequence of $\sim_\alpha$'s is continuous, i.e., that $\sim_\alpha = \bigcap_{\beta<\alpha} \sim_\beta$ for all limit ordinals $\alpha < \omega_1$.

By a result of Silver [Sil80], since each $\sim_\alpha$ is Borel and does not contain perfectly many classes, $\sim_\alpha$ can only have countably many classes. Thus each $\sim_\alpha$ partitions the reals into countably many degree-invariant Borel parts. By Martin's Turing determinacy, one of those parts contains a cone. On the other hand, every cone is partitioned by some equivalence relation $\sim_\alpha$, since we are assuming $\sim$ is not trivial on any cone. Therefore, for every $X \in 2^\omega$, there is an ordinal $\alpha < \omega_1$ such that $Y \not\sim_\alpha X$ for some $Y \geq_T X$. Let $f(X) \in \omega_1$ be the least such α. By our observation above, f cannot be constant on any cone, because for each $\sim_\alpha$, there is a cone of reals all $\sim_\alpha$-equivalent to each other, and hence $f(X) > \alpha$ on that cone.

We claim that $f(X) \geq \omega_1^X$ on a cone. Suppose it is not and hence that the set

$$\mathcal{S}_1 = \{X \in 2^\omega : (\forall Y \geq X) \ \ Y \sim_{\omega_1^X} X\}$$

contains no cone. $\mathcal{S}_1$ is a degree-invariant Π^1_2 set of reals, and hence by Martin's Turing determinacy, it must be disjoint from a cone. Restricted to this cone, the function f has a Σ^1_2 presentation (define $f(X)$ to be the $\Pi^{1,X}_1$ initial segment of $\mathcal{H}^X$, the Harrison linear ordering [Har68] relative to X, of all β satisfying $(\forall Y \geq_T X)\ Y \sim_\beta X$). Notice that the function f not only is degree invariant, but also preserves order in the sense that $X \leq_T Z$ implies $f(X) \leq f(Z)$. By Martin's Lemma 2, f must be constant on a cone, which we have already stated is a contradiction. Thus $\mathcal{S}_1$ contains a cone $\mathcal{C}_1$.

We now claim that, for every $X, Y \in \mathcal{C}_1$,

$$Y \geq_T X \quad \& \quad \omega_1^Y = \omega_1^X \qquad \Rightarrow \qquad Y \sim X. \tag{1}$$

The reason is that for such X and Y, $\omega_1^{X \oplus Y} = \omega_1^Y = \omega_1^X$, and hence $Y \sim_{\omega_1^X} X$ implies $Y \sim_{\omega_1^{X\oplus Y}} X$, which implies $X \sim Y$.

By a result of Harrington [Har78, Lemma 2.10] (see also [Mon13, Lemma 3.6]), for every X, Y with $\omega_1^X = \omega_1^Y$, there is a G such that

$$\omega_1^X = \omega_1^{X\oplus G} = \omega_1^G = \omega_1^{G\oplus Y} = \omega_1^Y.$$

If $X, Y \in \mathcal{C}_1$, we can assume $G \in \mathcal{C}_1$ too by relativizing to the base of $\mathcal{C}_1$. Using (1), we then get that

$$X \sim (X \oplus G) \sim G \sim (G \oplus Y) \sim Y. \tag{2}$$

We have shown that

$$\omega_1^X = \omega_1^Y \Rightarrow X \sim Y \qquad \text{for every } X, Y \in \mathcal{C}_1. \tag{3}$$

The second part of the proof is to show the reversal on some cone.

Let $\mathcal{A}_{\mathcal{C}_1} = \{\omega_1^X : X \in \mathcal{C}_1\} \subseteq \omega_1$, which, by a result of Sacks [Sac76, Corollary 3.16], is the set of all ordinals that are admissible relative to the base of the cone $\mathcal{C}_1$. By (3), we can view $\sim$ as an equivalence relation on $\mathcal{A}_{\mathcal{C}_1}$. We say that $\alpha \in \mathcal{A}_{\mathcal{C}_1}$ is $\sim$-*new* if, for $\beta < \alpha$ with $\beta \in \mathcal{A}_{\mathcal{C}_1}$, we have $\beta \not\sim \alpha$ or, in other words, if α is the least element of its $\sim$-equivalence class. Consider

$$\mathcal{S}_2 = \{X \in \mathcal{C}_1 : \omega_1^X \text{ is } \sim\text{-new}\} = \{X \in \mathcal{C}_1 : (\forall Y \in \mathcal{C}_1)\ Y \sim X \rightarrow \omega_1^Y \geq \omega_1^X\}.$$

Note that $\mathcal{S}_2$ is Π^1_2. Thus, by Martin's Turing determinacy, it either contains or is disjoint from a cone $\mathcal{C}_2$. We claim that it cannot be disjoint from $\mathcal{C}_2$: If it was, consider the map $g\colon \mathcal{C}_2 \to \omega_1$ such that $g(X)$ is the least $\alpha \in \mathcal{A}_{\mathcal{C}_1}$ such that $\alpha \sim \omega_1^X$. (Note that g has a Σ^1_2 representation: Let $g(X)$ be the initial segment of $\mathcal{H}^X$ of all β such that there exists $Y \in \mathcal{A}_{\mathcal{C}_1}$ with $Y \sim X$ and $\beta < \omega_1^Y$.) Using Martin's Lemma 2 again, g must be constant on a cone, but then $\sim$ would be trivial on that cone. Thus $\mathcal{S}_2$ must contain a cone $\mathcal{C}_2$. For $X, Y \in \mathcal{C}_2$, we then have that $X \sim Y \iff \omega_1^X = \omega_1^Y$. $\square$

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Department of Mathematics, University of California, Berkely, California 94720

E-mail address: `antonio@math.berkeley.edu`

URL: `www.math.berkeley.edu/∼antonio`