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ON ISOPERIMETRIC INEQUALITIES FOR SINGLE LAYER POTENTIALS

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ABSTRACT. We show that among all rectangles of given perimeter, the square is a minimizer of the Schatten $p-$norms of the single layer potentials. We also prove that the equilateral triangle is a minimizer of the Schatten $p-$norms of the single layer potentials over triangles of equal perimeter.

1. INTRODUCTION

The best known isoperimetric inequality is the classical isoperimetric inequality

$$A \leq \frac{P^2}{4\pi},$$

relating the area A enclosed by a closed curve of perimeter P. It is well-known that among all plane domains of given area the circle has the smallest perimeter. We refer the reader to [2] for extensive discussions on isoperimetric inequalities. In the present paper we discuss isoperimetric inequalities for Schatten $p-$norms of so-called single layer potentials. Let D be a bounded domain in the plane with piecewise smooth boundary. The single layer potential acting on $L^2(\partial D, ds)$ is defined by

$$\mathcal{S}_{\partial D} f(z) = -\frac{1}{2\pi} \int_{\partial D} f(w) \ln |z-w| \, ds_w,$$

where ds = arc-length. It is well-known that $\mathcal{S}_{\partial D}$ is a Hilbert-Schmidt operator acting on $L^2(\partial D, ds)$ and hence it is compact. If the boundary curve is a circle of radius r, then the corresponding single layer potential is injective if and only if $r \neq 1$ (see [5, Lemma 8.23]). The analog of single layer potential defined on D is often referred to as logarithmic potential and is defined by

$$(\mathcal{L}_D f)(z) = -\frac{1}{2\pi} \int_D f(w) \log |z-w| dA_w, \ f \in L^2(D, dA),$$

where dA denotes the area measure. It is easy to show that $\mathcal{L}_D$ is also self-adjoint and compact. We refer the interested reader to [1], [3] and [8] for details on spectral properties of logarithmic potentials.

Recently Ruzhansky and Suragan in [7] established an isoperimetric inequality for the Schatten $p-$norms of logarithmic potentials over bounded domains of a given area. We recall that a compact self-adjoint operator T on a separable Hilbert space is said to belong to the Schatten $p-$class if its set of singular numbers belongs

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to ℓ^p. The Schatten $p-$norm of T is simply the ℓ^p-norm of its singular numbers. Ruzhansky's results are similar to isoperimetric inequalities for the Laplace operator (see [4], [6], and [10]). Among other things, Ruzhansky and Suragan in [7] proved that the Schatten $p-$norm is maximized on the equilateral triangle centered at the origin among all triangles of a given area. They also conjectured the following:

"For any integer p, the maximizer of Schatten $p-$norms of the logarithmic potential over all convex $n-$gons of a given area, is the regular $n-$gon."

We showed in [11] that the conjecture is true for $n = 4$, namely for quadrilaterals. It seems fair to say that the Ruzhansky-Suragan conjecture should have an affirmative answer. In the present paper we establish similar isoperimetric inequalities for Schatten $p-$norms of single layer potentials over triangles and rectangles. We shall apply the so-called Purkiss principle. We recall the Purkiss principle (see, for instance, [9]):

Let $f(x_1, \cdots, x_n)$ and $g(x_1, \cdots, x_n)$ be two symmetric functions, i.e. $f(X) = f(\sigma(X))$ and $g(X) = g(\sigma(X))$ where $X = (x_1, \cdots, x_n)$, for all $\sigma \in \mathrm{Symm}\{x_1, \cdots, x_n\}$. Suppose f and g have continuous second derivatives in a neighborhood of a point $P = (r, \cdots, r)$. On the set where $g = g(P)$, the function f will have a local minimum or maximum at P except in the degenerate case, i.e. where $\nabla g \equiv 0$.

2. Main results

It is our intention to establish the following à la Ruzhansky (cf. [7] and [11]) isoperimetric inequalities:

Theorem A. *Let S be a square and R be a rectangle with $|\partial S| = |\partial R|$. Assume that the single layer potential is positive for ∂R and ∂S. Then*

$$\|\mathcal{S}_{\partial S}\|_p \leq \|\mathcal{S}_{\partial R}\|_p, \quad \textit{for any integer } 3 \leq p < \infty.$$

Proof. Without loss of generality we may assume that $R = [0, x] \times [0, y]$ with $|\partial R| = 2x + 2y = \ell > 0$. For the Hilbert-Schmidt norm, one finds that

$$\begin{aligned}
\|\mathcal{S}_{\partial R}\|_2^2 &= \frac{1}{4\pi^2} \int_{\partial R} \int_{\partial R} \log^2 |z - w| ds_z ds_w \\
&= \frac{1}{4\pi^2} \bigg\{ 2 \int_0^x \int_0^x \log^2 |t - s| dt ds + 2 \int_0^y \int_0^y \log^2 |t - s| dt ds \\
&+ 2 \int_0^x \int_0^x \log^2(|t - s|^2 + 1) dt ds + 2 \int_0^y \int_0^y \log^2(|t - s|^2 + 1) dt ds \\
&+ 6 \int_0^x \int_0^y \log^2(t^2 + s^2) dt ds \bigg\}.
\end{aligned}$$

Clearly $\|\mathcal{S}_{\partial R}\|_2^2$ defines a symmetric, but certainly not twice differentiable, function in x and y. For an integer $p \geq 3$, let $\Psi_p(x, y) = \|\mathcal{S}_{\partial R}\|_p^p$. Then Ψ_p is symmetric. In the integral representation of Schatten $p-$norms with $p \geq 3$, each x and y appear at least three times as the upper limit of integration. Then by the fundamental theorem of calculus, Ψ_p is twice differentiable in x and y. For instance for $p = 3$, we have

$$\Psi_p(x, y) = (\frac{-1}{2\pi})^3 \int_{\partial R} \int_{\partial R} \int_{\partial R} \log |t_1 - t_2| \log |t_1 - t_3| \log |t_2 - t_3| ds_{t_1} ds_{t_2} ds_{t_3}.$$

The function $L(x,y) = 2x + 2y$ for $(x,y) \in \mathbb{R}^+ \times \mathbb{R}^+$ is obviously symmetric and has continuous second derivatives. By the Purkiss principle Ψ_p has a maximum or minimum at $x_0 = y_0$, but since $L(x_0, y_0) = \ell$, then $x_0 = y_0 = \frac{\ell}{4}$. It is easy to show that $(\frac{\ell}{4}, \frac{\ell}{4})$ cannot be a maximum. Therefore $(\frac{\ell}{4}, \frac{\ell}{4})$ is a minimizer. $\square$

Theorem B. *Suppose Δ_e is an equilateral triangle and Δ is a triangle with $|\partial\Delta| = |\partial\Delta_e|$. Assume that the single layer potential is positive for $\partial\Delta_e$ and $\partial\Delta$. Then*

$$\|\mathcal{S}_{\partial\Delta_e}\|_p \le \|\mathcal{S}_{\partial\Delta}\|_p, \quad \textit{for any integer } 3 \le p < \infty.$$

Proof. Assume Δ is a triangle of sides x, y, z and interior angles α, β, γ. For the Hilbert-Schmidt norm of single layer potential over $\partial\Delta$ we have

$$\begin{aligned}
\|\mathcal{S}_{\partial\Delta}\|_2^2 &= \frac{1}{4\pi^2}\int_{\partial\Delta}\int_{\partial\Delta} \log^2|z-w|ds_z ds_w \\
&= \frac{1}{4\pi^2}\Bigg\{\int_0^x\int_0^x \log^2|t-s|dtds + \int_0^y\int_0^y \log^2|t-s|dtds \\
&+ \int_0^z\int_0^z \log^2|t-s|dtds + 2\int_0^x\int_0^y \log^2\sqrt{t^2+s^2-2ts\cos\alpha}dtds \\
&+ 2\int_0^x\int_0^z \log^2\sqrt{t^2+s^2-2ts\cos\beta}dtds \\
&+ 2\int_0^y\int_0^z \log^2\sqrt{t^2+s^2-2ts\cos\gamma}dtds\Bigg\}.
\end{aligned}$$

The above expression is symmetric in x, y, z. For $p \ge 3$, let Ψ_p be the Schatten $p-$norm of single layer potential over $\partial\Delta$. Then Ψ_p defines a symmetric, at least twice differentiable function of x, y and z. The perimeter function $L(x,y,z) = x + y + z$ is symmetric and has continuous second derivatives for $(x,y,z) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$. If $L(x,y,z) = \ell > 0$, by the Purkiss principle Ψ_p attains a minimum at $x_0 = y_0 = z_0$. Therefore $(\frac{\ell}{3}, \frac{\ell}{3}, \frac{\ell}{3})$ is a minimizer. $\square$

Remark 2.1. The computer generated graph below illustrates symmetricity and twice differentiability of the defining function $\Psi_2(x,y)$ on a rectangle of perimeter one. Apparently Theorem A holds for $p = 2$, even though it does not follow from our proof. We suspect that Theorem B holds for $p = 2$ as well.

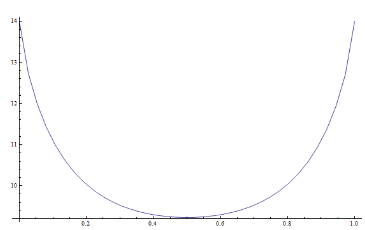

FIGURE 1. $y = \Psi_2^2(x, 1-x)$ for $0 < x < 1$.

Remark 2.2. The Schatten $p-$norms of single layer potentials on the boundary of $n-$gons with $n \ge 5$ do not yield symmetric functions of side length. As a result, our method is inconclusive for $n-$gons with $n \ge 5$.

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