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ON THE NECESSITY OF BUMP CONDITIONS FOR THE TWO-WEIGHTED MAXIMAL INEQUALITY

LENKA SLAVÍKOVÁ

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Abstract. We study the necessity of bump conditions for the boundedness of the Hardy-Littlewood maximal operator between weighted L^p spaces with different weights. The conditions in question are obtained by replacing the $L^{p'}$-average of $\sigma^{\frac{1}{p'}}$ in the Muckenhoupt A_p-condition by an average with respect to a stronger Banach function norm, and are known to be sufficient for the two-weighted maximal inequality. We show that these conditions are in general not necessary for such an inequality to be true.

1. Introduction and statement of the result

The Hardy–Littlewood maximal operator M is defined for every measurable function f on $\mathbb{R}^n$ by

$$Mf(x) = \sup_{Q:x\in Q} \frac{1}{|Q|}\int_Q |f|, \quad x \in \mathbb{R}^n,$$

where the supremum is taken over all cubes Q containing x. By a "cube" we always mean a compact cube with sides parallel to coordinate axes.

Assume that $1 < p < \infty$. A longstanding open problem in harmonic analysis is to characterize those couples (w, σ) of nonnegative locally integrable functions, called weights in the sequel, which satisfy the inequality

$$\int_{\mathbb{R}^n} w(M(f\sigma))^p \le C \int_{\mathbb{R}^n} \sigma |f|^p \tag{1.1}$$

for all measurable functions f and some positive constant C.

In the special case when $\sigma = w^{1-p'}$, where $p' = \frac{p}{p-1}$, inequality (1.1) was characterized by Muckenhoupt [14]. He showed that the correct necessary and sufficient condition is the A_p-condition

$$\sup_Q \left(\frac{1}{|Q|}\int_Q w\right)\left(\frac{1}{|Q|}\int_Q \sigma\right)^{p-1} < \infty. \tag{1.2}$$

We note that throughout this paper, the notation $\sup_Q$ means that the supremum is taken over all cubes Q in $\mathbb{R}^n$.

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The situation is much more complicated in the two-weighted case, when we do not assume any relationship between w and σ. It is well known that the A_p-condition (1.2) is still necessary for (1.1) in this setting, but it is not sufficient anymore (see, e.g., [9, Chapter 4, Example 1.15]). A solution to the two-weighted problem was given by Sawyer [23], who showed that (1.1) holds if and only if there is a positive constant C such that

$$\int_Q w(M(\chi_Q\sigma))^p \leq C\int_Q \sigma \tag{1.3}$$

for every cube Q. This characterizing condition, however, still involves the operator M itself, and hence does not give a quite satisfactory answer to the above-mentioned problem.

Another approach to the two-weighted problem (1.1) consists in finding sufficient conditions for (1.1) that are close in form to the A_p-condition (1.2). These conditions are called "bump conditions" in the literature. They are more explicit than (1.3), and thus more appropriate for the use in applications. On the other hand, as we will show in the present paper, these conditions are not necessary for (1.1) - at least not in their currently available form.

To introduce the bump theory, let us first observe that the A_p-condition (1.2) can be written in the form

$$\sup_Q \|w^{\frac{1}{p}}\|_{L^p,Q}\|\sigma^{\frac{1}{p'}}\|_{L^{p'},Q} < \infty, \tag{1.4}$$

where, for any $q \in (1,\infty)$ and any cube Q, $\|\cdot\|_{L^q,Q}$ denotes the L^q-norm on Q with respect to the normalized Lebesgue measure $dx/|Q|$.

Neugebauer [17] showed that if the norms in (1.4) are replaced by stronger Lebesgue norms, namely, if

$$\sup_Q \|w^{\frac{1}{p}}\|_{L^{pr},Q}\|\sigma^{\frac{1}{p'}}\|_{L^{p'r},Q} < \infty \tag{1.5}$$

holds for some $r > 1$, then the two-weighted maximal inequality (1.1) is fulfilled.

Pérez [20] found a way how to weaken the sufficient condition (1.5). He noticed that in order to obtain (1.1) one just needs to "bump" in a suitable way the $L^{p'}$-norm of $\sigma^{\frac{1}{p'}}$ in (1.4). He also showed that more general norms than just those of Lebesgue can be used in this connection. For instance, if L^B denotes the Orlicz space induced by the Young function B and $\|\cdot\|_{L^B,Q}$ stands for the normalized Orlicz norm on a cube Q, then the bump condition

$$\sup_Q \|w^{\frac{1}{p}}\|_{L^p,Q}\|\sigma^{\frac{1}{p'}}\|_{L^B,Q} < \infty \tag{1.6}$$

was proved in [20] to be sufficient for (1.1) provided that the complementary Young function $\overline{B} \in B_p$, that is, $\overline{B}$ satisfies the B_p-condition

$$\int_1^\infty \frac{\overline{B}(t)}{t^p}\frac{dt}{t} < \infty. \tag{1.7}$$

(See Section 2 for definitions regarding Orlicz spaces.) We point out that condition (1.7) is sharp in the sense that whenever L^B is an Orlicz space such that (1.6) implies (1.1) for every couple (w,σ), then (1.7) has to be fulfilled.

A basic example of an Orlicz space for which this result can be applied is the Lebesgue space L^q with $q > p'$. The strength of the result lies, however, in Orlicz

spaces that are "closer to $L^{p'}$", such as, for instance, the space $L^{p'}(\log L)^\gamma$ with $\gamma > p' - 1$.

The last result can be further improved if yet more general spaces of measurable functions, the so-called "Banach function spaces" (see Section 2 for the definition), are brought into play. For any Banach function space X, we define the normalized X-norm on a cube Q by

$$\|f\|_{X,Q} = \|\tau_{\ell(Q)} f \chi_Q\|_X,$$

where τ_δ denotes, for $\delta > 0$, the dilation operator $\tau_\delta f(x) = f(\delta x)$, and $\ell(Q)$ stands for the sidelength of the cube Q. The maximal operator M_X is then given by

$$M_X f(x) = \sup_{Q: x \in Q} \|f\|_{X,Q}, \quad x \in \mathbb{R}^n.$$

Notice that if $X = L^1$, then M_X coincides with the classical Hardy-Littlewood maximal operator M.

A sufficient condition for (1.1), proved in [20] again, has the form

$$\sup_Q \|w^{\frac{1}{p}}\|_{L^p,Q} \|\sigma^{\frac{1}{p'}}\|_{X,Q} < \infty, \tag{1.8}$$

where X is any Banach function space whose associate space X' fulfills

$$\int_{\mathbb{R}^n} (M_{X'} f)^p \le C \int_{\mathbb{R}^n} |f|^p \tag{1.9}$$

for all measurable functions f and some positive constant C. Condition (1.9) can be reduced to the B_p-condition if X is an Orlicz space.

Condition (1.9) can be weakened if we allow it to depend on σ. Namely, the following implication holds: if X is a Banach function space such that (1.8) is fulfilled and there is a positive constant C for which

$$\int_Q (M_{X'}(\sigma^{\frac{1}{p}} \chi_Q))^p \le C \int_Q \sigma \tag{1.10}$$

for every cube Q, then (1.1) holds. This was proved by Pérez and Rela [21] as a consequence of the Sawyer characterization of the two-weighted maximal inequality. We note that the result in [21] is restricted only to Orlicz spaces, however, it is easy to observe that the proof given there works equally well for an arbitrary Banach function space over $\mathbb{R}^n$. Moreover, the paper [21] gives even a quantitative version of this result which is shown to hold, at least for Orlicz spaces, not only in the Euclidean setting, but also in the more general context of spaces of homogeneous type.

It is worth noticing that condition (1.10) is in many situations considerably weaker than (1.9). For instance, one can easily observe that (1.9) is not valid when $X' = L^p$, while (1.10) holds with $X' = L^p$ if and only if

$$\int_Q (M_{L^p}(\sigma^{\frac{1}{p}} \chi_Q))^p = \int_Q M(\sigma \chi_Q) \le C \int_Q \sigma \tag{1.11}$$

for all cubes Q. It was shown by Fujii [8] and rediscovered later by Wilson [24] that the validity of condition (1.11) is equivalent to the fact that σ is an A_∞-weight, that is, a weight which satisfies the one-weighted A_p-condition for some $p > 1$.

Let us mention that the bump theory is an active area of research not only in connection with the two-weighted inequality for the Hardy-Littlewood maximal operator, but also in connection with a similar inequality for other operators.

A power bump condition for fractional integrals appeared in [22] and its extension into the setting of Banach function spaces was found in [18]. A very popular line of research is nowadays the study of bump conditions for singular integral operators. An early contribution to the investigation of this topic constitutes, e.g., the paper [19]. More recently, it has been shown that the bump condition

$$\sup_Q \|w^{\frac{1}{p}}\|_{L^A,Q}\|\sigma^{\frac{1}{p'}}\|_{L^B,Q} < \infty \tag{1.12}$$

is sufficient for the two-weighted inequality for singular integral operators provided that $\overline{A} \in B_{p'}$ and $\overline{B} \in B_p$. The proof in full generality was found by Lerner [13] and independently by Nazarov, Reznikov, Treil and Volberg [15] (for $p = 2$), completing the series of several partial results [3–6, 12]. It was conjectured that the weaker condition

$$\sup_Q \|w^{\frac{1}{p}}\|_{L^p,Q}\|\sigma^{\frac{1}{p'}}\|_{L^B,Q} < \infty \quad \& \quad \sup_Q \|w^{\frac{1}{p}}\|_{L^A,Q}\|\sigma^{\frac{1}{p'}}\|_{L^{p'},Q} < \infty$$

with $\overline{A} \in B_{p'}$ and $\overline{B} \in B_p$ might be sufficient as well, however, only partial results have been proved so far - see, e.g., [1, 7, 10, 11, 16].

The principal question we shall discuss in this paper is the necessity of bump conditions for the two-weighted maximal inequality. As we have seen, several versions of bump conditions are now available in the literature. We shall focus on the one due to Pérez and Rela [21], which has been the weakest so far.

Question 1.1. Given a couple (w, σ) of weights satisfying (1.1), is it true that there is a Banach function space X fulfilling (1.8) and (1.10)?

We notice that the answer to this question is positive whenever σ is an A_∞-weight. Indeed, in this situation it suffices to take $X = L^{p'}$. We already know that (1.10) is then fulfilled (see (1.11)). Further, condition (1.8) is in this case just the standard A_p-condition, which is well known to be necessary for (1.1). In fact, according to the reverse Hölder inequality (see, e.g., [9, Chapter 4, Lemma 2.5]), condition (1.1) implies even (1.8) with $X = L^{p'+\varepsilon}$ for some $\varepsilon > 0$, depending on σ. Since the space $X = L^{p'+\varepsilon}$ satisfies not only (1.10), but also the stronger condition (1.9) (or, equivalently, condition (1.7) with $B(t) = t^{p'+\varepsilon}$), one can obtain even a better conclusion in this case.

The interesting problem is whether a similar result holds without the A_∞-assumption. We show that this is not the case in general.

Given $x \in \mathbb{R}^n$, we shall denote by $|x|_{\max}$ the maximum norm of x, that is, if $x = (x_1, \dots, x_n)$, then $|x|_{\max} = \max_{i=1,\dots,n} |x_i|$. We shall also use the notation $\log_+ x = \max\{\log x, 0\}$, $x > 0$.

Theorem 1.2. *Let $1 < p < \infty$, and let*

$$w(x) = \frac{|x|_{\max}^{n(p-1)}}{(1 + \log_+ |x|_{\max}^n)^p},$$
$$\sigma(x) = \frac{1}{|x|_{\max}^n (1 + \log_+ \frac{1}{|x|_{\max}^n})^{p'}}, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

Then the couple (w, σ) fulfills (1.1), but there is no Banach function space X for which (1.8) and (1.10) hold simultaneously.

Remark 1.3. Assume that $\alpha \in (0,n)$ and $\beta \in \mathbb{R}$, and set

$$\sigma(x) = \frac{1}{|x|_{\max}^{\alpha}(1+\log_+ \frac{1}{|x|_{\max}^n})^{\beta}}, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

Then the answer to Question 1.1 is positive, regardless of what w is. This follows from the fact that σ is an A_∞-weight, combined with our previous observations.

2. Preliminaries

In this section we collect necessary prerequisities from the theory of Banach function spaces. An interested reader can find more details in [2].

Let $n \in \mathbb{N}$. We denote by $\mathcal{M}$ the set of all Lebesgue measurable functions on $\mathbb{R}^n$ having their values in $[-\infty,\infty]$. If F is a measurable subset of $\mathbb{R}^n$, then $|F|$ denotes the Lebesgue measure of F.

We say that a functional $\|\cdot\|_X : \mathcal{M} \to [0,\infty]$ is a *Banach function norm* if, for all functions f, $g \in \mathcal{M}$, for all sequences $(f_k)_{k=1}^{\infty}$ in $\mathcal{M}$ and for all constants $a \in \mathbb{R}$, the following properties hold:

(P1) $\|f\|_X = 0$ if and only if $f = 0$ a.e.; $\|af\|_X = |a|\|f\|_X$; $\|f+g\|_X \le \|f\|_X + \|g\|_X$;
(P2) $|f| \le |g|$ a.e. implies $\|f\|_X \le \|g\|_X$;
(P3) $|f_k| \nearrow |f|$ a.e. implies $\|f_k\|_X \nearrow \|f\|_X$;
(P4) if $F \subseteq \mathbb{R}^n$ with $|F| < \infty$, then $\|\chi_F\|_X < \infty$;
(P5) if $F \subseteq \mathbb{R}^n$ with $|F| < \infty$, then $\int_F |f(x)|\,dx \le C_F\|f\|_X$ for some constant C_F depending on F but independent of f.

The collection of all $f \in \mathcal{M}$ for which $\|f\|_X < \infty$ is denoted by X and is called a *Banach function space*.

To every Banach function norm $\|\cdot\|_X$ there corresponds another functional on $\mathcal{M}$, denoted by $\|\cdot\|_{X'}$ and defined, for $g \in \mathcal{M}$, by

$$\|g\|_{X'} = \sup_{\|f\|_X \le 1} \int_{\mathbb{R}^n} |f(x)g(x)|\,dx. \tag{2.1}$$

It turns out that $\|\cdot\|_{X'}$ is also a Banach function norm, we call it the *associate norm* of $\|\cdot\|_X$. The Banach function space X' built upon the Banach function norm $\|\cdot\|_{X'}$ is called the *associate space* of X. It is known (see, e.g., [2, Chapter 1, Theorem 2.7]) that $(X')' = X$.

Let us now mention particular examples of Banach function spaces. The basic examples are the *Lebesgue spaces* L^p, given by

$$\|f\|_{L^p} = \begin{cases} \left(\int_{\mathbb{R}^n} |f(y)|^p\,dy\right)^{\frac{1}{p}}, & 1 \le p < \infty; \\ \operatorname{esssup}_{y\in\mathbb{R}^n} |f(y)|, & p = \infty, \quad f \in \mathcal{M}. \end{cases}$$

A generalization of Lebesgue spaces is provided by the notion of *Orlicz spaces*. Given a *Young function* B, namely, a nonnegative continuous increasing convex function on $[0,\infty)$ such that $\lim_{t\to 0_+} \frac{B(t)}{t} = 0$ and $\lim_{t\to\infty} \frac{B(t)}{t} = \infty$, the Orlicz norm $\|\cdot\|_{L^B}$ is given by

$$\|f\|_{L^B} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} B\left(\frac{|f(y)|}{\lambda}\right) dy \le 1\right\}, \quad f \in \mathcal{M}. \tag{2.2}$$

It can be shown that $\|\cdot\|_{L^B}$ is indeed a Banach function norm and, for any cube Q, the normalized Orlicz norm on Q can be expressed in the form

$$\|f\|_{L^B,Q} = \inf\left\{\lambda > 0 : \frac{1}{|Q|}\int_Q B\left(\frac{|f(y)|}{\lambda}\right)\,dy \le 1\right\}, \quad f \in \mathcal{M}.$$

The associate norm to $\|\cdot\|_{L^B}$ is equivalent to another Orlicz norm induced by the *complementary Young function* $\overline{B}$ defined by

$$\overline{B(t)} = \sup_{s\ge 0}(st - B(s)), \quad t \in [0,\infty).$$

For any $p \in (1,\infty)$, the particular choice of $B(t) = t^p$ in (2.2) yields the Lebesgue space L^p. We note that $(L^p)' = L^{p'}$, where we employ the usual notation $p' = \frac{p}{p-1}$. The Orlicz space induced by the Young function B equivalent to $t^p \log^\gamma(e+t)$ for $p \in (1,\infty)$ and $\gamma \in \mathbb{R}$ is denoted by $L^p(\log L)^\gamma$ and one has $(L^p(\log L)^\gamma)' = L^{p'}(\log L)^{-\gamma}$.

3. Proof of Theorem 1.2

We devote this section to the proof of Theorem 1.2. Throughout the proof, we shall denote

$$Q_r = \{x \in \mathbb{R}^n : |x|_{\max} \le r\}, \quad r > 0;$$

in other words, Q_r will stand for the cube centered at 0 and with sidelength $2r$. We shall write "$\approx$" in order to express that the two sides of an equation are equivalent up to multiplicative constants independent of appropriate quantities.

Proof of Theorem 1.2. We first prove that

$$M\sigma(x) \approx \frac{1 + \left|\log \frac{1}{|x|^n_{\max}}\right|}{|x|^n_{\max}(1+\log_+ \frac{1}{|x|^n_{\max}})^{p'}}, \quad x \in \mathbb{R}^n \setminus \{0\}, \tag{3.1}$$

up to multiplicative constants depending on p and n.

Consider the function

$$f(t) = \frac{1}{t(1+\log_+ \frac{1}{t})^{p'}}, \quad t > 0.$$

Since $\lim_{t\to 0_+} f(t) = \infty$ and f is nonincreasing on some neighbourhood of 0, we can find $a \in (0,1)$ such that $f(a) \ge 1$ and f is nonincreasing on $(0,a)$. Let us set

$$g(t) = \begin{cases} \frac{1}{t(1+\log\frac{1}{t})^{p'}}, & t \in (0,a), \\ 1, & t \in [a,1], \\ \frac{1}{t}, & t \in (1,\infty). \end{cases}$$

Then g is nonincreasing on $(0,\infty)$ and $f \approx g$ on $(0,\infty)$, since $f(t) = g(t)$ unless $t \in [a,1]$, and $c_1 \le f(t) \le c_2$ for some $c_1 > 0$, $c_2 > 0$ and every $t \in [a,1]$. Therefore,

$$\sigma(x) = f(|x|^n_{\max}) \approx g(|x|^n_{\max}) =: h(x), \quad x \in \mathbb{R}^n \setminus \{0\},$$

and, by the coarea formula,

$$
\begin{aligned}
M\sigma(x) \approx Mh(x) &= \frac{1}{|Q_{|x|_{\max}}|}\int_{Q_{|x|_{\max}}} h(y)\,dy \\
&= \frac{1}{2^n|x|_{\max}^n}\int_0^{|x|_{\max}}\int_{\{y\in\mathbb{R}^n:|y|_{\max}=r\}} h(y)\,d\mathcal{H}^{n-1}(y)\,dr \\
&\approx \frac{1}{|x|_{\max}^n}\int_0^{|x|_{\max}} g(r^n)r^{n-1}\,dr \approx \frac{1}{|x|_{\max}^n}\int_0^{|x|_{\max}^n} g(s)\,ds \\
&\approx \frac{1}{|x|_{\max}^n}\int_0^{|x|_{\max}^n} f(s)\,ds,
\end{aligned}
\tag{3.2}
$$

where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure.

Given $t\in(0,1]$, we have

$$
\int_0^t f(s)\,ds = \int_0^t \frac{ds}{s(1+\log\frac{1}{s})^{p'}} \approx \frac{1}{(1+\log\frac{1}{t})^{p'-1}} = \frac{1+|\log\frac{1}{t}|}{(1+\log_+\frac{1}{t})^{p'}}. \tag{3.3}
$$

Also, if $t\in(1,\infty)$, then

$$
\begin{aligned}
\int_0^t f(s)\,ds &= \int_0^1 f(s)\,ds + \int_1^t f(s)\,ds = C + \int_1^t \frac{ds}{s} \\
&= C + \log t \approx 1 + \log t = \frac{1+|\log\frac{1}{t}|}{(1+\log_+\frac{1}{t})^{p'}}.
\end{aligned}
\tag{3.4}
$$

A combination of (3.2), (3.3) and (3.4) yields (3.1).

Using (3.1), we obtain

$$
\begin{aligned}
(M\sigma)^p(x)w(x) &\approx \frac{\left(1+\left|\log\frac{1}{|x|_{\max}^n}\right|\right)^p}{|x|_{\max}^n(1+\log_+\frac{1}{|x|_{\max}^n})^{p'p}(1+\log_+|x|_{\max}^n)^p} \\
&= \begin{cases} \frac{1}{|x|_{\max}^n(1+\log\frac{1}{|x|_{\max}^n})^{p'}}, & |x|_{\max}\le 1; \\ \frac{1}{|x|_{\max}^n}, & |x|_{\max}>1 \end{cases} \\
&= \frac{1}{|x|_{\max}^n(1+\log_+\frac{1}{|x|_{\max}^n})^{p'}} = \sigma(x), \quad x\in\mathbb{R}^n\setminus\{0\}.
\end{aligned}
$$

Hence, for any cube Q,

$$
\int_Q (M(\chi_Q\sigma))^p(x)w(x)\,dx \le \int_Q (M\sigma)^p(x)w(x)\,dx \approx \int_Q \sigma(x)\,dx,
$$

and Sawyer's characterization (1.3) of the two-weighted maximal inequality yields that the couple (w,σ) satisfies (1.1).

Let X be any Banach function space. Given $b \in (0,1)$, we have

$$\begin{aligned}
\left\|\sigma^{\frac{1}{p}}\right\|_{X',Q_b} &= \left\|\left(\sigma^{\frac{1}{p}}\chi_{Q_b}\right)(2by)\right\|_{X'} \\
&= \frac{1}{(2b)^{\frac{n}{p}}}\left\|\frac{\chi_{Q_{\frac{1}{2}}}(y)}{|y|_{\max}^{\frac{n}{p}}\left(1+\log_+\frac{1}{(2b|y|_{\max})^n}\right)^{\frac{p'}{p}}}\right\|_{X'} \\
&\geq \frac{1}{(2b)^{\frac{n}{p}}}\left\|\frac{\chi_{Q_{\frac{1}{2}}\setminus Q_{\frac{b}{2}}}(y)}{|y|_{\max}^{\frac{n}{p}}\left(1+\log_+\frac{1}{b^{2n}}\right)^{\frac{p'}{p}}}\right\|_{X'} \\
&= \frac{1}{(2b)^{\frac{n}{p}}\left(1+2\log_+\frac{1}{b^n}\right)^{\frac{p'}{p}}}\left\|\frac{\chi_{Q_{\frac{1}{2}}\setminus Q_{\frac{b}{2}}}(y)}{|y|_{\max}^{\frac{n}{p}}}\right\|_{X'} \\
&\geq \frac{1}{2^{\frac{n+p'}{p}}b^{\frac{n}{p}}(1+\log_+\frac{1}{b^n})^{\frac{p'}{p}}}\left\|\frac{\chi_{Q_{\frac{1}{2}}\setminus Q_{\frac{b}{2}}}(y)}{|y|_{\max}^{\frac{n}{p}}}\right\|_{X'}.
\end{aligned}$$

Thus, for any $a \in (0,1)$,

$$\begin{aligned}
\int_{Q_a}\left(M_{X'}(\sigma^{\frac{1}{p}}\chi_{Q_a})\right)^p(x)\,dx &\geq \int_{Q_a}\left\|\sigma^{\frac{1}{p}}\right\|^p_{X',Q_{|x|_{\max}}}dx \\
&\geq \int_{Q_a}\frac{1}{2^{p'+n}|x|_{\max}^n(1+\log_+\frac{1}{|x|_{\max}^n})^{p'}}\left\|\frac{\chi_{Q_{\frac{1}{2}}\setminus Q_{\frac{|x|_{\max}}{2}}}(y)}{|y|_{\max}^{\frac{n}{p}}}\right\|^p_{X'}dx \\
&\geq \frac{1}{2^{p'+n}}\left\|\frac{\chi_{Q_{\frac{1}{2}}\setminus Q_{\frac{a}{2}}}(y)}{|y|_{\max}^{\frac{n}{p}}}\right\|^p_{X'}\int_{Q_a}\frac{dx}{|x|_{\max}^n(1+\log_+\frac{1}{|x|_{\max}^n})^{p'}} \\
&= \frac{1}{2^{p'+n}}\left\|\frac{\chi_{Q_{\frac{1}{2}}\setminus Q_{\frac{a}{2}}}(y)}{|y|_{\max}^{\frac{n}{p}}}\right\|^p_{X'}\int_{Q_a}\sigma(x)\,dx.
\end{aligned} \tag{3.5}$$

Assume that X fulfills (1.10). Then there is a constant $C > 0$, independent of $a \in (0,1)$, such that

$$\int_{Q_a}\left(M_{X'}(\sigma^{\frac{1}{p}}\chi_{Q_a})\right)^p(x)\,dx \leq C\int_{Q_a}\sigma(x)\,dx. \tag{3.6}$$

Since $\int_{Q_a}\sigma(x)\,dx$ is positive and finite, a combination of (3.5) and (3.6) yields that

$$\left\|\frac{\chi_{Q_{\frac{1}{2}}\setminus Q_{\frac{a}{2}}}(y)}{|y|_{\max}^{\frac{n}{p}}}\right\|_{X'} \leq 2^{\frac{p'+n}{p}}C^{\frac{1}{p}} =: D.$$

Passing to limit when a tends to 0 and using the property (P3) of $\|\cdot\|_X$, we obtain

$$\left\|\frac{\chi_{Q_{\frac{1}{2}}}(y)}{|y|_{\max}^{\frac{n}{p}}}\right\|_{X'} \leq D. \tag{3.7}$$

To get a contradiction, assume that condition (1.8) is satisfied as well. Since

$$\left(\int_{Q_{\frac{1}{2}}} w(x)\,dx\right)^{\frac{1}{p}} \|\sigma^{\frac{1}{p'}}\|_{X,Q_{\frac{1}{2}}} = \|w^{\frac{1}{p}}\|_{L^p,Q_{\frac{1}{2}}} \|\sigma^{\frac{1}{p'}}\|_{X,Q_{\frac{1}{2}}}$$
$$\leq \sup_{Q} \|w^{\frac{1}{p}}\|_{L^p,Q} \|\sigma^{\frac{1}{p'}}\|_{X,Q} < \infty$$

and $\int_{Q_{\frac{1}{2}}} w(x)\,dx$ is clearly positive, we deduce that $\|\sigma^{\frac{1}{p'}}\|_{X,Q_{\frac{1}{2}}} < \infty$. However, by (3.7) and by the identity $X = (X')'$, we have

$$\begin{aligned}
\|\sigma^{\frac{1}{p'}}\|_{X,Q_{\frac{1}{2}}} &= \sup_{\|u\|_{X'}\leq 1} \int_{Q_{\frac{1}{2}}} \sigma^{\frac{1}{p'}}(x)|u(x)|\,dx \\
&\geq \frac{1}{D}\int_{Q_{\frac{1}{2}}} \frac{\sigma^{\frac{1}{p'}}(x)}{|x|_{\max}^{\frac{n}{p}}}\,dx \\
&= \frac{1}{D}\int_{Q_{\frac{1}{2}}} \frac{dx}{|x|_{\max}^{n}(1+\log_+ \frac{1}{|x|_{\max}^{n}})} = \infty,
\end{aligned}$$

a contradiction. Thus, conditions (1.8) and (1.10) cannot be fulfilled simultaneously. The proof is complete. □

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Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Praha 8, Czech Republic

Current address: Department of Mathematics, University of Missouri, Columbia, MO, 65211, USA

E-mail address: slavikoval@missouri.edu