

## BRAIDED $\mathbb{Z}_q$ -EXTENSIONS OF POINTED FUSION CATEGORIES

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ABSTRACT. We classify braided  $\mathbb{Z}_q$ -extensions of pointed fusion categories, where  $q$  is a prime number. As an application, we classify modular categories of Frobenius-Perron dimension  $q^3$ .

### 1. INTRODUCTION

Let  $k$  be an algebraically closed field of characteristic 0. By definition, a fusion category is a  $k$ -linear semisimple rigid tensor category with finitely many isomorphism classes of simple objects, finite-dimensional spaces of morphisms, and such that the unit object  $\mathbf{1}$  is simple. We refer the reader to [3] for main notions and basic results on fusion categories.

Let  $\mathcal{C}$  be a fusion category and let  $G$  be a finite group with identity element 0. A  $G$ -grading on  $\mathcal{C}$  is a decomposition  $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$  as a direct sum of full Abelian subcategory such that the dual functor  $*$  sends  $\mathcal{C}_g$  into  $\mathcal{C}_{g^{-1}}$  and the tensor product  $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  maps  $\mathcal{C}_g \times \mathcal{C}_h$  into  $\mathcal{C}_{gh}$ .

The grading  $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$  is called faithful if  $\mathcal{C}_g \neq 0$  for all  $g \in G$ . If  $\mathcal{C}$  has a faithful  $G$ -grading and  $\mathcal{C}_0 = \mathcal{D}$ , then  $\mathcal{C}$  is called a  $G$ -extension of  $\mathcal{D}$ .

Group extensions of fusion categories play important roles in classifying fusion categories and have been intensively studied by several authors [3–5]. An important class of group extensions is the  $\mathbb{Z}_q$ -extensions of pointed fusion categories, where  $q$  is a prime number. By a pointed fusion category we mean a fusion category whose simple objects are all invertible. Several typical examples of  $\mathbb{Z}_q$ -extensions of pointed fusion categories are recalled in Section 2.

The main work of this paper is to classify braided  $\mathbb{Z}_q$ -extensions of pointed fusion categories. Let  $\mathcal{C}$  be a braided  $\mathbb{Z}_q$ -extension of a pointed fusion category. Suppose that  $\mathcal{C}$  is not pointed. We prove that  $\mathcal{C}$  is equivalent to a Deligne tensor product  $\mathcal{B} \boxtimes \mathcal{E}$ , where  $\mathcal{B}$  is a pointed fusion category and  $\mathcal{E}$  is a fusion category of  $q$  power dimension. In particular,  $\mathcal{E}$  is of type  $(1, m; \alpha, n)$  for some positive integers  $m, n$  and  $\alpha^2$  is a power of  $q$ . Suppose further that  $\mathcal{C}$  is modular. Then we prove that  $\mathcal{C}$  is equivalent to a Deligne tensor product  $\mathcal{B} \boxtimes \mathcal{I}$ , where  $\mathcal{B}$  is a pointed modular category and  $\mathcal{I}$  is an Ising category. Finally, we apply these results to modular categories of Frobenius-Perron (FP) dimension  $q^3$  and prove that this class of modular categories

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is equivalent to  $\mathcal{B} \boxtimes \mathcal{I}$ , where  $\mathcal{B}$  is a pointed modular category of FP dimension 2 and  $\mathcal{I}$  is an Ising category.

2. PRELIMINARIES AND EXAMPLES

Let  $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$  be a faithful grading on  $\mathcal{C}$ . Then the FP dimensions of  $\mathcal{C}_g$  are all equal [3, Proposition 8.20], and hence we have  $\text{FPdim}(\mathcal{C}) = |G| \text{FPdim}(\mathcal{C}_0)$ , where we denote by  $\text{FPdim}(\mathcal{C})$  the FP dimension of  $\mathcal{C}$ .

We denote by  $\text{Irr}(\mathcal{C})$  the set of non-isomorphic simple objects of  $\mathcal{C}$ , and by  $\text{Irr}_\alpha(\mathcal{C})$  the set of non-isomorphic simple objects of FP dimension  $\alpha$ . The adjoint subcategory  $\mathcal{C}_{ad}$  is the full tensor subcategory of  $\mathcal{C}$  generated by simple objects in  $X \otimes X^*$ , for all  $X \in \text{Irr}(\mathcal{C})$ . The rank of  $\mathcal{C}$  is the cardinality of the set  $\text{Irr}(\mathcal{C})$ .

Every fusion category  $\mathcal{C}$  has a unique faithful grading  $\mathcal{C} = \bigoplus_{g \in \mathcal{U}(\mathcal{C})} \mathcal{C}_g$  such that  $\mathcal{C}_0 = \mathcal{C}_{ad}$ . This grading is called the universal grading of  $\mathcal{C}$ , and the group  $\mathcal{U}(\mathcal{C})$  is called the universal grading group of  $\mathcal{C}$ . This grading is universal because any faithful grading  $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$  comes from a surjective group homomorphism  $\mathcal{U}(\mathcal{C}) \rightarrow G$ . This universal property implies the following result:

**Lemma 2.1.** *Let  $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$  be a faithful grading on  $\mathcal{C}$ . Then  $\mathcal{C}_{ad} \subseteq \mathcal{C}_0$ .*

A fusion category  $\mathcal{C}$  is called integral if  $\text{FPdim}(X)$  is an integer for all objects  $X$  in  $\mathcal{C}$ , where  $\text{FPdim}(X)$  denotes the FP dimension of  $X$ . A fusion category  $\mathcal{C}$  is called weakly integral if  $\text{FPdim}(\mathcal{C})$  is an integer. Let  $\mathcal{C}$  be a  $G$ -extension of a pointed fusion category  $\mathcal{D}$ . Then  $\text{FPdim}(\mathcal{C}) = |G| \text{FPdim}(\mathcal{D})$ , and hence  $\mathcal{C}$  is weakly integral since  $\text{FPdim}(\mathcal{D})$  is an integer.

Let  $\mathcal{C}_{pt}$  denote the fusion subcategory generated by all invertible simple objects of  $\mathcal{C}$ . Then  $\mathcal{C}_{pt}$  is the largest pointed fusion subcategory of  $\mathcal{C}$ . All non-isomorphic invertible objects of  $\mathcal{C}$  form a group with multiplication given by tensor product. We denote this group by  $G(\mathcal{C})$ . The group  $G(\mathcal{C})$  acts on the set  $\text{Irr}(\mathcal{C})$  by left tensor multiplication, and this action preserves FP dimension. For  $X \in \text{Irr}(\mathcal{C})$ , we use  $G[X]$  to denote the stabilizer of  $X$  under this action.

We now discuss some examples of  $\mathbb{Z}_q$ -extensions of pointed fusion categories.

**Example 2.2** (Generalized Tambara-Yamagami fusion categories). Let  $\mathcal{C}$  be a fusion category. If  $\mathcal{C}$  is not pointed and  $X \otimes Y$  is a direct sum of invertible objects, for all non-invertible simple objects  $X, Y \in \mathcal{C}$ , then  $\mathcal{C}$  is a generalized Tambara-Yamagami fusion category.

Generalized Tambara-Yamagami fusion categories were classified in [7], up to equivalence of tensor categories, and then were further studied in [9]. By [9], there exists a normal subgroup  $N$  of  $G(\mathcal{C})$  such that  $G[X] = N$ , for all non-invertible simple objects  $X$ . This implies that  $\text{FPdim}(X) = \sqrt{|N|}$  for all non-invertible simple objects  $X$ . The rank of  $\mathcal{C}$  is  $[G(\mathcal{C}) : N](1 + |N|)$ , and hence  $\text{FPdim}(\mathcal{C}) = 2|G(\mathcal{C})|$ . This implies that  $\mathcal{C}$  is a  $\mathbb{Z}_2$ -extension of a pointed fusion category generated by  $G(\mathcal{C})$ .

**Example 2.3** (Tambara-Yamagami fusion categories). Let  $\mathcal{C}$  be a generalized Tambara-Yamagami fusion category. If  $N = G(\mathcal{C})$ , then the rank of  $\mathcal{C}_1$  is 1. Then we can write  $\text{Irr}(\mathcal{C}) = G(\mathcal{C}) \cup \{X\}$ , where  $X$  is the unique non-invertible simple object of  $\mathcal{C}$ . In this case,  $\mathcal{C}$  is called a Tambara-Yamagami fusion category. This class of fusion categories was classified in [10].

**Example 2.4** (Ising fusion category). Let  $\mathcal{C}$  be a generalized Tambara-Yamagami fusion category. If the order of  $G(\mathcal{C})$  is 2 and  $N = G(\mathcal{C})$ , then  $\mathcal{C}$  is called an Ising fusion category. It is well known that any Ising category admits a structure of braided category. This class of fusion categories was classified in [2, Appendix B].

**Example 2.5.** A fusion category  $\mathcal{C}$  is called nilpotent if there is a sequence of fusion categories  $\text{Vec}_k = \mathcal{C}_0 \subseteq \mathcal{C}_1 \subseteq \dots \subseteq \mathcal{C}_n = \mathcal{C}$  and a sequence of finite groups  $G_1, \dots, G_n$  such that  $\mathcal{C}_i$  is obtained from  $\mathcal{C}_{i-1}$  by a  $G_i$ -extension, for all  $1 \leq i \leq n$ . If the groups  $G_1, \dots, G_n$  can be chosen to be cyclic of prime order, then  $\mathcal{C}$  is called cyclically nilpotent.

Let  $\mathcal{C}$  be a cyclically nilpotent fusion category, and let  $\mathcal{C}_0, \mathcal{C}_1, \dots, \mathcal{C}_n, G_1, \dots, G_n$  be the corresponding fusion subcategories and finite groups. Since  $\mathcal{C}_0$  is the trivial fusion category and  $\mathcal{C}_1$  is a  $\mathbb{Z}_p$ -extension of  $\mathcal{C}_0$  for some prime number  $p$ ,  $\mathcal{C}_1$  is a pointed fusion category. It follows that  $\mathcal{C}_2$  is a  $\mathbb{Z}_q$ -extension of a pointed fusion category ( $\mathcal{C}_1$ ) for some prime number  $q$ .

**Lemma 2.6.** *Let  $q$  be a prime number and let  $\mathcal{C} = \bigoplus_{g \in \mathbb{Z}_q} \mathcal{C}_g$  be a faithful grading of  $\mathcal{C}$ . Assume that the trivial component  $\mathcal{C}_0$  is pointed. Then*

- (1) *the adjoint subcategory  $\mathcal{C}_{ad}$  is pointed;*
- (2)  *$\mathcal{C}$  is pointed or  $\mathcal{C}_0 = \mathcal{C}_{pt}$  is the largest pointed fusion subcategory of  $\mathcal{C}$ .*

*Proof.* (1) By Lemma 2.1,  $\mathcal{C}_{ad}$  is contained in  $\mathcal{C}_0$ . Hence,  $\mathcal{C}_{ad}$  is pointed.

(2) Since  $\mathcal{C}_{pt}$  is the unique largest pointed fusion subcategory of  $\mathcal{C}$ ,  $\mathcal{C}_{pt}$  contains  $\mathcal{C}_0$  as a fusion subcategory. This fact shows that  $\text{FPdim}(\mathcal{C}_0)$  divides  $\text{FPdim}(\mathcal{C}_{pt})$  [3, Proposition 8.15]. On the other hand,  $\text{FPdim}(\mathcal{C}) = q \text{FPdim}(\mathcal{C}_0)$ . These facts imply that  $\text{FPdim}(\mathcal{C}_{pt}) = \text{FPdim}(\mathcal{C}_0)$  or  $q \text{FPdim}(\mathcal{C}_0)$ . The first case means that  $\mathcal{C}_0 = \mathcal{C}_{pt}$ , and the second case means that  $\mathcal{C} = \mathcal{C}_{pt}$  is pointed. □

**Lemma 2.7.** *Let  $\mathcal{C}$  be a  $\mathbb{Z}_q$ -extension of a pointed fusion category. Assume that  $\mathcal{C}$  is not pointed. Then  $\mathcal{C}$  is a generalized Tambara-Yamagami fusion category if and only if  $q = 2$ .*

*Proof.* Let  $\mathcal{C} = \bigoplus_{g \in \mathbb{Z}_q} \mathcal{C}_g$  be the  $\mathbb{Z}_q$ -extension. If  $q = 2$ , then Lemma 2.6 shows that  $\mathcal{C}_0 = \mathcal{C}_{pt}$  is the largest pointed fusion subcategory of  $\mathcal{C}$ , and hence all non-invertible simple objects are contained in  $\mathcal{C}_1$ . Let  $X, Y$  be non-invertible simple objects of  $\mathcal{C}$ . Then  $X \otimes Y$  is contained in  $\mathcal{C}_0$ , which means that  $X \otimes Y$  is a direct sum of invertible simple objects. Hence,  $\mathcal{C}$  is a generalized Tambara-Yamagami fusion category.

Conversely, if  $\mathcal{C}$  is a generalized Tambara-Yamagami fusion category, then  $\text{FPdim}(\mathcal{C}) = 2|G(\mathcal{C})|$  by Example 2.2. On the other hand,  $\text{FPdim}(\mathcal{C}) = q \text{FPdim}(\mathcal{C}_0) = q|G(\mathcal{C})|$  by Lemma 2.6. Hence  $q = 2$  as claimed. □

### 3. MAIN RESULTS

Recall that a fusion category  $\mathcal{C}$  is braided if it has a natural isomorphism  $c_{X,Y} : X \otimes Y \rightarrow Y \otimes X$ , for all  $X, Y$  in  $\mathcal{C}$ , which satisfies the hexagon axioms [6].

Let  $1 = d_0 < d_1 < \dots < d_s$  be positive real numbers, and let  $n_0, n_1, \dots, n_s$  be positive integers. A fusion category is said to be of type  $(d_0, n_0; d_1, n_1; \dots; d_s, n_s)$  if  $n_i$  is the number of the non-isomorphic simple objects of Frobenius-Perron dimension  $d_i$ , for all  $0 \leq i \leq s$ .

**Theorem 3.1.** *Let  $\mathcal{C}$  be a braided  $\mathbb{Z}_q$ -extension of a pointed fusion category. Suppose that  $\mathcal{C}$  is not pointed. Then  $\mathcal{C} \cong \mathcal{B} \boxtimes \mathcal{E}$ , where  $\mathcal{B}$  is a pointed fusion category and*

$\mathcal{E}$  is a fusion category of  $q$  power dimension. In particular,  $\mathcal{E}$  is of type  $(1, m; \alpha, n)$  for some positive integers  $m, n$  and  $\alpha^2$  is a power of  $q$ .

*Proof.* Let  $X_1, X_2, \dots, X_s$  be a list of all non-isomorphic simple objects of  $\mathcal{C}$  such that  $1 < \text{FPdim}(X_1) \leq \dots \leq \text{FPdim}(X_s)$ . We may say that  $\text{FPdim}(X_1) = \alpha$  for some positive real number  $\alpha$ . By Lemma 2.1 and Lemma 2.6,  $\mathcal{C}_{ad} \subseteq \mathcal{C}_0 = \mathcal{C}_{pt}$ . So  $X_1 \otimes X_1^*$  is contained in  $\mathcal{C}_{pt}$ , and hence the stabilizer  $G[X_1]$  of  $X_1$  under the action of  $G[\mathcal{C}]$  is of order  $\alpha^2$ . Let  $\mathcal{D}$  be the fusion subcategory generated by simple objects in  $G[X_1]$ . It is a pointed fusion subcategory of  $\mathcal{C}$  with FP dimension  $\alpha^2$ .

Let  $\mathcal{D}^{co}$  be the commutator of  $\mathcal{D}$  in  $\mathcal{C}$ ; that is,  $\mathcal{D}^{co}$  is the fusion subcategory of  $\mathcal{C}$  generated by all simple objects  $X$  of  $\mathcal{C}$  such that  $X \otimes X^*$  is contained in  $\mathcal{D}$  [5]. Clearly, all invertible objects and  $X_1$  are contained in  $\mathcal{D}^{co}$ , which means that  $\text{FPdim}(\mathcal{D}^{co}) \geq \text{FPdim}(\mathcal{C}_{pt}) + \alpha^2$ . On the other hand,  $\text{FPdim}(\mathcal{C}_{pt})$  divides  $\text{FPdim}(\mathcal{D}^{co})$  and  $\text{FPdim}(\mathcal{D}^{co})$  divides  $\text{FPdim}(\mathcal{C}) = q \text{FPdim}(\mathcal{C}_{pt})$ . This implies that  $\text{FPdim}(\mathcal{D}^{co}) = \text{FPdim}(\mathcal{C})$ . Hence, we must have that  $\mathcal{D}^{co} = \mathcal{C}$ .

By [5, Lemma 4.15],  $\mathcal{C}_{ad} = (\mathcal{D}^{co})_{ad} \subseteq \mathcal{D}$ . On the other hand,  $\mathcal{C}_{ad}$  has FP dimension at least  $\alpha^2$  since  $X_1 \otimes X_1^*$  is contained in it. It follows that  $\mathcal{C}_{ad} = \mathcal{D}$ . Since  $\mathcal{C}_{ad}$  has FP dimension  $\alpha^2$ ,  $\mathcal{C}$  cannot have simple objects with FP dimension greater than  $\alpha$ . In other words, the FP dimensions of simple objects of  $\mathcal{C}$  can only be 1 or  $\alpha$ .

Since  $\mathcal{C}$  is braided and nilpotent, [1, Theorem 1.1] shows that we have a decomposition  $\mathcal{C} \cong \boxtimes_{p_i} \mathcal{C}_{p_i}$ , where  $\mathcal{C}_{p_i}$  is a fusion subcategory of prime power dimension. The simple object  $X$  of  $\mathcal{C}$  has the form  $X \cong \bigotimes_{p_i} X_{p_i}$ , where  $X_{p_i}$  is a simple object of  $\mathcal{C}_{p_i}$ . So if there exist  $\mathcal{C}_{p_i}$  and  $\mathcal{C}_{p_k}$  such that they both contain non-invertible simple objects, then the number of distinct FP dimensions of simple objects of  $\mathcal{C}$  is at least 3. This contradicts the results obtained above. Therefore, there is only one subcategory in the decomposition of  $\mathcal{C}$  such that it contains non-invertible simple objects. So  $\mathcal{C}$  has the decomposition  $\mathcal{C} \cong \mathcal{B} \boxtimes \mathcal{E}$ , where  $\mathcal{B}$  is a pointed fusion category and  $\mathcal{E}$  is a fusion category of a prime power dimension. In particular,  $\mathcal{E}$  is of type  $(1, m; \alpha, n)$  for some positive integers  $m, n$ . We shall prove that  $\text{FPdim}(\mathcal{E})$  is a power of  $q$ .

Assume that  $\text{FPdim}(\mathcal{E}) = p^a$  for some prime number  $p$ , and hence we may assume that  $\text{FPdim}(\mathcal{E}_{pt}) = p^i$  for some  $0 \leq i \leq a - 1$ . Lemma 2.6(2) and our assumption show that  $\mathcal{C}_0 = \mathcal{C}_{pt}$ , and hence  $\mathcal{C}_0 = \mathcal{B} \boxtimes \mathcal{E}_{pt}$ .

On the one hand,  $\text{FPdim}(\mathcal{C}) = q \text{FPdim}(\mathcal{C}_0)$  since  $\mathcal{C}$  is a  $\mathbb{Z}_q$ -extension of  $\mathcal{C}_0$ . On the other hand,  $\text{FPdim}(\mathcal{C}) = \text{FPdim}(\mathcal{B}) \text{FPdim}(\mathcal{E})$  since  $\mathcal{C}$  has the decomposition  $\mathcal{B} \boxtimes \mathcal{E}$ . Hence, we have

$$\begin{aligned} \text{FPdim}(\mathcal{B}) \text{FPdim}(\mathcal{E}) &= q \text{FPdim}(\mathcal{C}_0); \\ \text{FPdim}(\mathcal{B}) p^a &= q \text{FPdim}(\mathcal{B}) p^i; \\ p^a &= q \cdot p^i. \end{aligned}$$

This means that  $p = q$  and  $i = a - 1$ . So  $\text{FPdim}(\mathcal{E})$  is a power of  $q$ . Finally, [4, Theorem 2.11] shows that  $\alpha^2$  is a power of  $q$ . □

**Corollary 3.2.** *Let  $\mathcal{C}$  be a braided  $\mathbb{Z}_q$ -extension of a pointed fusion category. Then:*

- (1) *If  $\mathcal{C}$  is not integral, then  $\mathcal{C}$  is a generalized Tambara-Yamagami fusion category. In this case  $\mathcal{C} \cong \mathcal{B} \boxtimes \mathcal{E}$ , where  $\mathcal{B}$  is a pointed fusion category and  $\mathcal{E}$  has FP dimension  $2^k$  for some  $k$ .*
- (2) *If  $q > 2$ , then  $\mathcal{C}$  is integral.*

*Proof.* (1) Let  $\mathcal{C} = \bigoplus_{g \in \mathcal{U}(\mathcal{C})} \mathcal{C}_g$  be the universal grading of  $\mathcal{C}$ . Then  $\alpha^2 = \text{FPdim}(\mathcal{C}_g)$  for all  $g \in \mathcal{U}(\mathcal{C})$ , by the proof of Theorem 3.1. It follows that every component  $\mathcal{C}_g$  either contains only one simple object with FP dimension  $\alpha$  or contains  $\alpha^2$  non-isomorphic invertible simple objects. This hence implies that, for any  $X, Y \in \text{Irr}_\alpha(\mathcal{C})$ ,  $X \otimes Y$  is either a direct sum of  $\alpha^2$  invertible simple objects or a direct sum of  $\alpha$  copies of a simple object with FP dimension  $\alpha$ . Note that the latter case holds true if and only if  $\mathcal{C}$  is integral. Hence, if  $\mathcal{C}$  is not integral, then  $X \otimes Y$  is a direct sum of invertible simple objects, so  $\mathcal{C}$  is a generalized Tambara-Yamagami fusion category.

Let  $\mathcal{E}$  be the braided fusion category of  $q$  power dimension in Theorem 3.1, and let  $\text{FPdim}(\mathcal{E}) = q^k$  for some  $k$ . Since  $\mathcal{C}$  is not integral,  $\mathcal{E}$  is not integral. This happens only if  $q = 2$  [5, Corollary 3.11].

(2) Suppose on the contrary that  $\mathcal{C}$  is not integral. Then  $\mathcal{C}$  is a generalized Tambara-Yamagami fusion category by part (1), and hence  $q = 2$ . This is a contradiction.  $\square$

Recall that a fusion category is called group-theoretical if it is Morita equivalent to a pointed fusion category.

*Remark 3.3.* If  $\mathcal{C}$  is integral, then  $\mathcal{C}$  is a group-theoretical fusion category by [1, Theorem 6.10], since  $\mathcal{C}$  is braided and nilpotent of nilpotency class 2.

Let  $\mathcal{C}$  be a braided fusion category, and let  $\mathcal{D}$  be a fusion subcategory of  $\mathcal{C}$ . The Müger centralizer of  $\mathcal{D}$  in  $\mathcal{C}$  is the fusion subcategory

$$\mathcal{D}' = \{X \in \mathcal{C} \mid c_{Y,X}c_{X,Y} = id_{X \otimes Y}, \text{ for all } Y \in \mathcal{D}\}.$$

The Müger center  $\mathcal{Z}_2(\mathcal{C})$  of  $\mathcal{C}$  is the Müger centralizer of  $\mathcal{C}$  itself. A braided fusion category  $\mathcal{C}$  is non-degenerate if its Müger center  $\mathcal{Z}_2(\mathcal{C})$  is trivial. A braided fusion category is premodular if it has a spherical structure. By [3, Propositions 8.23, 8.24], any weakly integral fusion category is premodular. Hence a weakly integral braided fusion category is modular if and only if it is non-degenerate. In particular, if  $\mathcal{C}$  is a braided  $G$ -extension of a pointed fusion category, then  $\mathcal{C}$  is modular if and only if it is non-degenerate.

**Theorem 3.4.** *Let  $\mathcal{C}$  be a  $\mathbb{Z}_q$ -extension of a pointed fusion category. Assume in addition that  $\mathcal{C}$  is modular. Then  $\mathcal{C}$  fits into one of the following classes:*

- (1)  $\mathcal{C}$  is pointed;
- (2)  $\mathcal{C} \cong \mathcal{B} \boxtimes \mathcal{I}$ , where  $\mathcal{B}$  is a pointed modular category and  $\mathcal{I}$  is an Ising category.

*Proof.* We may assume that  $\mathcal{C}$  is not pointed. Since  $\mathcal{C}$  is modular, we have  $\mathcal{U}(\mathcal{C}) \cong G(\mathcal{C})$  [5, Theorem 6.2]. It follows that the order of  $\mathcal{U}(\mathcal{C})$  is equal to  $|G(\mathcal{C})| = \text{FPdim}(\mathcal{C}_{pt})$ . Let  $\mathcal{C} = \bigoplus_{g \in \mathbb{Z}_q} \mathcal{C}_g$  be the  $\mathbb{Z}_q$ -extension. By Lemma 2.6,

$$\text{FPdim}(\mathcal{C}_{pt}) = \text{FPdim}(\mathcal{C}_0),$$

and hence

$$\text{FPdim}(\mathcal{C}) = q \text{FPdim}(\mathcal{C}_0) = q \text{FPdim}(\mathcal{C}_{pt}).$$

On the other hand,

$$\text{FPdim}(\mathcal{C}) = |\mathcal{U}(\mathcal{C})| \text{FPdim}(\mathcal{C}_g) = \text{FPdim}(\mathcal{C}_{pt}) \text{FPdim}(\mathcal{C}_g),$$

for any  $g \in \mathcal{U}(\mathcal{C})$ . Hence, every component of the universal grading has FP dimension  $q$ . In particular,  $\mathcal{C}_{ad}$  has FP dimension  $q$ .

Let  $X$  be a non-invertible simple object of  $\mathcal{C}$ . Then the proof of Theorem 3.1 shows that  $\text{FPdim}(\mathcal{C}_{ad}) = \text{FPdim}(X)^2$ , which means that  $\text{FPdim}(X) = \sqrt{q}$ . Since  $q$  is a prime number,  $\sqrt{q}$  is not an integer, and hence  $\mathcal{C}$  is not integral. By Corollary 3.2,  $\mathcal{C}$  is a generalized Tambara-Yamagami fusion category. Therefore  $\mathcal{C} \cong \mathcal{B} \boxtimes \mathcal{I}$  as described above, by [9, Theorem 5.4].  $\square$

**Corollary 3.5.** *Let  $q$  be a prime number and let  $\mathcal{C}$  be a modular category of FP dimension  $q^3$ . Then*

- (1)  $\mathcal{C}$  is pointed, or
- (2)  $\mathcal{C} \cong \mathcal{B} \boxtimes \mathcal{I}$ , where  $\mathcal{B}$  is a pointed modular category of FP dimension 2 and  $\mathcal{I}$  is an Ising category.

*Proof.* [3, Theorem 8.28] shows that  $\mathcal{C}$  is a  $\mathbb{Z}_q$ -extension of a fusion category  $\mathcal{D}$ , where  $\mathcal{D}$  has FP dimension  $q^2$ . By [3, Proposition 8.32],  $\mathcal{D}$  is either pointed or an Ising fusion category.

If  $\mathcal{D}$  is pointed, then Theorem 3.4 shows that  $\mathcal{C}$  is either pointed or equivalent to  $\mathcal{B} \boxtimes \mathcal{I}$ , where  $\mathcal{B}$  is a pointed modular category of FP dimension 2 and  $\mathcal{I}$  is an Ising category.

If  $\mathcal{D}$  is an Ising fusion category, then  $\mathcal{D}$  is a modular category by [2, Corollary B.12]. By Müger's Theorem [8, Theorem 4.2],  $\mathcal{C}$  is equivalent to  $\mathcal{D} \boxtimes \mathcal{D}'$ , where  $\mathcal{D}'$  is the Müger centralizer of  $\mathcal{D}$ . Again by Müger's Theorem,  $\mathcal{D}'$  is a modular category since  $\mathcal{C}$  is modular. Finally,  $\mathcal{D}'$  is a pointed fusion category by [3, Corollary 8.30] since its FP dimension is 2.  $\square$

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