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# ADJOINT GROUPS OVER $\mathbb{Q}_p(X)$ AND R-EQUIVALENCE - REVISITED

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ABSTRACT. We obtain a class of examples of non-rational adjoint classical groups of type ${}^2A_n$ and a group of type ${}^2D_3$ over the function field $F$ of a smooth geometrically integral curve over a $p$-adic field with $p \neq 2$. We also show that for any group of type $C_n$ over $F$, the group of rational equivalence classes of $G$ over $F$ is trivial, i.e., $G(F)/R = (1)$.

## 1. INTRODUCTION

Let $F$ be the function field of a smooth, geometrically integral curve over a $p$-adic field with $p \neq 2$ and $G$ be an absolutely simple adjoint algebraic group over $F$. In [PrS] we show that if $G$ is an adjoint algebraic group over $F$ of type ${}^2A_n^*$, $C_n$ or $D_n$ ($D_4$ non-trialitarian) such that the associated hermitian form has even rank, trivial discriminant (if $G$ is of type ${}^2A_n^*$ or $D_n$) and trivial Clifford invariant (if $G$ is of type $D_n$), then the group of rational equivalence classes, $G(F)/R$ is trivial. In this paper we show that the hypotheses on the hermitian forms associated to $G$ are necessary for groups of type $A_n$ and we extend the result in (Theorem 6.1, [PrS]) to any group of type $C_n$. Further, for a group $G$ of type $D_3$ with $h$ being an associated hermitian form, we show that if $\operatorname{disc}(h)$ is non-trivial, then $G(F)/R$ need not be trivial. For general groups of outer type $A_n$ and ${}^1D_n$, the triviality of $G(F)/R$ remains open. The main results in this paper are:

**Theorem 1.1.** *Let $p$ be a prime such that $p \neq 2$. Let $F = \mathbb{Q}_p(t)$ be the rational function field in one variable over the $p$-adic field $\mathbb{Q}_p$. Then for every positive integer $n \geq 2$, there exist absolutely simple adjoint algebraic groups $G$ of type ${}^2A_{2n-1}$ over $F$ such that, the group of rational equivalence classes over $F$ is non-trivial, i.e., $G(F)/R \neq (1)$.*

**Theorem 1.2.** *Let $F$ be the function field of a smooth, geometrically integral curve over a $p$-adic field with $p \neq 2$. For any absolutely simple adjoint classical group $G$ of type $C_n$, the group of rational equivalence classes over $F$ is trivial, i.e., $G(F)/R = (1)$.*

*Remark* 1.3. Using the exceptional isomorphisms for algebraic groups of low rank, a group of type ${}^2A_3$ can be identified with a group of type ${}^2D_3$. Hence for $p \neq 2$, by Theorem 1.1 we know that there exists a group $G$ of type ${}^2D_3$ over $\mathbb{Q}_p(t)$ such that $G(\mathbb{Q}_p(t))/R \neq (1)$. However in Example 6.1 we present a direct construction of such a group of type $D_3$ over $\mathbb{Q}_p(t)$ which has non-trivial R-equivalence classes.

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The triviality of $G(F)/R$ is closely related to $G$ being rational over $F$. For $F$ as above we note that the cohomological dimension of $F$ is $\leq 3$ (see [Se1]). Over some fields of cohomological dimension $\leq 3$, Merkurjev (Theorem 3, [M]) has shown that there exist groups of type ${}^2D_3$ which are non-rational and Gille (Theorem 3, [G1]) has shown that there exist groups of type ${}^1D_4$ which are non-rational. However, those examples do not yield information on the triviality of $G(F)/R$ over the function field $F$ of a $p$-adic curve. Furthermore, the existence of such non-rational groups over the function field of a $p$-adic curve $F$, was not known earlier. As an immediate corollary of Theorem 1.1 and Remark 1.3 we get examples of non-rational adjoint classical groups over $F$.

**Corollary 1.4.** *Let $F = \mathbb{Q}_p(t)$ with $p \neq 2$, be a rational function field over $\mathbb{Q}_p$ in one variable.*

(1) *If $p \neq 2$, for every positive integer $n \geq 2$, there exist absolutely simple adjoint algebraic groups $G$ of type ${}^2A_{2n-1}$ defined over $F$ which are non-rational.*
(2) *For $p \neq 2$, there exist groups $G$ of type ${}^2D_3$ defined over $F$ which are non-rational.*

We summarise below the known results, including the ones in this paper, along with the remaining open cases for convenience. Let $F$ be the function field of a smooth, geometrically integral curve over a $p$-adic field with $p \neq 2$.

| **Type of group $G$** | |
|---|---|
| ${}^1A_{n-1}$, $(n \geq 2)$ | $\mathbf{G}(F)/R = (1)$ and $G$ is rational (§2, [M]). |
| ${}^2A_{n-1}$ | $\mathbf{G}(F)/R = (1)$; when $n$ is odd due to Merkurjev (§2, [M]). |
| | $\mathbf{G}(F)/R = (1)$; when associated central simple algebra has square-free index and associated hermitian form has even rank and trivial discriminant (Theorem 5.3, [PrS]). |
| | $\mathbf{G}(F)/R$ need not be (1); when $m \geq 2$ and $n = 2m$ and $p \neq 2$ (Theorem 1.1) in this paper. |
| | Triviality of $\mathbf{G}(F)/R$ is not known when underlying central simple algebra has index divisible by square of a prime. |
| $B_n$ | $\mathbf{G}(F)/R = (1)$ and $G$ is rational (§2, [M]). |
| $C_n$ | $\mathbf{G}(F)/R = (1)$, for $n = 2$ and $n$ odd by Merkurjev (§2, Lemma 3, [M]) and for other cases (Theorem 6.1, [PrS]) and Theorem 1.2 in this paper. |
| ${}^1D_n$ | $\mathbf{G}(F)/R = (1)$; when $n = 2$ ($D_4$ non-trialitarian) and $n = 3$ due to Merkurjev (Proposition 5, [M]). |
| | $\mathbf{G}(F)/R = (1)$; when associated hermitian form has even rank, trivial discriminant and trivial Clifford invariant (Theorem 7.2, [PrS]). |
| | Not known when associated hermitian form has even rank, trivial discriminant and non-trivial Clifford invariant. |
| ${}^2D_n$ | $\mathbf{G}(F)/R$ need not be (1); when $n = 3$ (Theorem 1.3 this paper). |

## 2. Preliminaries

In this section, we recall some basic notions on hermitian forms over algebras with involutions.

2.1. **Notation and basic definitions.** Let $K$ be a field of characteristic different from 2. Let $(A, \sigma)$ denote a central simple algebra over a field $K$ with an involution, that is, $\sigma : A \to A$ is an anti-automorphism of order 2. Let $E = K^\sigma$ denote the fixed field of $K$ under $\sigma$. Then either $E = K$ or $K$ is a quadratic field extension of $E$.

For $\epsilon = \pm 1$, we denote by $(V, h)$ an $\epsilon$-hermitian form over $(A, \sigma)$. We denote by $W(A, \sigma)$ the Witt group of non-degenerate hermitian forms over $(A, \sigma)$. We refer to [L] and [Sc] for basic facts on quadratic and hermitian forms and their Witt groups. If $A = E$ and $\sigma$ is the identity, then $W(A, \sigma) = W(E)$, the usual Witt group of non-degenerate quadratic forms over $E$. For $a, b \in E^*$, we denote the associated quaternion algebra over $E$ by $(a, b)$. This is a central simple algebra over $E$ of degree 2 with basis $\{1, i, j, ij\}$ subject to the relations $i^2 = a, j^2 = b, ij = -ji$. We denote an $n$-fold Pfister form by $\langle\langle a_1, \ldots, a_n\rangle\rangle = \langle 1, -a_1\rangle \otimes \cdots \otimes \langle 1, -a_n\rangle$, for $a_1, \ldots, a_n \in E^*$.

2.2. **Invariants of hermitian forms over** $(A, \sigma)$**.** We recall some Galois cohomological invariants of hermitian forms over $(A, \sigma)$. We refer the reader to [BP1], [BP2], [PP] and [P] for more details. Let $D$ be a central simple algebra over $K$ which is a *division* algebra and $\sigma$ be an involution on $D$. Let $(V, h)$ be a non-degenerate hermitian form over $(D, \sigma)$.

(1) Rank $rank(h)$ : The *rank* of $(V, h)$ is defined as the dimension of the underlying $D$-vector space $V$, say $n = dim_D(V)$.
(2) Discriminant $\mathrm{disc}(h)$ : Given a basis of the $D$-module $V$, the hermitian form $h$ is given by some matrix $M(h)$ in this basis. Let $A = M_n(D)$ and let $m^2 = dim_K(A)$. The *discriminant*, $\mathrm{disc}(h)$ of $(V, h)$ is defined as:
$\mathrm{disc}(h) = (-1)^{m(m-1)/2}\ \mathrm{Nrd}_A(M(h)) \in E^*/E^{*2}$ if $\sigma$ is of first kind
$\mathrm{disc}(h) = (-1)^{m(m-1)/2}\ \mathrm{Nrd}_A(M(h)) \in E^*/N_{K|E}(K^*)$ if $\sigma$ is of second kind.
(3) Clifford invariant $c(h)$ : Suppose that $(D, \sigma)$ is of the first kind and orthogonal type over $E$. Let $(V, h)$ and $(V, h')$ be non-degenerate hermitian forms over $(D, \sigma)$ of the same rank such that $\mathrm{disc}(h) = \mathrm{disc}(h')$. The *relative Clifford invariant*, $Cl_h(h') \in {}_2Br(E)/(D)$ is defined by Bartels [B]. Let $H_{2n}$ be a hyperbolic form of rank $2n$ over $(D, \sigma)$. Let $(V, h)$ be a hermitian form such that $rank(h) = 2n$ and $\mathrm{disc}(h)$ is trivial. Then the *Clifford invariant*, $c(h) := Cl_{H_{2n}}(h)$ (see §2 [BP1] for more details). If $D = E$, then this invariant is the usual Clifford invariant of the quadratic form $h$.
(4) Rost invariant $R(h)$ : We refer to the relevant sections in [BP1] and [PP] for the definition of this invariant.

2.3. **Multipliers of similitudes.** Let $K$ be a field of characteristic not equal to 2 and $(A, \sigma)$ be a central simple algebra over $K$ with an involution. Let $E = K^\sigma$ be the fixed subfield under $\sigma$ in $K$. A similitude of $(A, \sigma)$ is an element $g \in A$ such that $\sigma(g)g \in E^*$. The scalar $\sigma(g)g$ is called the multiplier of the similitude $g$ and it is denoted by $\mu(g)$. The set of all similitudes of $(A, \sigma)$ is a subgroup of $A^*$ which is denoted by $\mathrm{Sim}(A, \sigma)$, and the map $\mu$ is a group homomorphism

$\mu : \mathrm{Sim}(A,\sigma) \to E^*$. We refer to §12.B and §12.C of [KMRT] for more details on similitudes of $(A,\sigma)$. The image of the map $\mu$ is denoted by $G(A,\sigma)$. Suppose $\sigma$ is an involution of orthogonal type on a central simple algebra $A$ of even degree $2m$ over a field $K$. For every similitude $g \in \mathrm{Sim}(A,\sigma)$ we have $\mathrm{Nrd}_A(g) = \pm\mu(g)^m$. A similitude is called proper if $\mathrm{Nrd}_A(g) = +\mu(g)^m$, otherwise it is called an improper similitude. In this case we write $G_+(A,\sigma)$ for the group of multipliers corresponding to proper similitudes. By convention, for a symplectic involution or an involution of the second kind, we set $G_+(A,\sigma) = G(A,\sigma)$. Consider the algebraic group $\mathbf{PSim}(A,\sigma)$ defined by

$$\mathbf{PSim}(A,\sigma) = \mathbf{Sim}(A,\sigma)/R_{K/E}(G_m),$$

where $R_{K/E}$ is the Weil restriction from $K$ to $E$ and $K$ is the center of $A$ (see [KMRT] for details). The connected component of identity of the algebraic group $\mathbf{PSim}(A,\sigma)$ is denoted by $\mathbf{PSim}_+(A,\sigma)$. Following the usual notation (§12.B [KMRT]), we denote the algebraic groups $\mathbf{PSim}(A,\sigma)$ according to the type of $\sigma$ as:

$$\mathbf{PSim}(A,\sigma) = \begin{cases} \mathbf{PGO}(A,\sigma) & \text{if } \sigma \text{ is of orthogonal type,} \\ \mathbf{PGSp}(A,\sigma) & \text{if } \sigma \text{ is of symplectic type,} \\ \mathbf{PGU}(A,\sigma) & \text{if } \sigma \text{ is of unitary type.} \end{cases}$$

We consider the groups $\mathrm{Sim}(A,\sigma)$ and $\mathrm{PSim}(A,\sigma)$ as the group of $F$-points of the corresponding algebraic groups $\mathbf{Sim}(A,\sigma)$ and $\mathbf{PSim}(A,\sigma)$ respectively. When the involution $\sigma$ is of unitary or symplectic type $\mathbf{PSim}(A,\sigma)$ is a connected group (see §2 [M]). In the case of an orthogonal involution $\sigma$ we denote the connected component of $\mathbf{PSim}(A,\sigma)$ by $\mathbf{PGO}_+(A,\sigma)$.

For $(A,\sigma)$, a central simple algebra over $K$ with an involution, let $E = K^\sigma$. Set $NK^* := \{\sigma(z)z : z \in K^*\}$. If $\sigma$ is an involution of the first kind, then $NK^* = E^{*2}$ and if $\sigma$ is of the second kind, then $NK^*$ is the group of norms $N_{K/E}(K^*)$ of the quadratic field extension $K/E$. We denote by $\mathrm{Hyp}(A,\sigma)$ the subgroup of $E^*$ generated by the norms of all finite extensions $M/E$ such that $\sigma_M$ is a hyperbolic involution. Further, for $(D,\sigma)$ a central division algebra over $K$ with an involution $\sigma$, and $h$ a non-degenerate hermitian form over $(D,\sigma)$ of rank $m$, we denote by $G(h)$ and $G_+(h)$ the groups $G(M_m(D),\sigma_h)$ and $G_+(M_m(D),\sigma_h)$ respectively, where $\sigma_h$ is the adjoint involution corresponding to $h$. We also denote by $\mathrm{Hyp}(h)$ the group $\mathrm{Hyp}(M_m(D),\sigma_h)$.

## 3. Some known results

In this subsection we recall some results which are used in the proofs of the main theorems. We refer to the earlier section for notation and terminology. We start with the following result due to Merkurjev.

**Theorem 3.1** (Theorem 1, [M])**.** *With notation as in section* 2.3 *above, there is a natural isomorphism*

$$\boldsymbol{PSim}_+(A,\sigma)(E)/R \simeq G_+(A,\sigma)/(NK^* \cdot \mathit{Hyp}(A,\sigma)).$$

Recall that for a field $E$ the $u$-invariant, $u(E)$ of $E$, is the largest dimension of anisotropic quadratic forms over $E$ and is $\infty$ if such an integer does not exist for $E$. Let $F$ be the function field of a smooth, geometrically integral curve over a $p$-adic field with $p \neq 2$. We next state a well-known result due to Parimala-Suresh on $u(F)$.

**Theorem 3.2** (Theorem 4.6, [PS2])**.** *Let $F$ be the function field of a curve over a $p$-adic field. If $p \neq 2$, then $u(F) = 8$.*

We now list adjoint classical groups over $F$ for which $G(F)/R$ is known to be trivial (see [PrS]). We start with the following theorem on groups of type $A_n$ (Theorem 5.3, [PrS]).

**Theorem 3.3.** *Let $F$ be the function field of a smooth, geometrically integral curve over a $p$-adic field with $p \neq 2$. Let $Z/F$ be a quadratic field extension and $(Q, \tau)$ a quaternion algebra over $Z$ with a unitary $Z/F$ involution. Let $h$ be a hermitian form over $(Q, \tau)$ of even rank $2n$ and trivial discriminant. Then for the adjoint group $\boldsymbol{PGU}(M_{2n}(Q), \tau_h)$, the group of rational equivalence classes is trivial, i.e.,*

$$\boldsymbol{PGU}(M_{2n}(Q), \tau_h)(F)/R = (1).$$

*In fact,*

$$G(h) = Hyp(h) = F^*.$$

For groups of type $C_n$ we have the following theorem (Theorem 6.1, [PrS]).

**Theorem 3.4.** *Let $F$ be the function field of a smooth, geometrically integral curve over a $p$-adic field with $p \neq 2$. Let $(A, \sigma)$ be a central simple algebra over $F$ with a symplectic involution. Let $h$ be a hermitian form over $(A, \sigma)$ of even rank $2n$. Then for the adjoint group $\boldsymbol{PGSp}(M_{2n}(A), \sigma_h)$, the group of rational equivalence classes is trivial, i.e.,*

$$\boldsymbol{PGSp}(M_{2n}(A), \sigma_h)(F)/R = (1).$$

*In fact*

$$G(h) = Hyp(h) = F^*.$$

For groups of type $D_n$, we have the following theorem (Theorem 7.2, [PrS]).

**Theorem 3.5.** *Let $F$ be the function field of a smooth, geometrically integral curve over a $p$-adic field with $p \neq 2$. Let $(A, \sigma)$ be a central simple algebra over $F$ with an orthogonal involution. Let $h$ be a hermitian form over $(A, \sigma)$ of even rank $2n$, trivial discriminant and trivial Clifford invariant. Then for the adjoint group $\boldsymbol{PGO}_+(M_{2n}(A), \sigma_h)$, the group of rational equivalence classes is trivial, i.e.,*

$$\boldsymbol{PGO}_+(M_{2n}(A), \sigma_h)(F)/R = (1).$$

*In fact,*

$$G_+(h) = Hyp(h) = F^*.$$

## 4. Adjoint groups of type ${}^2A_n$

Let $F$ be the function field of a smooth, geometrically integral curve over a $p$-adic field with $p \neq 2$. In this section we show that the hypothesis of Theorem 5.3, [PrS] is necessary. For every positive integer $n \geq 2$ we give examples of absolutely simple adjoint algebraic groups $G$ of type ${}^2A_{2n-1}$ over the rational function field $F$ over $\mathbb{Q}_p$ with $p \neq 2$ such that the group of rational equivalence classes over $F$ is non-trivial, i.e., $G(F)/R \neq (1)$.

For a central simple algebra $(B, \tau)$ of even degree with a unitary $Z/F$ involution, let $D(B, \tau)$ denote its *discriminant algebra* over $F$. Recall that $D(B, \tau)$ has a canonical involution of the first kind (see §10, [KMRT] for details on the discriminant algebra). We start with the following proposition.

**Proposition 4.1.** *Let $F$ be the function field of a smooth, geometrically integral curve over a $p$-adic field with $p \neq 2$. Let $Z/F$ be a quadratic field extension and $(Q, \tau)$ a quaternion algebra over $Z$ with unitary $Z/F$ involution. Let $h$ be a non-degenerate hermitian form of even rank $2n$ over $(Q, \tau)$. Then for the associated discriminant algebra $D(M_{2n}(Q), \tau_h)$ we have*

$$\mathrm{Nrd}(D(M_{2n}(Q), \tau_h)) = Hyp(h).$$

*Proof.* Let $d \in F^*$ be such that $Z = F(\sqrt{d})$. We claim that $D(M_{2n}(Q), \tau_h) \sim (d, \mathrm{disc}(h))_F$. If $Q$ is split, then by Corollary 10.35, [KMRT] we have $D(M_{2n}(Q), \tau_h) \sim (d, \mathrm{disc}(h))_F$. If $Q$ is non-split let $K = F(R_{Z/F}(SB(M_{2n}(Q))))$ be the function field of the Weil transfer of the Severi-Brauer variety of $M_{2n}(Q)$. Then $M_{2n}(Q)_K$ is split over $K$. Therefore $D(M_{2n}(Q), \tau_h)_K \sim (d, \mathrm{disc}(h))_K$. As the map $Br(F) \to Br(K)$ is injective (see Corollary 2.12, [MT]), we have $D(M_{2n}(Q), \tau_h) \sim (d, \mathrm{disc}(h))_F$.

To prove the inclusion $\mathrm{Nrd}(D(M_{2n}(Q), \tau_h)) \subset \mathrm{Hyp}(h)$, we use results of [PrS]. Let $L/F$ be a finite field extension such that $D(M_{2n}(Q), \tau_h)$ is split over $L$. Then $(d, \mathrm{disc}(h))$ splits over $L$, which implies $\mathrm{disc}(h) \in N_{LZ/L}((LZ)^*)$. Thus, over $L$, $h$ has even rank and trivial discriminant in $L^*/N_{LZ/L}((LZ)^*)$. By Theorem 5.3, [PrS], $\mathrm{Hyp}(h_L) = L^*$. Hence $N_{L/F}(L^*) = N_{L/F}(\mathrm{Hyp}(h_L)) \subset \mathrm{Hyp}(h)$. Thus, $\mathrm{Nrd}(D(M_{2n}(Q), \tau_h)) \subset \mathrm{Hyp}(h)$.

Conversely, if $h$ is hyperbolic over a finite field extension $M$ of $F$, then $h_M$ has trivial discriminant. Therefore, $D(M_{2n}(Q), \tau_h) \sim (d, \mathrm{disc}(h))$ splits over $M$. Hence, $\mathrm{Hyp}(h) \subset \mathrm{Nrd}(D(M_{2n}(Q), \tau_h))$. This inclusion has been proved in [BMT]. □

*Proof of Theorem* 1.1. Let $F = \mathbb{Q}_p(t)$ with $p \neq 2$. Let $b \in \mathbb{Z}_p^*$ be a non-square unit. Then $\langle\langle b, p\rangle\rangle$ is an anisotropic 2-fold Pfister form over $\mathbb{Q}_p$ (see VI, 2.2, [L]). Let $H = (b, t)_F$ be a quaternion algebra over $F$.

Let $K = \mathbb{Q}_p(\sqrt{p})$. As $K$ is a totally ramified field extension of $\mathbb{Q}_p$, the residue field of $K$ is the same as the residue field of $\mathbb{Q}_p$. Hence by Hensel's lemma, $b \notin K^{*2}$. By taking residues in the Laurent series field $K((t))$ and using (1.9, Chapter VI, [L]) we see that $H$ does not split over $F(\sqrt{p})$, i.e., the norm form of $H$, $n_H = \langle 1, -t, -b, tb\rangle$ is anisotropic over $F(\sqrt{p})$.

As $b$ is a unit in $\mathbb{Z}_p^*$, by Chapter VI, 2.5, [L], $(-1, b)$ splits over $\mathbb{Q}_p$. Hence $\langle 1, 1\rangle \cdot n_H = 0$ in the Witt group $W(F)$ of $F$. Hence $-1 \in \mathrm{Nrd}(H)$ (cf. Chapter III, 2.3 and 2.4′, [L]). Let $u \in H^*$ be such that $-1 = \mathrm{Nrd}(u)$. We have the following two cases.

*Case* 1. $n = 2m$. We choose a quaternion basis $1, i, j, ij$ of $H$ such that $i$ commutes with $u$. So $i^2 = b$, $j^2 = t$, $i \cdot j = -j \cdot i$ and $i \cdot u = u \cdot i$. Now consider the involution

$$\sigma = \mathrm{Int}\begin{pmatrix} j/t & 0 & \dots & 0 \\ 0 & i/b & \dots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \dots & i/b \end{pmatrix} \circ^{-} t$$

on $M_{2m}(H)$, where $^{-}$ denotes the canonical involution on $H$ and $t$ denotes the transpose of a matrix.

As $\sigma(\mathrm{diag}(j, i, \cdots, i)) = -\mathrm{diag}(j, i, \cdots, i)$, $\sigma$ is an orthogonal involution and

$$\mathrm{disc}(\sigma) = \mathrm{Nrd}(j) \cdot \mathrm{Nrd}(i) \equiv t \cdot b \not\equiv 1 \pmod{F^{*2}}.$$

Let $(B,\tau) = (M_{2m}(H) \otimes_F F(\sqrt{p}), \sigma \otimes \gamma)$, where $\sigma$ is the orthogonal involution constructed above and $\gamma$ is the non-trivial automorphism of $F(\sqrt{p})/F$. So $(B,\tau)$ is a central simple algebra over $F(\sqrt{p})$ with a unitary $F(\sqrt{p})/F$ involution $\tau$. Let $g = \operatorname{diag}(j, uj, \cdots, uj)$, where $u \in H^*$ is the element chosen above. As $i \cdot u = u \cdot i$, we have $\sigma(g) = \operatorname{diag}(-j, j\bar{u}, \cdots, j\bar{u})$. Hence

$$\mu(g) = \sigma(g) \cdot g = -t.$$

Thus $-t \in G(B,\tau)$.

By Proposition 10.33, [KMRT], as $D(B,\tau) \sim (p, \operatorname{disc}(\sigma))_F = (p, t\cdot b)_F$ we have

$$\begin{aligned} \mu(g) \cup D(B,\tau) &= (-t) \cup (p) \cup (t \cdot b) && \text{in } H^3(F,\mu_2) \\ &= (-t) \cup (p) \cup (b) && \text{since } \langle\langle -t, t\cdot b\rangle\rangle \simeq \langle\langle -t, b\rangle\rangle \\ &= (t) \cup (p) \cup (b) && \text{since } (-1,b)_F \text{ is split.} \end{aligned}$$

We claim that $(t) \cup (p) \cup (b) \neq 0 \in H^3(F,\mu_2)$. If $(t)\cup(p)\cup(b) = 0 \in H^3(F,\mu_2)$, then $\langle 1, -t\rangle \cdot \langle\langle p, b\rangle\rangle = 0 \in I^3(F)$. But by taking residues in $\mathbb{Q}_p((t))$ and noting that $\langle\langle b, p\rangle\rangle$ is anisotropic over $\mathbb{Q}_p$ we have $\langle 1, -t\rangle \cdot \langle\langle p, b\rangle\rangle \neq 0$. Thus, $(t)\cup(p)\cup(b) \neq 0$. Hence $\mu(g) \notin \operatorname{Nrd}(D(B,\tau))$ (see Chapter III, 2.3 and 2.4$'$, [L]). By Proposition 4.1 above, $\mu(g) \notin \operatorname{Hyp}(B,\tau)$. Hence by §2, [M], $\mathbf{PGU}(B,\tau)(F)/R \neq (1)$.

*Case* 2. $n = 2m+1$. We choose a quaternion basis $1, i, j, ij$ of $H$ such that $j$ commutes with $u$. So $i^2 = b$, $j^2 = t$, $i\cdot j = -j \cdot i$ and $j \cdot u = u \cdot j$. Now consider the involution $\sigma$,

$$\sigma = \operatorname{Int}\begin{pmatrix} i & 0 & \dots & 0 \\ 0 & j & \dots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \dots & j \end{pmatrix} \circ^{-} t$$

on $M_{2m+1}(H)$ with $m \geq 1$, where $^{-}$ denotes the canonical involution on $H$ and $t$ denotes the transpose of a matrix.

As $\sigma(\operatorname{diag}(i,j,\cdots,j)) = -\operatorname{diag}(i,j,\cdots,j)$, $\sigma$ is an orthogonal involution and $\operatorname{disc}(\sigma) = (-1)^{2m+1}\cdot \operatorname{Nrd}(i) \cdot \operatorname{Nrd}(j)^{2m} \equiv b \pmod{F^{*2}}$.

Let $(B,\tau) = (M_{2m+1}(H) \otimes_F F(\sqrt{p}), \sigma \otimes \gamma)$, where $\sigma$ is the orthogonal involution constructed above and $\gamma$ is the non-trivial automorphism of $F(\sqrt{p})/F$. So $(B,\tau)$ is a central simple algebra over $F(\sqrt{p})$ with a unitary $F(\sqrt{p})/F$ involution $\tau$. Let $g = \operatorname{diag}(j, j\bar{u}, \cdots, j\bar{u})$, where $u \in H^*$ is the element chosen above. As $\sigma(g) = \operatorname{diag}(j, -ju, \cdots, -ju)$, we have

$$\mu(g) = \sigma(g)\cdot g = t.$$

Thus $t \in G(B,\tau)$.

Further, by Proposition 10.33, [KMRT],

$$\begin{aligned} D(B,\tau) &\sim \lambda^{2m+1} M_{2m+1}(H) \otimes_F (p, \operatorname{disc}(\sigma))_F \\ &\sim H \otimes_F (p,b) \\ &\sim (b,t) \otimes_F (p,b) \\ &\sim (b, t\cdot p). \end{aligned}$$

Therefore, we have

$$\begin{aligned}\mu(g) \cup D(B,\tau) &= (t) \cup (b) \cup (t \cdot p) && \text{in } H^3(F,\mu_2)\\ &= (t) \cup (b) \cup (-p) && \text{since } \langle\langle t, t\cdot p\rangle\rangle \simeq \langle\langle t, -p\rangle\rangle\\ &= (t) \cup (b) \cup (p) && \text{since } (-1,b)_F \text{ is split.}\end{aligned}$$

Hence arguing exactly as in Case 1, we have, $\mu(g) \cup D(B,\tau) \neq 0$ in $H^3(F,\mu_2)$. Hence $\mu(g) \notin \mathrm{Nrd}(D(B,\tau))$ (see Chapter III, 2.3 and 2.4′, [L]). By Lemma 10, [BMT], $\mu(g) \notin \mathrm{Hyp}(B,\tau)$. Hence by §2, [M], $\mathbf{PGU}(B,\tau)(F)/R \neq (1)$.

In view of the above Theorem 1.1 we get the following result.

**Corollary 4.2.** *Let $F$ be the rational function field in one variable over a $p$-adic field with $p \neq 2$. For each positive integer $n \geq 2$, there exists an absolutely simple adjoint algebraic group of type ${}^2A_{2n-1}$ defined over $F$ which is non-rational.*

*Proof.* For $F$ as in the corollary, let $(B,\tau)$ be a central simple algebra constructed as in Theorem 1.1 above. We have seen in Theorem 1.1 that the group of rational equivalence classes of $\mathbf{PGU}(B,\tau)(F)/R$ is non-trivial. Hence the group $\mathbf{PGU}(B,\tau)$ is not rational (Proposition 1, [M]). □

## 5. Adjoint groups of type $C_n$

Let $F$ be the function field of a smooth, geometrically integral curve over a $p$-adic field with $p \neq 2$. In this section we extend Theorem 6.1, [PrS] to an arbitrary group of type $C_n$ over $F$. We start with the following.

**Theorem 5.1.** *Let $F$ be the function field of a smooth, geometrically integral curve over a $p$-adic field with $p \neq 2$. Let $(Q_1 \otimes_F Q_2, \sigma)$ be a biquaternion division algebra over $F$ with symplectic involution. Let $h$ be a hermitian form over $(Q_1 \otimes_F Q_2, \sigma)$ of odd rank $r$. Then for the adjoint group $\boldsymbol{PGSp}(M_r(Q_1 \otimes Q_2), \sigma_h)$, the group of rational equivalence classes is trivial, i.e.,*

$$\boldsymbol{PGSp}(M_r(Q_1 \otimes_F Q_2), \sigma_h)(F)/R = (1).$$

*In fact,*

$$G(h) = Hyp(h) = F^*.$$

*Proof.* Let $q_A$ be an Albert form for $Q_1 \otimes_F Q_2$. So $q_A$ is a 6-dimensional quadratic form over $F$ with trivial discriminant. As the $u$-invariant of $F$, $u(F) = 8$ (see Theorem 4.6, [PS2]), the group of spinor norms of $q_A$, $Sn(q_A) = F^*$ (see proof of the Theorem 4.1, [PrS]). Let $\lambda \in F^*$. Then $\lambda \in Sn(q_A)$. Thus there exists a finite field extension $L/F$ such that $q_A$ is isotropic over $L$ and $\lambda = N_{L/F}(x)$, for some $x \in L^*$. Over $L$, $Q_1 \otimes Q_2 \sim H$ for some quaternion algebra. By Morita correspondence $M_r(Q_1 \otimes Q_2, \sigma_h)$ will correspond to $(M_{2r}(H), \sigma_{h_L})$. By Theorem 6.1, [PrS], $\mathrm{Hyp}(h_L) = L^*$ and thus $x \in \mathrm{Hyp}(h_L)$. Therefore, $\lambda = N_{L/F}(x) \in N_{L/F}(\mathrm{Hyp}(h_L)) \subset \mathrm{Hyp}(h)$. Hence $F^* = \mathrm{Hyp}(h) = G(h)$. Thus, the group of rational equivalence classes, $\mathbf{PGSp}(M_r(Q_1 \otimes_F Q_2), \sigma_h)(F)/R = (1)$. □

*Proof of Theorem* 1.2. Let $G$ be an absolutely simple adjoint algebraic group of type $C_n$ over $F$. By Weil's classification results ([We]), the group $G$ is associated to a central simple algebra $A$ over $F$ with a symplectic involution. Thus $\exp(A) \leq 2$ and by Corollary 2.2, [PS1] its index, $\mathrm{ind}(A) \leq 4$. The corollary follows by combining the above Theorem 5.1 along with Theorem 6.1 and Corollary 6.2 from [PrS].

## 6. Adjoint groups of type $D_n$

Let $F$ be the function field of a smooth, geometrically integral curve over a $p$-adic field with $p \neq 2$. For an adjoint classical group $G$ of type $D_n$ over $F$ we consider separately the cases when the associated hermitian form $h$ has trivial discriminant (that is, $G$ is of type ${}^1D_n$) and $h$ has non-trivial discriminant (that is, $G$ is of type ${}^2D_n$).

6.1. **Adjoint groups of type ${}^2D_n$.** Let $F$ be the function field of a smooth, geometrically integral curve over a $p$-adic field with $p \neq 2$. In this section we give an example of an adjoint group $G$ of type ${}^2D_3$ over a field $F$ for which the group of R-equivalence classes is not trivial, as mentioned in Remark 1.3 . Hence such a group $G$ is non-rational over $F$.

**Example 6.1.** Let $F = \mathbb{Q}_p(t)$ be a rational function field in one variable over $\mathbb{Q}_p$ with $p \neq 2$, where $t$ is an indeterminate. Let $p$ denote a uniformizing parameter of $\mathbb{Q}_p$ and $(p, u)$ be the unique quaternion division algebra over $\mathbb{Q}_p$. Let $Q_1 = (p, u) \otimes_{\mathbb{Q}_p} F$, $Q_2 = (t, u)$ and $Q = Q_1 \otimes_F Q_2 = (p \cdot t, u)$ be quaternion algebras over $F$. Let $^-$ denote the canonical involution on $Q$. Let $\sigma_{h'}$ be the adjoint involution on $M_2(Q)$ corresponding to the skew-hermitian form $h' := \langle 1, -p\rangle \cdot \langle j\rangle = \langle j, -pj\rangle$ over $(Q, ^-)$. In other words, $(M_2(Q), \sigma_{h'}) = (M_2(F), \sigma_{\langle 1,-p\rangle}) \otimes (Q, \sigma_{\langle j\rangle})$, where $\sigma_{\langle 1,-p\rangle}$ and $\sigma_{\langle j\rangle}$ are the adjoint involutions corresponding to $\langle 1, -p\rangle$ and $\langle j\rangle$. Note that $\sigma_{\langle 1,-p\rangle}$ and $\sigma_{\langle j\rangle}$ are orthogonal involutions on $M_2(F)$ and $Q$ respectively. Moreover, $\mathrm{disc}(\sigma_{\langle 1,-p\rangle}) = p \in F^*/F^{*2}$ and $\mathrm{disc}(\sigma_{\langle j\rangle}) = u \in F^*/F^{*2}$. Therefore, one of the components of the Clifford algebra $C(M_2(Q), \sigma_{h'})$ is Brauer-equivalent to $(\mathrm{disc}(\sigma_{\langle 1,-p\rangle}), \mathrm{disc}(\sigma_{\langle j\rangle})) = (p, u)$ and the other one to $Q \otimes_F (p, u) \sim (t, u)$, (see Tao's result, page 150, [KMRT]).

Let $h := h' + \langle i\rangle = \langle j, -pj, i\rangle$ be a skew-hermitian form over $(Q, ^-)$ and let $\sigma_h$ be the corresponding adjoint involution. Thus, we have $\mathrm{disc}(\sigma_h) = p \cdot t$. Set $L = F(\sqrt{p \cdot t})$. We show that the element $-p \cdot t$ is a non-trivial element in the group $\mathbf{PSim}_+(M_3(Q), \sigma_h)(F)/R$. Clearly $-p \cdot t$ belongs to $N_{L/F}(L^*)$. As $(-1, u)$ is split over $\mathbb{Q}_p$ we can find in $Q$ a pure quaternion $i'$ with $i'^2 = -pt$ that anti-commutes with $j$. Then $g = \mathrm{diag}(i', i', i)$ is a similitude with multiplier $\mu(g) = -pt$. Hence $-pt \in G_+(M_3(Q), \sigma_h)$.

Observe that $(-p \cdot t) \cup (p) \cup (u) = (t) \cup (p) \cup (u)$ and $(-p \cdot t) \cup (t) \cup (u) = (t) \cup (p) \cup (u)$. Hence $(-p \cdot t) \cup Q_i = (t) \cup (p) \cup (u)$, for $i = 1, 2$. We claim that $(t) \cup (p) \cup (u) \neq 0 \in H^3(F, \mu_2)$. Consider the Pfister form $q = \langle 1, -t\rangle \cdot \langle\langle p, u\rangle\rangle$ corresponding to the symbol $(t) \cup (p) \cup (u)$. Here we use the notation $\langle\langle a\rangle\rangle = \langle 1, -a\rangle$ for $a \in F^*$. We write $q = \langle\langle p, u\rangle\rangle \perp \langle -t\rangle \cdot \langle\langle p, u\rangle\rangle$. We consider the quadratic form $q$ in the field of formal Laurent series $\mathbb{Q}_p((t))$ with uniformizing parameter $t$. By VI, 1.9 (2), [L], $q$ is anisotropic over $\mathbb{Q}_p((t))$ as $\langle\langle p, u\rangle\rangle$ is anisotropic over $\mathbb{Q}_p$. Hence, $(-p \cdot t) \cup \mathrm{Nrd}(Q_i) = (t) \cup (p) \cup (u) \neq 0 \in H^3(F, \mu_2)$, for $i = 1, 2$. By Chapter III, 2.3 and 2.4$'$, [L], $-p \cdot t \notin \mathrm{Nrd}(Q_i)$. Thus, $-p \cdot t$ is a non-trivial element in the group $\mathbf{PSim}_+(M_3(Q), \sigma_h)(F)/R$ (by Proposition 9 [M]).

As an immediate consequence we have:

**Theorem 6.2.** *Let $F$ be the rational function field in one variable over a $p$-adic field with $p \neq 2$. Then there exists an absolutely simple adjoint algebraic group of type ${}^2D_3$ defined over $F$ which is non-rational.*

*Proof.* For $F$ as in the theorem, consider the central simple algebra with involution $(M_3(Q), \sigma_h)$ constructed as in Example 6.1 above. We have seen in Example 6.1 that the group of rational equivalence classes of $\mathbf{PSim}_+(M_3(Q), \sigma_h)(F)/R$ is non-trivial. Hence the group $\mathbf{PSim}_+(M_3(Q), \sigma_h)$ is not rational (Proposition 1, [M]). □

6.2. **Groups of type $^1D_n$.** Let $F$ be the function field of a smooth, geometrically integral curve over a $p$-adic field with $p \neq 2$. Let $G$ be an absolutely simple adjoint algebraic group of inner type $^1D_n$ defined over $F$. Let $(A, \sigma)$ be a central simple algebra with orthogonal involution over $F$ associated to the group $G$. The group $G$ being of type $^1D_n$ translates into $(A, \sigma)$ having even rank and trivial discriminant.

If we assume further that $(A, \sigma)$ has trivial Clifford invariant, then the group of rational equivalence classes, $G(F)/R = (1)$ by Theorem 7.2, [PrS]. Combining this with the results in this paper leaves only one case open where the behaviour of $G(F)/R$ is not known, namely when $(A, \sigma)$ has even rank, trivial discriminant and *non-trivial* Clifford invariant.

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