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ON THE GLOBAL ATTRACTIVITY AND ASYMPTOTIC STABILITY FOR AUTONOMOUS SYSTEMS OF DIFFERENTIAL EQUATIONS ON THE PLANE

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Dedicated to the memory of Czesław Olech

ABSTRACT. The autonomous system of differential equations

$$x' = f(x), \qquad (x = (x_1, x_2)^T \in \mathbb{R}^2,\ f(x) = (f_1(x), f_2(x))^T),$$

is considered, and sufficient conditions are given for the global attractivity of the unique equilibrium $x = 0$. This property means that all solutions tend to the origin as $t \to \infty$. The two cases (a) $\operatorname{div} f(x) \le 0$ ($x \in \mathbb{R}^2$) and (b) $\operatorname{div} f(x) \ge 0$ ($x \in \mathbb{R}^2$) are treated, where $\operatorname{div} f(x) := \partial f_1(x)/\partial x_1 + \partial f_2(x)/\partial x_2$. Earlier results of N. N. Krasovskiĭ and C. Olech about case (a) are improved and generalized to case (b). Three types of assumptions are required: certain stability properties of the origin (local attractivity, stability), boundedness above in some sense for $\operatorname{div} f(x)$, and assumptions that $|f(x)|$ is not as small as $|x| \to \infty$. The conditions of the second and third types are connected with each other.

1. Introduction

Consider the system of differential equations

$$x' = f(x), \qquad (x = (x_1, x_2)^T \in \mathbb{R}^2), \tag{1.1}$$

where $f : \mathbb{R}^2 \to \mathbb{R}^2$ is of class C^1, and $f(0) = 0$ (the origin of $x = 0$ is a rest point). We will use the standard notation $|x| := (x_1^2 + x_2^2)^{1/2}$ and

$$J(x) := \frac{\partial f(x)}{\partial x} \text{ (Jacobi matrix)}, \quad \operatorname{div} f(x) := \frac{\partial f_1}{\partial x_1}(x) + \frac{\partial f_2}{\partial x_2}(x).$$

If $P \in \mathbb{R}^2$, then $x(\cdot, P) : [0, \infty) \to \mathbb{R}^2$ denotes the solution of (1.1) satisfying the initial condition $x(0, P) = P$. The *positive trajectory* $\gamma^+(P)$ and the *positive limit set* $\Lambda^+(P)$ of P are defined by

$$\gamma^+(P) := \{x(t, P) : 0 \le t < \infty\},$$
$$\Lambda^+(P) := \{x \in \mathbb{R}^2 : \exists \{t_n\},\ \lim_{n\to\infty} t_n = \infty,\ \lim_{n\to\infty} x(t_n, P) = x\};$$

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the negative trajectory $\gamma^-(P)$ and the negative limit set $\Lambda^-(P)$ are defined analogously. The orbit $o(P)$ through P is $\gamma^+(P) \cup \gamma^-(P)$.

Let us review the necessary stability concepts [18]:

- the origin is *stable* if for every $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that $|P| < \delta$, $t \geq 0$ imply $|x(t, P)| < \varepsilon$;
- the origin *attracts* $Q \in \mathbb{R}^2$ if $\lim_{t\to\infty} |x(t, Q)| = 0$;
- the *basin* of the origin is the set of the points attracted by the origin;
- the origin is an *attractor* if its basin contains one of its neighborhoods;
- the origin is *asymptotically stable* if it is stable and it is an attractor;
- the origin is a *global attractor* (GA) if its basin is equal to $\mathbb{R}^2$;
- the origin is *globally asymptotically stable* if it is stable and it is a global attractor.

We will use three groups of assumptions. The assumptions (A1) and (A2) express stability properties of the origin:

(A1) the origin is an attractor;
(A2) the origin is stable.

Assumptions (B1)-(B4) require that $\mathrm{div} f(x)$ is bounded above in some sense:

(B1) there is a $\beta > 0$ such that $\mathrm{div} f(x) \leq -\beta$ for all $x \in \mathbb{R}^2$ (Krasovskiĭ [12]);
(B2) there is a continuous strictly increasing function $a : [0, \infty) \to [0, \infty)$ with $a(0) = 0$ such that

$$\mathrm{div} f(x) \leq -a(|x|), \quad (x \in \mathbb{R}^2);$$

(B3) $\mathrm{div} f(x) \leq 0$ $(x \in \mathbb{R}^2)$ (Olech [15]);
(B4) $\mathrm{div} f(x) \geq 0$ $(x \in \mathbb{R}^2)$, and

$$\int_{|x|\leq r} \mathrm{div} f(x)\,\mathrm{d}x = o(r^{\alpha+1}), \quad (r \to \infty),$$

with some $\alpha \geq 0$.

Assumptions (C2)-(C5) say that $|f(x)|$ is not small; (C1) is very natural being necessary for GA:

(C1) $x = 0$ is the unique rest point of (1.1) in $\mathbb{R}^2$;
(C2) there is an $l > 0$ such that $|f(x)| \geq l|x|$ $(x \in \mathbb{R}^2)$ (Krasovskiĭ [12]);
(C3) there are $\kappa > 0$ and $\varepsilon > 0$ such that $|f(x)| \geq \varepsilon$ $(|x| \geq \kappa)$ (Olech [15]);
(C4) $\int_0^\infty \left[\min_{|x|=r} |f(x)|\right] \mathrm{d}r = \infty$ (Hartman-Olech [8]);
(C5) there are $\kappa > 0$ and $\varepsilon > 0$ such that $|f(x)| \geq \varepsilon |x|^\alpha$ $(|x| \geq \kappa)$ with α in assumption (B4).

Krasovskiĭ [12] proved that (A1), (B1), and (C2) imply GA. In 1960 Markus and Yamabe [13] supposed that (i) $\mathrm{div} f(x) < 0$, (ii) $\det J(x) > 0$ (i.e., the eigenvalues of $J(x)$ have negative real parts for all $x \in \mathbb{R}^2$), and conjectured GA. (They proved their conjecture in the case when either one of the partial derivatives $\partial f_i/\partial x_k$ $(i, k = 1, 2)$ vanishes identically on $\mathbb{R}^2$.) Many papers [1,3,5,7,14] and monographs [2, 9] have dealt with the Markus-Yamabe conjecture. Relaxing (i) to (B3) and dropping (ii), Olech [15, Theorem 5] returned to Krasovskiĭ's approach and proved GA under the additional conditions (A1) and (C3). Hartman and Olech [8] relaxed condition (C3) to (C4) and extended this result to $\mathbb{R}^n$ in the special case when $J(x)$ is symmetric. The original conjecture was solved affirmatively in the case $n = 2$ by Meisters and Olech [14] for polynomial vector fields, and later on independently by

Fessler [5] and Gutiérrez [7] for general vector fields. Finally, Cima et al. [3] gave polynomial counterexamples for each $n \geq 3$.

In this paper we follow Olech's direction: we omit (ii) from the Markus-Yamabe conjecture and search for substituting conditions on f such that these conditions and (i) together guarantee GA. We develop Krasovskiĭ's and Olech's technique further making it "global" and combining it with the phase volume method [10] (see Lemma 2.1 in the present paper). We show that condition (C4) can replace (C3) also in the general asymmetric case. Meanwhile, thanks to our new technique, the proof becomes essentially simpler (Theorem 3.1). Then we extend our method to the case of $\operatorname{div} f(x) \geq 0$ ($x \in \mathbb{R}^2$) (Theorem 3.2). Finally, we show that in Olech's theorem the attractivity condition (A1) can be replaced with (A2), i.e., with the stability of the origin, that results in a sufficient condition for the global asymptotic stability of the origin (Theorem 3.3).

The famous Jacobian Conjecture [11] is the following: if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a polynomial map with a constant non-zero Jacobian determinant, then f is a polynomial automorphism (i.e., there exists f^{-1} and f^{-1} is also a polynomial map). This conjecture is still open. It is worth noticing that the Jacobian Conjecture and the Markus-Yamabe conjecture are related [4].

2. Lemmas

For a solution $x(\cdot, x_0)$ of (1.1) consider the mapping

$$x_0 \mapsto x(t, x_0), \quad (x_0 \in \mathbb{R}^2,\ t \in \mathbb{R}).$$

The first lemma describes the change of the area during this mapping. The proof can be an exercise in a graduate course on ODE theory, so it is omitted.

Lemma 2.1. *For any measurable set $F_0 \subset \mathbb{R}^2$ we have*

$$\mu(x(t, F_0)) = \int_{F_0} \exp\left[\int_0^t \operatorname{div} f(x(s, x_0))\, \mathrm{d}s\right] \mathrm{d}x_0, \tag{2.1}$$

where μ denotes the Lebesgue measure.

The following lemma characterizes the positive limit set of a trajectory. It is a direct consequence of the Poincaré-Bendixson Theorem [2, 9] and Lemma 2.1.

Lemma 2.2. *Suppose (C1). If $\overline{\gamma^+(P)}$ is compact, then the positive limit set $\Lambda^+(P)$ is either*

- **(a)** *the rest point 0,*
- **(b)** *a homoclinic orbit to the origin, i.e., an orbit whose positive and negative limit sets are the origin, or*
- **(c)** *a closed periodic orbit (a cycle) around the origin.*

If $\operatorname{div} f(x)$ does not change sign in $\mathbb{R}^2$, then case (c) is possible only if $\operatorname{div} f(x)$ vanishes identically inside the cycle.

The last lemma excludes the compactness of $\overline{\gamma^+(P)}$ for some points P of the boundary of the basin. In what follows we will use the notation $\operatorname{int} H$, $\operatorname{ext} H$, and $\operatorname{bd} H$ for the interior, exterior, and boundary of any set $H \subset \mathbb{R}^2$.

Lemma 2.3. *Suppose that (A1), (C1) are satisfied and $\operatorname{div} f(x)$ is of the same sign on the whole plane. If Ω denotes the basin of the origin, then for every $P \in \operatorname{bd} \Omega$*

we have

$$\limsup_{t\to\infty} |x(t,P)| = \infty. \tag{2.2}$$

Proof. Let $P \in \operatorname{bd}\Omega$ be arbitrary and suppose that (2.2) is not true. Then $\overline{\gamma^+(P)}$ is compact, and there are only the three possibilities (a), (b), and (c) formulated in Lemma 2.2. We prove that all of them are impossible.

The set $\operatorname{bd}\Omega$ is closed and invariant with respect to (1.1), so $\overline{\gamma^+(P)} \subset \operatorname{bd}\Omega$. Since $\Lambda^+(P) \subset \overline{\gamma^+(P)}$, $\Lambda^+(P)$ also consists of boundary points. This excludes possibilities (a) and (b) because the origin is attractive. In case (c) we show that the interior of the Jordan curve [9] $\Lambda^+(P)$ is a subset of the basin Ω. (Here and in what follows, according to [9, p. 146], the "interior" of a plane Jordan curve means the bounded connected open plane set surrounded by the curve.) In fact, suppose the opposite, i.e., there is an $R \notin \Omega$ in the interior of $\Lambda^+(P)$. Since $0 \notin \Lambda^+(R)$, by Lemma 2.2 we know that $\Lambda^+(R)$ is a cycle in the interior of the Jordan curve $\Lambda^+(P)$, so we have an annulus around the origin whose boundary consists of two trajectories. But this is impossible because $\Lambda^+(P) \subset \operatorname{bd}\Omega$. Therefore $\Lambda^+(P)$ is a Jordan curve around the origin whose interior is a subset of Ω.

Now we need a further development of Krasovskiĭ's and Olech's technique [12,15], whose central idea is the application of the Green Formula [16]. At first we construct the closed path for the line integral in the formula. To this end, besides the basic system (1.1), we need the associated system

$$x' = g(x), \quad g(x) := (-f_2(x), f_1(x))^T, \tag{2.3}$$

whose solutions give trajectories orthogonal to the trajectories of (1.1). The solution and the positive trajectory through P will be denoted by $x^*(\cdot, P)$ and $\gamma^{*+}(P)$, respectively.

Take a point $P_0 \in \Lambda^+(P)$ arbitrarily. Then $x(\cdot, P_0)$ is a periodic solution of (1.1); the smallest positive period will be denoted by T. For the sake of the definiteness, suppose that the orientation of the Jordan curve $\Lambda^+(P)$ defined by the parametrization $t \mapsto x(t, P_0)$ is positive. Let the first segment of the path be $P_0P_T := \{x(t,P_0) : 0 \le t \le T\}$, that is, nothing else but the cycle $\Lambda^+(P)$ itself. Then we go along the segment P_TQ_0 of the trajectory $\gamma^{*+}(P_T)$ of the associated system (2.3) from P_T to a point Q_0 near $P_T = P_0$ in the interior of $\Lambda^+(P)$: $Q_0 = x^*(t_0^*, P_T)$, $t_0^* > 0$ is small (see Figure 1). The following segment Q_0Q_1 of the path lies on the trajectory $\gamma^-(Q_0)$ of the original system (1.1) so that $Q_1 = x(-t_1, Q_0)$ ($t_1 > 0$) is the point where $\gamma^-(Q_0)$ intersects $\gamma^{*+}(P_0) = \gamma^{*+}(P_T)$ the first time. Such a point Q_1 (i.e., a value $t_1 > 0$) exists provided that Q_0 is sufficiently near P_T (i.e., $t_0^* > 0$ is sufficiently small). Since $Q_0 \in \Omega$, Q_1 is located on the trajectory $\gamma^{*+}(P_T)$ strictly between P_T and Q_0, which means that $Q_1 = x^*(t_1^*, P_T)$ and $0 < t_1^* < t_0^*$. Finally the closing piece Q_1P_0 of the path is the segment of $\gamma^{*-}(Q_1)$ from Q_1 to P_0: $P_0 = x^*(-t_1^*, Q_1)$.

Let us apply the Green Formula [16] to the closed curve $G := P_0P_TQ_0Q_1P_0$ and to its interior D:

$$\oint_G (-f_2(x))\,\mathrm{d}x_1 + f_1(x)\,\mathrm{d}x_2 = \int_D \operatorname{div} f(x)\,\mathrm{d}x. \tag{2.4}$$

Now we estimate the contour integral on the left-hand side

$$\oint_G \ldots = \oint_{P_0P_T} \ldots + \oint_{P_TQ_0} \ldots + \oint_{Q_0Q_1} \ldots + \oint_{Q_1P_0} \ldots .$$

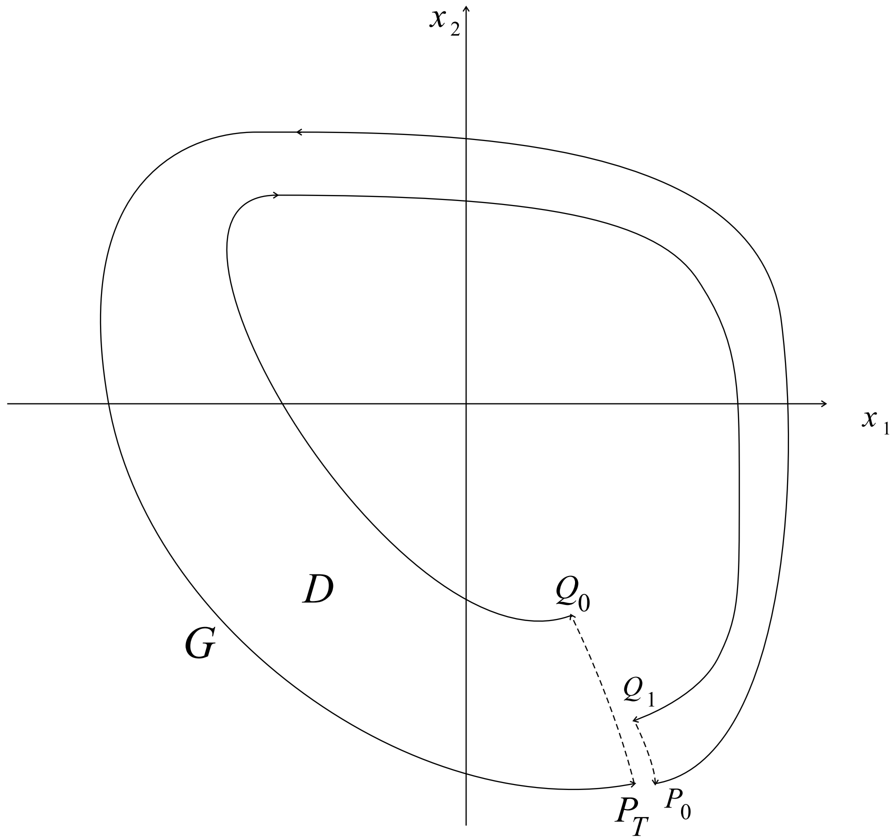

FIGURE 1. The path for the Green Formula in the proof of Lemma 2.3

The first and third members of this sum are equal to zero and

$$\oint_{Q_1P_0} \dots = -\oint_{P_0Q_1} \dots = -\oint_{P_TQ_1} \dots,$$

therefore

$$\oint_G \dots = \oint_{Q_1Q_0} \dots = \int_{t_1^*}^{t_0^*} |f(x^*(\tau, P_T))|^2 \,\mathrm{d}\tau > 0$$

because $t_1^* < t_0^*$. On the other hand, by Lemma 2.2 $\operatorname{div} f(x)$ vanishes in the interior of D. Consequently, the left-hand side of equality (2.4) is positive, and the right-hand side of the same equality equals zero, which is a contradiction. □

3. The main theorems

The first theorem deals with the case of $\operatorname{div} f(x) \leq 0$ $(x \in \mathbb{R}^2)$.

Theorem 3.1. *If (A1), (B3), (C1) and (C4) are satisfied, then the origin is globally attractive.*

Proof. Suppose that the statement is not true and let P_0 be one of the nearest points of $\operatorname{bd}\Omega$ to the origin. If $C(r_0)$ denotes the circle of radius $r_0 := |P_0| > 0$, then we can show that the interior of the Jordan curve $C(r_0)$ is a subset of Ω. In fact, using the method of contradiction suppose that there exists a $Q \notin \Omega$ with $|Q| < r_0$. Then $Q \notin \mathrm{bd}\,\Omega$, consequently $Q \in \mathrm{ext}\,\Omega$, and the *straight line interval* $(0, Q)$ contains at least one point of $\mathrm{bd}\,\Omega$, which contradicts the definition of r_0.

Take a $Q_0 \in (0, P_0) \subset \Omega$; this will be the initial point of the closed path to be constructed for the Green Formula. For the sake of the definiteness we assume that the closed curve Q_0OQ_0 ($Q_0O = \overline{\gamma^+(Q_0)}$, $OQ_0 = (O, Q_0)$) is positively oriented by the parametrization $t \mapsto x(t, Q_0)$. Trajectory $\gamma^+(Q_0)$ is bounded; let $Q_{\max}$ denote one of the farthest points of $\gamma^+(Q_0)$ from the origin. If Q_0 is sufficiently near to P_0, then Lemma 2.3 guarantees that $\gamma^+(Q_0)$ crosses $C(r_0)$, i.e., $|Q_{\max}| > r_0$. The first segment of the path will be $Q_0Q_{\max} \subset \gamma^+(Q_0)$.

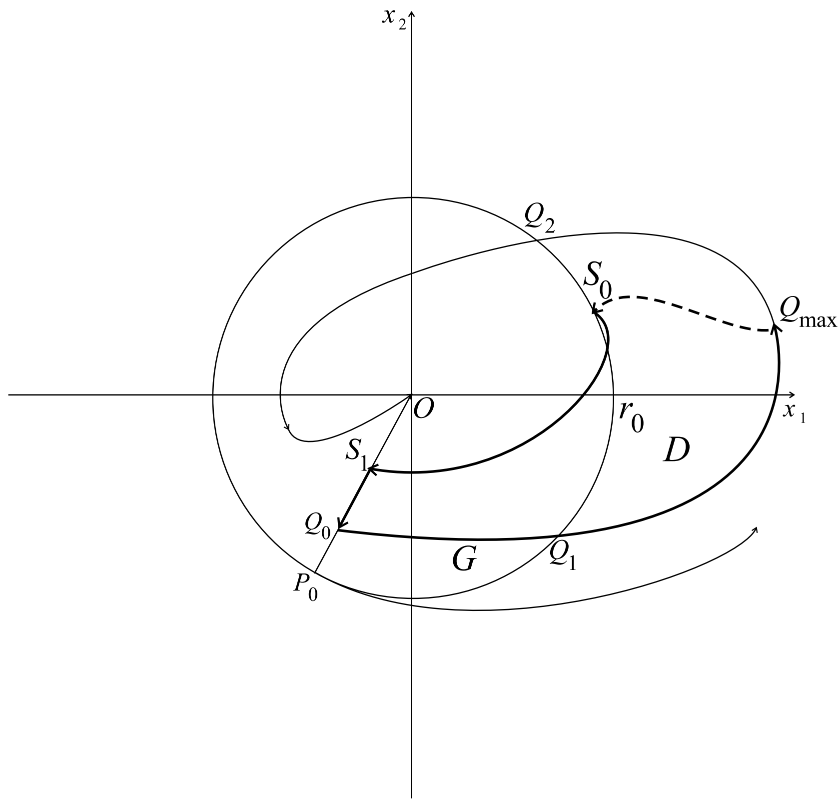

FIGURE 2. The path for the Green Formula in the proof of Theorem 3.1

To continue the path, let us start the positive trajectory $\gamma^{*+}(Q_{\max})$ of the associated system (2.3) into the interior of the closed curve Q_0OQ_0 (see the broken line on Figure 2). Let Q_1, and Q_2 denote the points where the trajectory $\gamma^+(Q_0)$ intersects the circle $C(r_0)$ the last time before $Q_{\max}$, and the first time after $Q_{\max}$, respectively, and consider the closed curve $Q_1Q_2Q_1$: $Q_1Q_2 \subset \gamma^+(Q_0)$, $Q_2Q_1 \subset C(r_0)$. Since the interior of $Q_1Q_2Q_1$ is precompact and $\gamma^{*+}(Q_{\max})$ must not intersect $\gamma^+(Q_0)$ at a further point different from $Q_{\max}$, Lemma 2.2 implies that $\gamma^{*+}(Q_{\max})$

must intersect the piece Q_2Q_1 of circle $C(r_0)$ at a point denoted by S_0. The second segment of the path will be $Q_{\max}S_0 \subset \gamma^{*+}(Q_{\max})$ $(S_0 = x^*(s^*, Q_{\max}))$(see Figure 2).

Let us start the negative trajectory $\gamma^-(S_0)$ of the original system (1.1) from S_0. Similarly to the reasonings above, by the use of Lemma 2.2 it can be seen that this trajectory intersects the half line connecting O and Q_0 at a point denoted by S_1. We choose the curve $S_0S_1 \subset \gamma^-(S_0)$ to be the third segment of the path. Finally, the path is closed by the straight line interval $[S_1, Q_0]$.

Introduce the notation $G := G(Q_0) := Q_0Q_{\max}S_0S_1Q_0$, and let $D := D(Q_0)$ denote the interior of G. Estimate the contour integral on the left-hand side of the Green Formula (2.4):

$$\oint_G \dots = \oint_{Q_0Q_{\max}} \dots + \oint_{Q_{\max}S_0} \dots + \oint_{S_0S_1} \dots + \oint_{S_1Q_0} \cdots . \tag{3.1}$$

The first and third members of this sum are equal to zero. In the second integral $I_2 = I_2(Q_0)$ we use the arc length parameter $l = u(\tau)$ on the curve $Q_{\max}S_0$ and obtain

$$\begin{aligned} I_2 &= \oint_{Q_{\max}S_0} (-f_2(x))\,\mathrm{d}x_1 + f_1(x)\,\mathrm{d}x_2 = \int_0^{s^*} |f(x^*(\tau, Q_{\max}))|^2\,\mathrm{d}\tau \\ &= \int_0^{u(s^*)} |f(x^*(u^{-1}(l), Q_{\max}))|\,\mathrm{d}l. \end{aligned} \tag{3.2}$$

By assumption (C4) we have

$$I_2 \geq \int_0^{u(s^*)} \left[\min_{|x|=|x^*(u^{-1}(l),Q_{\max})|} |f(x)|\right] \mathrm{d}l \geq \int_{r_0}^{Q_{\max}} \left[\min_{|x|=r} |f(x)|\right] \mathrm{d}r.$$

By Lemma 2.3 and condition (C4) the last side of these inequalities takes arbitrarily large values as $Q_0 \to P_0$. On the other hand, since $|Q_0 - S_1| < r_0$, the absolute value of the fourth member in (3.1) is less than a constant, independent of Q_0, so the left-hand side in (2.4) can be arbitrarily large as $Q_0 \to P_0$. But this is a contradiction because the right-hand side must not be positive. The contradiction came from the assumption that $\Omega \neq \mathbb{R}^2$, therefore the theorem is proved. $\square$

The second theorem is devoted to the case $\mathrm{div} f(x) \geq 0$ $(x \in \mathbb{R}^2)$.

Theorem 3.2. *If (A1), (B4), (C1) and (C5) are satisfied, then the origin is globally attractive.*

Proof. The proof coincides with that of Theorem 3.1 word for word until the formula (3.2). Let us denote by s^{**} the smallest number in $(0, s^*]$ such that $|x^*(s^{**}, Q_{\max})| = \max\{r_0; \kappa\} =: r_1$, where $\kappa > 0$ is the number in condition (C5). By Lemma 2.3 such an s^{**} exists, provided that Q_0 is near to P_0. Then from (3.2) and (C5) we obtain

$$\begin{aligned} I_2 &= \int_0^{u(s^*)} |f(x^*(u^{-1}(l), Q_{\max}))|\,\mathrm{d}l \geq \varepsilon \int_0^{u(s^{**})} |x^*(u^{-1}(l), Q_{\max})|^{\alpha}\,\mathrm{d}l \\ &\geq -\varepsilon \int_{|Q_{\max}|}^{r_1} r^{\alpha}\,\mathrm{d}r = \frac{\varepsilon}{\alpha+1}\left(|Q_{\max}|^{\alpha+1} - r_1^{\alpha+1}\right). \end{aligned}$$

On the other hand, by the Green Formula (2.4) and assumption (B4) with $R = |Q_{\max}|$ we have

$$\begin{aligned}\frac{\varepsilon}{\alpha+1}\left(R^{\alpha+1}-r_1^{\alpha+1}\right)-\text{const.} &\leq \oint_G \dots = \int_D \operatorname{div} f(x)\,\mathrm{d}x \\ &\leq \int_{|x|\leq R} \operatorname{div} f(x)\,\mathrm{d}x = o(R^{\alpha+1}) \quad (R\to\infty),\end{aligned} \tag{3.3}$$

that is a contradiction. □

In the last theorem, instead of local attractivity of the origin we suppose stability.

Theorem 3.3. *If (A2), (B2), (C1) and (C4) are satisfied, then the origin is globally asymptotically stable.*

Proof. At first we prove that the basin Ω of the origin is not empty. In accordance with the notation $C(\alpha)$ for the circle of radius α, we denote by $D(\alpha)$ the disc of radius α around the origin. Take the number $\delta(1) > 0$ from the definition of the stability of the origin. We show that there are $Z \in C(1)$ and $T > 0$ such that $|x(T, Z)| < \delta(1)$. In fact, if this assertion is not true, then from Lemma 2.1 we obtain

$$0 < \mu(D(\delta(1))) \leq \mu(x(t, D(1))) \leq \exp[-(a(\delta(1)))t]\mu(D(1)) \to 0 \quad (t\to\infty),$$

a contradiction, which means that the desired $Z \in C(1)$ exists.

From the definition of stability of the origin it follows that $|x(t, Z)| \leq 1$ for t large enough. Apply Lemma 2.2 to point Z. In general, assumption (B2) excludes the existence of any cycle or homoclinic orbit (see Lemma 2.1), therefore neither (b) nor (c) can be valid. So the only possible case is $\Lambda^+(Z) = \{0\}$, i.e., $\lim_{t\to\infty} |x(t, Z)| = 0$. This concludes the proof of the fact that Ω is not empty.

Now (A1) is not supposed, therefore Lemma 2.3 cannot be applied. In spite of this fact we can prove the statement of the lemma. In fact, take a point $P \in \operatorname{bd}\Omega$. If $\gamma^+(P)$ is bounded, then we again apply Lemma 2.2 and get $\lim_{t\to\infty} |x(t, P)| = 0$. But $P \in \operatorname{bd}\Omega$, so there is $R \notin \Omega$ with $|R| < \delta(1)$. By Lemma 2.2 we have $\Lambda^+(R) = \{0\}$, i.e., $\lim_{t\to\infty} |x(t, R)| = 0$, which contradicts $R \notin \Omega$. This means that $\gamma^+(P)$ cannot be bounded, and (2.2) is satisfied.

From this point we can repeat word for word the proof of Theorem 3.1. □

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