

RELATIONS ON $\overline{\mathcal{M}}_{g,n}$ VIA ORBIFOLD STABLE MAPS

EMILY CLADER

(Communicated by Lev Borisov)

ABSTRACT. Using the equivariant virtual cycle of the moduli space of stable maps to $[\mathbb{C}/\mathbb{Z}_r]$, or equivalently, the vanishing of high-degree Chern classes of a certain vector bundle over the moduli space of stable maps to $B\mathbb{Z}_r$, we derive relations in the Chow ring of $\overline{\mathcal{M}}_{g,n}(B\mathbb{Z}_r, 0)$. These push forward to yield tautological relations on $\overline{\mathcal{M}}_{g,n}$.

1. INTRODUCTION

The tautological ring of the moduli space of curves is the minimal family of subrings $R^*(\overline{\mathcal{M}}_{g,n})$ inside the Chow ring (or cohomology) of each $\overline{\mathcal{M}}_{g,n}$ that is stable under pullbacks and pushforwards via the forgetful and attaching maps between these moduli spaces. In particular, the kappa, psi, lambda, and boundary classes, which play important roles in Gromov-Witten theory, are all elements of the tautological ring.

Relations between tautological classes in \mathcal{M}_g were initially studied by Mumford [29] in the 1980s. More recently, ideas from topological string theory have been applied to develop a more complete picture. In 2000, Faber and Zagier conjectured the so-called FZ relations on \mathcal{M}_g , which were proved via the geometry of stable quotients in 2010 [30]. A remarkable generalization occurred in 2013, when Pandharipande-Pixton-Zvonkine [31] used Witten's 3-spin theory and the method of quantization developed by Givental and Teleman [17, 34] to prove that a set of relations previously conjectured by Pixton [32] holds in the cohomology of $\overline{\mathcal{M}}_{g,n}$. When restricted to \mathcal{M}_g , these relations recover the FZ relations, and indeed, all presently known tautological relations follow from those proved in [31].

The 3-spin relations [31] were cast in a new framework by Janda in [20, 21], which outline a general method for deriving tautological relations via the existence of certain limits when a semisimple Cohomological Field Theory moves toward a nonsemisimple point. This method can be used to prove that the 3-spin relations hold in the Chow ring of $\overline{\mathcal{M}}_{g,n}$, and that they are equivalent to relations that arise out of the geometry of equivariant stable maps to \mathbb{P}^1 .

Received by the editors March 2, 2016.

2010 *Mathematics Subject Classification*. Primary 14H10.

This work was partially supported by FRG grant DMS-1159265, RTG grant DMS-1045119, Dr. Max Rössler, the Walter Haefner Foundation, and the ETH Zürich Foundation.

The main result of the present paper is the following:

Theorem 1.1. *Suppose that $a_1, \dots, a_n \in \{0, 1, \dots, r-1\}$ satisfy $0 \neq \sum_{i=1}^n a_i \equiv 0 \pmod{r}$. Then, for any d such that*

$$\frac{1}{r} \sum_{i=1}^n a_i + g - 1 < d \leq 3g - 3 + n,$$

one has the following relations in $A^d(\overline{\mathcal{M}}_{g,\mathbf{a}}(B\mathbb{Z}_r, 0))$:

$$0 = \sum_{\substack{d_1 + \dots + d_m = d \\ d_i, m \geq 1}} \frac{1}{m! \prod_i d_i (d_i + 1)} \prod_{i=1}^m \left(B_{d_i+1}(0) \kappa_{d_i} - \sum_{j=1}^n B_{d_i+1} \left(\frac{a_j}{r} \right) \psi_j^{d_i} \right. \\ \left. + \frac{r}{2} \sum_{\substack{0 \leq l \leq g \\ I \subset [n]}} B_{d_i+1} \left(\frac{q_{l,I}}{r} \right) p^* i_{(l,I)*} (\gamma_{d_i-1}) + \frac{r}{2} \sum_{q=0}^{r-1} B_{d_i+1} \left(\frac{q}{r} \right) j_{(irr,q)*} (\gamma_{d_i-1}) \right).$$

If $a_1 = \dots = a_n = 0$, the same equations hold for $g > 0$ on components of $\overline{\mathcal{M}}_{g,\mathbf{0}}(B\mathbb{Z}_r, 0)$ corresponding to curves with a nontrivial r -torsion line bundle.

In particular, the pushforwards of these equations via the map

$$\rho : \overline{\mathcal{M}}_{g,\mathbf{a}}(B\mathbb{Z}_r, 0) \rightarrow \overline{\mathcal{M}}_{g,n}$$

give tautological relations $A^d(\overline{\mathcal{M}}_{g,n})$.

The definitions of the classes appearing in the theorem are reviewed in Section 3.2.

These relations can be viewed, in some sense analogously to the context of [21], as a consequence of the existence of the nonequivariant limit in the Cohomological Field Theory associated to the equivariant Gromov-Witten theory of the orbifold $[\mathbb{C}/\mathbb{Z}_r]$. Alternatively, they can be derived easily from the vanishing of high-degree Chern classes of a certain vector bundle on $\overline{\mathcal{M}}_{g,n}(B\mathbb{Z}_r, 0)$, which are expressible in terms of tautological classes by Toen's orbifold Grothendieck-Riemann-Roch formula [35]. In fact, the requisite application of Grothendieck-Riemann-Roch has previously been carried out by Tseng [36] in a more general setting, and, in precisely the setting needed here, by Chiodo [4]. We make use of Chiodo's formulation in what follows.

1.1. Relation to other work. We present the above relations from the point of view of the equivariant Gromov-Witten theory of $[\mathbb{C}/\mathbb{Z}_r]$, although the perspective via vanishing of high-degree Chern classes is more direct and leads naturally to a family of generalizations (see Section 3.4), because of the importance of $[\mathbb{C}/\mathbb{Z}_r]$ in connecting our results to other known tautological relations. In particular, in joint work [8] with Janda, we prove that the relations of Theorem 1.1 can be deduced, in a highly nontrivial way, from Pixton's 3-spin relations.¹ The main idea in the proof of this statement is a comparison between the graph-sum expression for the Gromov-Witten theory of (a compactification of) $[\mathbb{C}/\mathbb{Z}_r]$ obtained via Givental-Teleman reconstruction with that obtained via equivariant localization.

¹We remark, however, that this is only the case after pushforward to $\overline{\mathcal{M}}_{g,n}$. The initial equations in $\overline{\mathcal{M}}_{g,\mathbf{a}}(B\mathbb{Z}_r, 0)$ are not pulled back from the moduli space of curves; see Remark 3.1.

Also in [8], we study an important application of Theorem 1.1: it can be used to prove a conjecture of Pixton regarding the double ramification cycle $DR_{g,A} \in A^g(\overline{\mathcal{M}}_{g,n})$. Pixton proposed an explicit, mixed-degree tautological class $\Omega_{g,A} \in A^*(\overline{\mathcal{M}}_{g,n})$, conjecturing that (a) its component in degree g is equal to $DR_{g,A}$, and (b) its components in degrees greater than g vanish. The first of these statements was recently proved by Janda-Pandharipande-Pixton-Zvonkine [22]; the second is the main theorem of [8].

1.2. Outline of the paper. In Section 2, we review the basic definitions from orbifold Gromov-Witten theory that we will require. Most importantly, we recall how to express the equivariant virtual cycle for (most) components of the moduli space of stable maps to $[\mathbb{C}/\mathbb{Z}_r]$ as the Chern class of a vector bundle over $\overline{\mathcal{M}}_{g,n}(B\mathbb{Z}_r, 0)$. Via Chiodo's orbifold Grothendieck-Riemann-Roch computation [4], this affords us an expression for the equivariant virtual cycle in terms of tautological classes, which we present in Section 3 and use to prove Theorem 1.1. We conclude with a discussion of the alternative perspective mentioned above, which circumvents the equivariant theory and the orbifold $[\mathbb{C}/\mathbb{Z}_r]$ entirely, and which can be naturally generalized to other moduli spaces $\overline{\mathcal{M}}_{g,n}(BG, 0)$.

2. PRELIMINARIES ON $[\mathbb{C}/\mathbb{Z}_r]$ AND ITS GROMOV-WITTEN THEORY

The results of this section are well known to experts, but we recall them here for clarity.

2.1. The orbifold and its cohomology. By $[\mathbb{C}/\mathbb{Z}_r]$, we mean the stack quotient of \mathbb{C} by the action of \mathbb{Z}_r via multiplication by r th roots of unity.

The orbifold (or Chen-Ruan) cohomology of $[\mathbb{C}/\mathbb{Z}_r]$ is, as a vector space, the cohomology of the inertia stack $I[\mathbb{C}/\mathbb{Z}_r]$, whose objects are pairs (x, g) with $x \in [\mathbb{C}/\mathbb{Z}_r]$ and g an element of the isotropy group at x . Explicitly,

$$I[\mathbb{C}/\mathbb{Z}_r] = [\mathbb{C}/\mathbb{Z}_r] \sqcup \bigsqcup_{i=1}^{r-1} B\mathbb{Z}_r,$$

in which $B\mathbb{Z}_r$ denotes the stack consisting of a single point with \mathbb{Z}_r isotropy. Thus,

$$H_{CR}^*([\mathbb{C}/\mathbb{Z}_r]; \mathbb{C}) = \mathbb{C}\{\zeta_0, \zeta_1, \dots, \zeta_{r-1}\},$$

where ζ_0 is the constant function 1 on $[\mathbb{C}/\mathbb{Z}_r]$ and ζ_i is the constant function 1 on the i th copy of $B\mathbb{Z}_r$. The component of the inertia stack whose cohomology is generated by ζ_0 is called the *nontwisted sector*, while the other components are referred to as *twisted sectors*.

2.2. Stable maps to orbifolds. For an orbifold \mathcal{X} , let $\overline{\mathcal{M}}_{g,n}(\mathcal{X}, d)$ denote the moduli stack of n -pointed genus- g orbifold stable maps to \mathcal{X} of degree d . This stack parameterizes families of genus- g , n -pointed orbifold curves C equipped with a representable morphism $f : C \rightarrow \mathcal{X}$ for which the corresponding map between coarse underlying spaces is a stable map of degree d . The condition of representability means that the induced homomorphism on isotropy groups at every point is injective.

The precise definition of an orbifold curve can be found, for example, in [1]. For the present, we recall just two particularly important features. First, orbifold curves are allowed isotropy only at marked points and nodes. Second, the orbifold

structure at nodes is required to be *balanced*, which means that étale locally near each node, C has the form

$$[\{xy = 0\}/\mathbb{Z}_\ell]$$

with \mathbb{Z}_ℓ acting by $\xi(x, y) = (\xi x, \xi^{-1}y)$.

The moduli stack $\overline{\mathcal{M}}_{g,n}(\mathcal{X}, d)$ admits a perfect obstruction theory relative to the Artin stack of prestable pointed orbicurves, and this obstruction theory can be used to define a virtual fundamental class

$$[\overline{\mathcal{M}}_{g,n}(\mathcal{X}, d)]^{\text{vir}} \in H^*(\overline{\mathcal{M}}_{g,n}(\mathcal{X}, d); \mathbb{C}).$$

As in the nonorbifold case, the perfect obstruction theory is given by the object $(R^\bullet \pi_* f^* T\mathcal{X})^\vee$ in the derived category, where

$$\begin{array}{ccc} C & \xrightarrow{f} & \mathcal{X} \\ \pi \downarrow & & \\ \overline{\mathcal{M}}_{g,n}(\mathcal{X}, d) & & \end{array}$$

is the universal family over the moduli space.

Also analogously to the nonorbifold theory, there are evaluation morphisms, but in this case they map to the inertia stack:

$$\text{ev}_i : \overline{\mathcal{M}}_{g,n}(\mathcal{X}, d) \rightarrow I\mathcal{X}.$$

Specifically, the image of $(f : C \rightarrow \mathcal{X})$ can be viewed as the point $(x, g) \in I\mathcal{X}$, where x is the image of the i th marked point p_i and g is the image of the canonical generator of the isotropy group at p_i under the homomorphism of isotropy groups $G_{p_i} \rightarrow G_x$ induced by f .

There is a decomposition of $\overline{\mathcal{M}}_{g,n}(\mathcal{X}, d)$ according to the images of the evaluation maps:

$$(1) \quad \overline{\mathcal{M}}_{g,n}(\mathcal{X}, d) = \bigsqcup_{a_1, \dots, a_n \in \mathcal{I}} \text{ev}_1^{-1}(\mathcal{X}_{a_1}) \cap \dots \cap \text{ev}_n^{-1}(\mathcal{X}_{a_n}),$$

where

$$I\mathcal{X} = \bigsqcup_{i \in \mathcal{I}} \mathcal{X}_i$$

decomposes $I\mathcal{X}$ into twisted sectors. We denote

$$\overline{\mathcal{M}}_{g,\mathbf{a}}(\mathcal{X}, d) = \text{ev}_1^{-1}(\mathcal{X}_{a_1}) \cap \dots \cap \text{ev}_n^{-1}(\mathcal{X}_{a_n}),$$

where $\mathbf{a} = (a_1, \dots, a_n)$. Each of these components obtains a virtual fundamental class by restriction.

2.3. Stable maps to $[\mathbb{C}/\mathbb{Z}_r]$. In the special case where $\mathcal{X} = [\mathbb{C}/\mathbb{Z}_r]$, there are no stable maps of positive degree. The decomposition (1) of the moduli space $\overline{\mathcal{M}}_{g,n}([\mathbb{C}/\mathbb{Z}_r], 0)$ is written

$$\overline{\mathcal{M}}_{g,n}([\mathbb{C}/\mathbb{Z}_r], 0) = \bigsqcup_{a_1, \dots, a_n \in \mathbb{Z}_r} \overline{\mathcal{M}}_{g,\mathbf{a}}([\mathbb{C}/\mathbb{Z}_r], 0),$$

where $\overline{\mathcal{M}}_{g,\mathbf{a}}([\mathbb{C}/\mathbb{Z}_r], 0)$ is the component in which the i th marked point maps to the twisted sector indexed by ζ_i . (We identify \mathbb{Z}_r with the set $\{0, 1, \dots, r-1\}$ whenever it appears as an index set.)

We observe that the substack $\overline{\mathcal{M}}_{g,\mathbf{a}}([\mathbb{C}/\mathbb{Z}_r], 0)$ is:

- (1) Compact if and only if not every a_i is equal to zero.²
- (2) Nonempty if and only if $\sum_{i=1}^n a_i \equiv 0 \pmod{r}$.

The proof of the first of these assertions is a straightforward consequence of the compactness of $\overline{\mathcal{M}}_{g,n}(\mathcal{X}, 0)$ for compact targets, while the second is a standard fact in orbifold Gromov-Witten theory; alternatively, it will follow from the description of the moduli space in terms of line bundles given below.

Unless otherwise specified, we will always work only on components for which not every a_i is zero, to ensure compactness. In this case, $\overline{\mathcal{M}}_{g,\mathbf{a}}([\mathbb{C}/\mathbb{Z}_r], 0)$ is isomorphic to $\overline{\mathcal{M}}_{g,\mathbf{a}}(B\mathbb{Z}_r, 0)$. However, their obstruction theories are different; while there are obstructions to deformations of stable maps to $[\mathbb{C}/\mathbb{Z}_r]$ parameterized by $H^1(C, f^*N_{B\mathbb{Z}_r/[\mathbb{C}/\mathbb{Z}_r]})$, the moduli space of stable maps to $B\mathbb{Z}_r$ is smooth and unobstructed. This leads to the following relationship between virtual cycles:

$$(2) \quad [\overline{\mathcal{M}}_{g,\mathbf{a}}([\mathbb{C}/\mathbb{Z}_r], 0)]^{\text{vir}} = [\overline{\mathcal{M}}_{g,\mathbf{a}}(B\mathbb{Z}_r, 0)] \frown e(R^1\pi_*f^*N_{B\mathbb{Z}_r/[\mathbb{C}/\mathbb{Z}_r]}).$$

Another perspective on stable maps to $B\mathbb{Z}_r$ can be used to make equation (2) more explicit. The data of an orbifold stable map $C \rightarrow B\mathbb{Z}_r$ is equivalent to an orbifold line bundle L on C satisfying $L^{\otimes r} \cong \mathcal{O}_C$. The map $f : C \rightarrow B\mathbb{Z}_r$ corresponds to the line bundle f^*N , where N is the topologically trivial line bundle over $B\mathbb{Z}_r$ on which \mathbb{Z}_r acts by multiplication by r th roots of unity.

From this point of view, $f^*N_{B\mathbb{Z}_r/[\mathbb{C}/\mathbb{Z}_r]}$ corresponds to the universal line bundle \mathcal{L} on the universal curve over $\overline{\mathcal{M}}_{g,n}(B\mathbb{Z}_r, 0)$. Furthermore, the monodromy a_i becomes the *multiplicity* of L at the i th marked point—that is, a_i is the weight of the action of the isotropy group at the marked point p_i on the fiber of L .

Much more about orbifold line bundles and their tensor powers can be found, for example, in [3]. One crucial fact is the relationship between an orbifold line bundle L and its pushforward $|L|$ under the map $\epsilon : C \rightarrow |C|$ to the coarse underlying curve. Suppose that C_0 is an irreducible component of C with special points x_1, \dots, x_k at which L has multiplicities a_1, \dots, a_k . Then, if $L^{\otimes r} \cong \epsilon^*M$ for a line bundle M on $|C|$, the coarse underlying line bundle $|L| = \epsilon_*L$ satisfies

$$|L|^{\otimes r} \cong M \otimes \left(- \sum_{i=1}^k a_i [x_i] \right)$$

when restricted to $|C_0|$.

In the case of interest, $M = \mathcal{O}_C$, and the above formula implies that if C has no nodes, then

$$(3) \quad \deg(|L|) = -\frac{1}{r} \sum_{i=1}^n a_i,$$

and in general, an analogous formula holds on each component of the normalization of C . There are two important consequences of (3). First, since $|L|$ is a line bundle on a nonorbifold curve, it must have integral degree, so we recover the fact mentioned previously that $\sum_{i=1}^n a_i \equiv 0 \pmod{r}$. Second, because at least one a_i is nonzero, there is at least one component on which $\deg(|L|) < 0$. Since $\deg(|L|) \leq 0$ on every component, this implies that $H^0(L) = H^0(|L|) = 0$.

²There will, however, be at least one compact component of $\overline{\mathcal{M}}_{g,(0,\dots,0)}([\mathbb{C}/\mathbb{Z}_r], 0)$ when $g > 0$; see Remark 2.1.

Combining these observations, (2) gives the following expression for the virtual cycle of the moduli space of stable maps to $[\mathbb{C}/\mathbb{Z}_r]$ in terms of twisted theory over $B\mathbb{Z}_r$:

$$(4) \quad [\overline{\mathcal{M}}_{g,\mathbf{a}}([\mathbb{C}/\mathbb{Z}_r], 0)]^{\text{vir}} = [\overline{\mathcal{M}}_{g,\mathbf{a}}(B\mathbb{Z}_r, 0)] \frown e(-R^\bullet \pi_* \mathcal{L}).$$

This expression, and its equivariant version (5), will be central to the computations that follow.

Remark 2.1. If $a_1 = \cdots = a_n = 0$, then one has $\deg(L) = 0$ on every component of C , so the above argument that $H^0(L) = 0$ fails and (4) is no longer valid in general. However, in this case, the moduli space $\overline{\mathcal{M}}_{g,0}(B\mathbb{Z}_r, 0)$ of r -torsion line bundles splits into components, on each of which L is either always trivial or always nontrivial. On nontrivial components, which exist as long as $g > 0$, an r -torsion line bundle can have no global sections. Thus, (4) still holds on these components of the moduli space.

2.4. Equivariant theory. There is an action of \mathbb{C}^* on $[\mathbb{C}/\mathbb{Z}_r]$ via multiplication, and this induces an action on $\overline{\mathcal{M}}_{g,n}([\mathbb{C}/\mathbb{Z}_r], 0)$. The perfect obstruction theory is \mathbb{C}^* -equivariant, and as a result, one obtains an equivariant version of the virtual fundamental class.

In terms of the expression (4) on components where not every a_i is zero, the equivariant virtual fundamental class is

$$(5) \quad \overline{\mathcal{M}}_{g,\mathbf{a}}([\mathbb{C}/\mathbb{Z}_r], 0)]^{\mathbb{C}^*, \text{vir}} = [\overline{\mathcal{M}}_{g,\mathbf{a}}(B\mathbb{Z}_r, 0)]^{\mathbb{C}^*} \frown e^{\mathbb{C}^*}(-R^\bullet \pi_* \mathcal{L}),$$

in which \mathbb{C}^* acts trivially on $\overline{\mathcal{M}}_{g,\mathbf{a}}(B\mathbb{Z}_r, 0)$ and by multiplication on the fibers of the vector bundle $-R^\bullet \pi_* \mathcal{L}$. Despite the triviality of the action on these components, the above expression is useful, because the equivariant Euler class can easily be written in terms of Chern characters, which affords the possibility of applying the Grothendieck-Riemann-Roch formula.

3. PROOF OF THEOREM 1.1

In this section, we write (5) explicitly in terms of tautological classes on $\overline{\mathcal{M}}_{g,\mathbf{a}}(B\mathbb{Z}_r, 0)$ —following the Grothendieck-Riemann-Roch computations of Chiodo [3] and Tseng [36]—and we derive Theorem 1.1 via the existence of the nonequivariant limit.

3.1. Introducing Chern characters. Any multiplicative invertible characteristic class can be written in terms of Chern characters. For the \mathbb{C}^* -equivariant Euler class, one obtains

$$(6) \quad e^{\mathbb{C}^*}((-V)^\vee) = \exp\left(\sum_{d=0}^{\infty} s_d \text{ch}_d([V])\right)$$

for

$$s_d = \begin{cases} -\ln(\lambda) & d = 0, \\ \frac{(d-1)!}{\lambda^d} & d > 0, \end{cases}$$

where λ is the equivariant parameter.

The rank of the vector bundle $-R^\bullet \pi_* \mathcal{L}$ can be computed using the Riemann-Roch formula; it equals

$$\text{rank}(-R^\bullet \pi_* \mathcal{L}) = h^1(|L|) = -\deg(|L|) + g - 1 = \frac{1}{r} \sum_{i=1}^n a_i + g - 1$$

for any element (C, L) of $\overline{\mathcal{M}}_{g,\mathbf{a}}(B\mathbb{Z}_r, 0)$. Combining this with (5) and (6) expresses the equivariant virtual cycle of $\overline{\mathcal{M}}_{g,\mathbf{a}}([\mathbb{C}/\mathbb{Z}_r], 0)$ as the Poincaré dual of

$$(7) \quad e^{\mathbb{C}^*}(-R^\bullet \pi_* \mathcal{L}) = (-1)^{\deg} \lambda^{\frac{1}{r} \sum a_i + g - 1} \exp\left(\sum_{d=1}^{\infty} \frac{(d-1)!}{\lambda^d} \text{ch}_d(R^\bullet \pi_* \mathcal{L})\right).$$

Here, \deg denotes the nonequivariant degree (that is, the cohomological degree of an equivariant class after setting $\lambda = 1$), so $(-1)^{\deg}$ should be understood as an operator on $A_{\mathbb{C}^*}(\overline{\mathcal{M}}_{g,\mathbf{a}}([\mathbb{C}/\mathbb{Z}_r], 0))$.

3.2. GRR for the universal r th root. By applying the orbifold Grothendieck-Riemann-Roch formula (see [35] or Appendix A of [36]), the Chern characters of $R^\bullet \pi_* \mathcal{L}$ can be expressed in terms of tautological classes.

In [4], Chiodo carries out this computation explicitly and substantially simplifies the result. Strictly speaking, his computation is carried out on a slightly different moduli space from $\overline{\mathcal{M}}_{g,\mathbf{a}}(B\mathbb{Z}_r, 0)$, parameterizing r th roots of $\mathcal{O}(-\sum_i a_i [x_i])$ on curves with no orbifold structure at marked points and balanced \mathbb{Z}_r orbifold structure at every node. In particular, the representability assumption does not appear in his moduli space, so it will only agree with $\overline{\mathcal{M}}_{g,\mathbf{a}}(B\mathbb{Z}_r, 0)$ when all of the multiplicities at nodes are coprime to r . However, there is always a morphism between his moduli space $\overline{\mathcal{M}}_{g,n}^r$ and the moduli space $\overline{\mathcal{M}}_{g,\mathbf{a}}(B\mathbb{Z}_r, 0)$ appearing here, which is invertible on the interior and has explicitly-computable ramification indices on the boundary. Thus, we can state our formula on $\overline{\mathcal{M}}_{g,\mathbf{a}}(B\mathbb{Z}_r, 0)$, but all of the classes in the formula will be pulled back from $\overline{\mathcal{M}}_{g,n}^r$.

Corollary 3.1.8 of [4] states, in the present situation, that

$$\begin{aligned} \text{ch}_d(R^\bullet \pi_* \mathcal{L}) &= \frac{B_{d+1}(0)}{(d+1)!} \kappa_d - \sum_{i=1}^n \frac{B_{d+1}(\frac{a_i}{r})}{(d+1)!} \psi_i^d \\ &+ \frac{r}{2} \sum_{\substack{0 \leq l \leq g \\ I \subset [n]}} \frac{B_{d+1}(\frac{q_{l,I}}{r})}{(d+1)!} p^* i_{(l,I)*}(\gamma_{d-1}) + \frac{r}{2} \sum_{q=0}^{r-1} \frac{B_{d+1}(\frac{q}{r})}{(d+1)!} j_{(irr,q)*}(\gamma_{d-1}). \end{aligned}$$

Let us review the notation appearing in this formula. First, $B_{d+1}(x)$ are the Bernoulli polynomials, defined by the generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.$$

The kappa and psi classes are defined by pullback under the map

$$p : \overline{\mathcal{M}}_{g,n}(B\mathbb{Z}_r, 0) \rightarrow \overline{\mathcal{M}}_{g,n}(r),$$

where $\overline{\mathcal{M}}_{g,n}(r)$ is the moduli space of r -stable, n -pointed, genus- g curves. The notion of r -stability is defined precisely in Definition 2.1.1 of [4], but essentially, it refers to orbifold curves in which the marked points have trivial isotropy and the

nodes are balanced with \mathbb{Z}_r isotropy. On this moduli space, kappa and psi classes are defined as usual:

$$\kappa_d = \pi_*(c_1(\omega_{\log})^d), \quad \psi_i = c_1(\omega|_{p_i}),$$

where $\omega_{\log} = \omega_{C/X}(\sum_{i=1}^n [p_i])$.

Let Z be the codimension-two locus inside the universal curve over $\overline{\mathcal{M}}_{g,n}(r)$ consisting of nodes in singular fibers. There is a decomposition

$$(8) \quad Z = \bigsqcup_{\substack{0 \leq l \leq g \\ I \subset [n]}} Z_{(l,I)} \sqcup Z_{irr},$$

where $Z_{(l,I)}$ consists of nodes separating the curve C into a component of genus l containing the marked points in I and a component of genus $g-l$ containing the other marked points, while Z_{irr} consists of nonseparating nodes. Let $Z' \rightarrow Z$ be the two-fold cover given by nodes together with a choice of branch. The decomposition (8) induces an analogous decomposition of Z' , and the morphisms

$$i_{(l,I)} : Z'_{(l,I)} \rightarrow \overline{\mathcal{M}}_{g,n}(r)$$

are given by this two-fold cover, followed by the inclusion into the universal curve and the projection to $\overline{\mathcal{M}}_{g,n}(r)$.

The index $q_{l,I}$ is the multiplicity at the chosen branch of the node for any point in $Z'_{(l,I)}$. The equation (3) determines this multiplicity from l and I ; specifically, it is the unique number in $\{0, 1, \dots, r-1\}$ satisfying

$$q_{l,I} + \sum_{i \in I} a_i \equiv 0 \pmod{r}.$$

If ψ is the first Chern class of the line bundle over $Z'_{(l,I)}$ whose fiber is the cotangent line to the chosen branch of the node, and $\hat{\psi}$ is the first Chern class of the bundle whose fiber is the cotangent line to the opposite branch, then γ_d is defined by

$$\gamma_d = \sum_{i+j=d} (-\psi)^i \hat{\psi}^j.$$

Finally, let $Z'_{(irr,q)}$ be the locus inside the universal curve over $\overline{\mathcal{M}}_{g,n}^r$ consisting of nonseparating nodes with a choice of branch such that the multiplicity of the line bundle L at the chosen branch is equal to $q \in \{0, 1, \dots, r-1\}$. It should be noted that q is not determined by the topology of the underlying curve alone, as the multiplicity was in the case of separating nodes, so we must view $Z'_{(irr,q)}$ as lying over $\overline{\mathcal{M}}_{g,n}^r$ rather than over $\overline{\mathcal{M}}_{g,n}(r)$. We have morphisms

$$j_{(irr,q)} : Z'_{(irr,q)} \rightarrow \overline{\mathcal{M}}_{g,n}^r$$

given, as before, by the two-fold cover, inclusion into the universal curve, and projection. The class γ_d is defined by exactly the same formula as previously. By pullback under the birational map discussed above, the class $j_{(irr,q)*}(\gamma_{d-1})$ is viewed as lying on $\overline{\mathcal{M}}_{g,\mathbf{a}}(B\mathbb{Z}_r, 0)$, and, by abuse of notation, it is still denoted $\hat{j}_{(irr,q)*}(\gamma_{d-1})$.

3.3. Tautological relations. Equipped with this formula, the proof of Theorem 1.1 is immediate from (7):

Proof of Theorem 1.1. The existence of the nonequivariant limit in (7) implies that any term with a negative power of λ must vanish, which proves the theorem whenever not every a_i is zero. The case $a_1 = \cdots = a_n = 0$ follows from the same argument after applying Remark 2.1. \square

Remark 3.1. While pushforward yields relations in $A^d(\overline{\mathcal{M}}_{g,n})$, the original relations in $A^d(\overline{\mathcal{M}}_{g,\mathbf{a}}(B\mathbb{Z}_r, 0))$ do not arise by pullback from the moduli space of curves. The reason for this is the appearance of the classes $j_{(irr,q)*}(\gamma_{d-1})$, which are not pulled back because the multiplicity at a nonseparating node is not determined by the topology of the coarse underlying curve.

3.4. An alternative perspective. There is a different way to understand the proof of Theorem 1.1 that circumvents $[\mathbb{C}/\mathbb{Z}_r]$ entirely. Namely, let

$$M = \overline{\mathcal{M}}_{g,\mathbf{a}}(B\mathbb{Z}_r, 0),$$

and let V be the vector bundle $-R^\bullet\pi_*\mathcal{L}$ on this stack.

The total Chern class of a vector bundle can always be written in terms of its Chern characters:

$$c(V) = \exp\left(\sum_{d \geq 1} (-1)^{d+1} (d-1)! \cdot \text{ch}_d(V)\right).$$

While the exponential appears to have contributions in every degree, the total Chern class clearly vanishes beyond the rank of the vector bundle. Theorem 1.1 is a consequence of this vanishing when expanded in terms of the Chern characters on the right-hand side.

In particular, only two facts are necessary for the proof:

- (1) One has a moduli space M with a map to $\overline{\mathcal{M}}_{g,n}$.
- (2) On M , there is a vector bundle V whose Chern characters can be expressed in terms of classes whose pushforwards to $\overline{\mathcal{M}}_{g,n}$ are tautological.

Any such situation will yield an analogous collection of tautological relations. For example, consider the moduli space $\overline{\mathcal{M}}_{g,n}(BG, 0)$ for any finite group G . Fixing a character

$$\rho : G \rightarrow \mathbb{C}^*$$

yields an orbifold line bundle N_ρ on BG , which is topologically trivial and for which the isotropy group G acts on the fiber via ρ . If \mathcal{L}_ρ denotes the pullback of N_ρ to the universal curve over $\overline{\mathcal{M}}_{g,n}(BG, 0)$, then, for the same reason as above, $R^1\pi_*\mathcal{L}_\rho$ will be a vector bundle on any component $\overline{\mathcal{M}}_{g,\mathbf{a}}(BG, 0)$ on which at least one of the monodromies a_i is not in the kernel of ρ .

Thus, one can take M to be such a component of $\overline{\mathcal{M}}_{g,n}(BG, 0)$ and V to be the restriction of $R^1\pi_*\mathcal{L}_\rho$ to this component. By an application of the orbifold Grothendieck-Riemann-Roch theorem precisely analogous to Chiodo's, an expression for the Chern characters of $R^1\pi_*\mathcal{L}_\rho$ in terms of tautological classes can be obtained. In this way, any choice of G and ρ as above can be used to derive relations.

ACKNOWLEDGMENTS

The author is particularly indebted to Y. Ruan and F. Janda for numerous invaluable conversations and insights. The author would also like to thank A. Chiodo, R. Pandharipande, D. Petersen, A. Pixton, D. Ross, and D. Zvonkine for useful conversations and comments. A computer program by D. Johnson for calculating top intersections on $\overline{\mathcal{M}}_{g,n}$ was very helpful for studying cases of the main theorem.

REFERENCES

- [1] Dan Abramovich and Angelo Vistoli, *Compactifying the space of stable maps*, J. Amer. Math. Soc. **15** (2002), no. 1, 27–75 (electronic), DOI 10.1090/S0894-0347-01-00380-0. MR1862797
- [2] R. Cavalieri, *Hurwitz theory and the double ramification cycle*, arXiv: 1410.8550.
- [3] Alessandro Chiodo, *Stable twisted curves and their r -spin structures* (English, with English and French summaries), Ann. Inst. Fourier (Grenoble) **58** (2008), no. 5, 1635–1689. MR2445829
- [4] Alessandro Chiodo, *Towards an enumerative geometry of the moduli space of twisted curves and r th roots*, Compos. Math. **144** (2008), no. 6, 1461–1496, DOI 10.1112/S0010437X08003709. MR2474317
- [5] A. Chiodo and G. Farkas, *Singularities of the moduli space of level curves*, J. Eur. Math. Soc., to appear.
- [6] Alessandro Chiodo and Yongbin Ruan, *Landau-Ginzburg/Calabi-Yau correspondence for quintic three-folds via symplectic transformations*, Invent. Math. **182** (2010), no. 1, 117–165, DOI 10.1007/s00222-010-0260-0. MR2672282
- [7] A. Chiodo and D. Zvonkine, *Twisted r -spin potential and Givental’s quantization*, Adv. Theor. Math. Phys. **13** (2009), no. 5, 1335–1369. MR2672465
- [8] E. Clader and F. Janda, *Pixton’s double ramification cycle relations*, arXiv:1601.02871.
- [9] Tom Coates and Alexander Givental, *Quantum Riemann-Roch, Lefschetz and Serre*, Ann. of Math. (2) **165** (2007), no. 1, 15–53, DOI 10.4007/annals.2007.165.15. MR2276766
- [10] Carel Faber, *Algorithms for computing intersection numbers on moduli spaces of curves, with an application to the class of the locus of Jacobians*, New trends in algebraic geometry (Warwick, 1996), London Math. Soc. Lecture Note Ser., vol. 264, Cambridge Univ. Press, Cambridge, 1999, pp. 93–109, DOI 10.1017/CBO9780511721540.006. MR1714822
- [11] Carel Faber, *A conjectural description of the tautological ring of the moduli space of curves*, Moduli of curves and abelian varieties, Aspects Math., E33, Vieweg, Braunschweig, 1999, pp. 109–129. MR1722541
- [12] C. Faber and R. Pandharipande, *Relative maps and tautological classes*, J. Eur. Math. Soc. (JEMS) **7** (2005), no. 1, 13–49, DOI 10.4171/JEMS/20. MR2120989
- [13] Huijun Fan, Tyler Jarvis, and Yongbin Ruan, *The Witten equation, mirror symmetry, and quantum singularity theory*, Ann. of Math. (2) **178** (2013), no. 1, 1–106, DOI 10.4007/annals.2013.178.1.1. MR3043578
- [14] E. Getzler, *Topological recursion relations in genus 2*, Integrable systems and algebraic geometry (Kobe/Kyoto, 1997), World Sci. Publ., River Edge, NJ, 1998, pp. 73–106. MR1672112
- [15] E. Getzler, *Intersection theory on $\overline{\mathcal{M}}_{1,4}$ and elliptic Gromov-Witten invariants*, J. Amer. Math. Soc. **10** (1997), no. 4, 973–998, DOI 10.1090/S0894-0347-97-00246-4. MR1451505
- [16] Alexander B. Givental, *Gromov-Witten invariants and quantization of quadratic Hamiltonians* (English, with English and Russian summaries), Mosc. Math. J. **1** (2001), no. 4, 551–568, 645. Dedicated to the memory of I. G. Petrovskii on the occasion of his 100th anniversary. MR1901075
- [17] Alexander B. Givental, *Semisimple Frobenius structures at higher genus*, Internat. Math. Res. Notices **23** (2001), 1265–1286, DOI 10.1155/S1073792801000605. MR1866444
- [18] Eleny-Nicoleta Ionel, *Topological recursive relations in $H^{2g}(\mathcal{M}_{g,n})$* , Invent. Math. **148** (2002), no. 3, 627–658, DOI 10.1007/s002220100205. MR1908062
- [19] Eleny-Nicoleta Ionel, *Relations in the tautological ring of \mathcal{M}_g* , Duke Math. J. **129** (2005), no. 1, 157–186, DOI 10.1215/S0012-7094-04-12916-1. MR2155060
- [20] F. Janda, *Comparing tautological relations from the equivariant Gromov-Witten theory of projective spaces and spin structures*, arXiv: 1407.4778.

- [21] F. Janda, *Relations in the tautological ring and Frobenius manifolds near the discriminant*, arXiv: 1505.03419.
- [22] F. Janda, R. Pandharipande, A. Pixton, and D. Zvonkine, *Double ramification cycles on the moduli spaces of curves*, arXiv:1602.04705.
- [23] Paul Johnson, *Equivariant GW theory of stacky curves*, *Comm. Math. Phys.* **327** (2014), no. 2, 333–386, DOI 10.1007/s00220-014-2021-1. MR3183403
- [24] M. Kontsevich and Yu. Manin, *Gromov-Witten classes, quantum cohomology, and enumerative geometry*, *Comm. Math. Phys.* **164** (1994), no. 3, 525–562. MR1291244
- [25] Kefeng Liu and Hao Xu, *Computing top intersections in the tautological ring of \mathcal{M}_g* , *Math. Z.* **270** (2012), no. 3-4, 819–837, DOI 10.1007/s00209-010-0828-9. MR2892926
- [26] Xiaobo Liu and Rahul Pandharipande, *New topological recursion relations*, *J. Algebraic Geom.* **20** (2011), no. 3, 479–494, DOI 10.1090/S1056-3911-2010-00559-0. MR2786663
- [27] Eduard Looijenga, *On the tautological ring of \mathcal{M}_g* , *Invent. Math.* **121** (1995), no. 2, 411–419, DOI 10.1007/BF01884306. MR1346214
- [28] A. Marian, D. Oprea, R. Pandharipande, A. Pixton, and D. Zvonkine, *The Chern character of the Verlinde bundle over $M_{g,n}$* , arXiv: 1311.3028, 2014.
- [29] David Mumford, *Towards an enumerative geometry of the moduli space of curves*, *Arithmetic and geometry*, Vol. II, *Progr. Math.*, vol. 36, Birkhäuser Boston, Boston, MA, 1983, pp. 271–328. MR717614
- [30] R. Pandharipande and A. Pixton, *Relations in the tautological ring of the moduli space of curves*, arXiv: 1301.4561.
- [31] Rahul Pandharipande, Aaron Pixton, and Dimitri Zvonkine, *Relations on $\overline{\mathcal{M}}_{g,n}$ via 3-spin structures*, *J. Amer. Math. Soc.* **28** (2015), no. 1, 279–309, DOI 10.1090/S0894-0347-2014-00808-0. MR3264769
- [32] A. Pixton, *Conjectural relations in the tautological ring of $\overline{\mathcal{M}}_{g,n}$* , arXiv: 1207.1918.
- [33] Aaron Pixton, *The tautological ring of the moduli space of curves*, ProQuest LLC, Ann Arbor, MI, 2013. Thesis (Ph.D.)—Princeton University. MR3153424
- [34] Constantin Teleman, *The structure of 2D semi-simple field theories*, *Invent. Math.* **188** (2012), no. 3, 525–588, DOI 10.1007/s00222-011-0352-5. MR2917177
- [35] B. Toen, *Théorèmes de Riemann-Roch pour les champs de Deligne-Mumford* (French, with English and French summaries), *K-Theory* **18** (1999), no. 1, 33–76, DOI 10.1023/A:1007791200714. MR1710187
- [36] Hsian-Hua Tseng, *Orbifold quantum Riemann-Roch, Lefschetz and Serre*, *Geom. Topol.* **14** (2010), no. 1, 1–81, DOI 10.2140/gt.2010.14.1. MR2578300
- [37] Stephanie Yang, *Intersection numbers on $\overline{\mathcal{M}}_{g,n}$* , *J. Softw. Algebra Geom.* **2** (2010), 1–5, DOI 10.2140/jsag.2010.2.1. MR2881127

DEPARTMENT OF MATHEMATICS, SAN FRANCISCO STATE UNIVERSITY, THORNTON HALL 937,
1600 HOLLOWAY AVENUE, SAN FRANCISCO, CA 94132

E-mail address: eclader@sfsu.edu