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INEQUALITIES FOR THE RATIO OF COMPLETE ELLIPTIC INTEGRALS

HORST ALZER AND KENDALL RICHARDS

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Abstract. We present various inequalities for the complete elliptic integral of the first kind,

$$\mathcal{K}(r) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - r^2 \sin^2(t)}} dt \quad (0 < r < 1).$$

Among others, we prove that the inequalities

$$\frac{1}{1 + \frac{1}{4} r} < \frac{\mathcal{K}(r)}{\mathcal{K}(\sqrt{r})} \quad \text{and} \quad \frac{\mathcal{K}(\sqrt{1 - r^2})}{\mathcal{K}(\sqrt{1 - r})} < \frac{2}{1 + \sqrt{r}}$$

are valid for all $r \in (0, 1)$. These estimates refine results published by Anderson, Vamanamurthy, and Vuorinen in 1990.

1. Introduction

The complete elliptic integrals of the first and second kind are defined for $r \in (0, 1)$ by

$$\mathcal{K}(r) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - r^2 \sin^2(t)}} dt = \frac{\pi}{2} \sum_{n=0}^{\infty} a_n r^{2n} \tag{1.1}$$

and

$$\mathcal{E}(r) = \int_0^{\pi/2} \sqrt{1 - r^2 \sin^2(t)} dt = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{a_n}{1 - 2n} r^{2n}$$

with

$$a_n = \Big(\frac{(1/2)_n}{n!} \Big)^2 = \frac{1}{\pi} \Big(\frac{\Gamma(n + 1/2)}{\Gamma(n + 1)} \Big)^2. \tag{1.2}$$

Both functions can be expressed in terms of the Gaussian hypergeometric function, namely

$$\mathcal{K}(r) = \frac{\pi}{2} F(1/2, 1/2; 1; r^2) \quad \text{and} \quad \mathcal{E}(r) = \frac{\pi}{2} F(-1/2, 1/2; 1; r^2).$$

The associate complete elliptic integrals are given by

$$\mathcal{K}'(r) = \mathcal{K}(r') \quad \text{and} \quad \mathcal{E}'(r) = \mathcal{E}(r'), \quad \text{where} \quad r' = \sqrt{1 - r^2}.$$

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These functions have remarkable applications in the theory of quasiconformal mappings as well as in geometry, mechanics, geodesy and other fields. The main properties of $\mathcal{K}$ and $\mathcal{E}$ are collected, for instance, in the books [6] and [8]. We also refer to the research articles [2], [9], [10], [11], [14], [15], [16] and the references cited therein.

In the next sections we make use of the formulas

$$\mathcal{K}\Big(\frac{1-r}{1+r}\Big) = \frac{1+r}{2}\mathcal{K}'(r) \tag{1.3}$$

and

$$\mathcal{K}\Big(\frac{2\sqrt{r}}{1+r}\Big) = (1+r)\mathcal{K}(r) \tag{1.4}$$

which are known as Landen identities and we apply the limit relations

$$\lim_{r\to 1}\Big(\mathcal{K}(r) + \frac{1}{2}\log(1-r^2)\Big) = \log(4) \tag{1.5}$$

and

$$\lim_{r\to 0}\frac{\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)}{r^2} = \frac{\pi}{4}; \tag{1.6}$$

see [8, (164.02), (112.01)] [1, (17.3.27)].

The work on this paper has been inspired by the following elegant inequalities which were published by Anderson et al. [4] in 1990:

$$\frac{1}{1+r} < \frac{\mathcal{K}(r)}{\mathcal{K}(\sqrt{r})} \tag{1.7}$$

and

$$\frac{\mathcal{K}'(r)}{\mathcal{K}'(\sqrt{r})} < \frac{2}{1+r}. \tag{1.8}$$

Both inequalities are valid for all $r \in (0,1)$.

Here we present various new inequalities for the ratio of complete elliptic integrals. Among others, we prove that the following refinements of (1.7) and (1.8) hold for all $r \in (0,1)$:

$$\frac{1}{1+\frac{1}{4}r} < \frac{\mathcal{K}(r)}{\mathcal{K}(\sqrt{r})} \quad \text{and} \quad \frac{\mathcal{K}'(r)}{\mathcal{K}'(\sqrt{r})} < \frac{2}{1+\sqrt{r}}.$$

An application of the second inequality leads to an improvement of the well-known arithmetic mean - geometric mean inequality

$$0 < \frac{1+r}{2} - \sqrt{r} \quad (0 < r < 1).$$

We show that the lower bound 0 can be replaced by a positive expression written in terms of $\mathcal{K}$.

In the next section we provide three lemmas. They play a role in the proofs of our main results which are given in Section 3.

2. Lemmas

The first lemma provides properties of the logarithmic derivative of the gamma function; see [1, (6.3.8), (6.4.10)].

Lemma 2.1. *The function $\psi = \Gamma'/\Gamma$ is strictly concave on $(0,\infty)$ and satisfies for $x > 0$ the duplication formula*

$$\psi(2x) = \frac{1}{2}\psi(x) + \frac{1}{2}\psi\Big(x + \frac{1}{2}\Big) + \log(2).$$

Moreover, we need several limit relations. They are collected in the following lemmas.

Lemma 2.2. *We have*

$$\lim_{r\to 1} \frac{F(a,b;a+b;r^2)}{F(a,b;a+b;r)} = 1 \quad \textit{and} \quad \lim_{r\to 0} \frac{F(a,b;a+b;1-r^2)}{F(a,b;a+b;1-r)} = 2. \tag{2.1}$$

Proof. Using [12, eq. (2), p. 74] gives the asymptotic relation

$$F(a,b;a+b;x) \sim -\frac{1}{B(a,b)}\log(1-x) \quad (x \to 1),$$

where $B(a,b)$ denotes Euler's beta function. This leads to (2.1). □

Remark 2.3. Setting $a = b = 1/2$ in (2.1) yields

$$\lim_{r\to 1} \frac{\mathcal{K}(r)}{\mathcal{K}(\sqrt{r})} = 1 \tag{2.2}$$

and

$$\lim_{r\to 0} \frac{\mathcal{K}'(r)}{\mathcal{K}'(\sqrt{r})} = 2. \tag{2.3}$$

Lemma 2.4. *Let*

$$u(r) = \mathcal{K}'(\sqrt{r}), \quad v(r) = r\frac{d}{dr}u(r), \quad w(r) = u(r) + \frac{1}{2}\log(r)$$

and

$$A(r) = v(r)w(r^2) - 2v(r^2)w(r), \quad B(r) = v(r^2) - v(r), \tag{2.4}$$

$$C(r) = \Big(\frac{w(r^2)}{\log(r)} - 1\Big)\big(2w(r) - w(r^2)\big).$$

Then,

$$\lim_{r\to 0} A(r) = \log(2), \quad \lim_{r\to 0} B(r) = 0, \quad \lim_{r\to 0} C(r) = -\log(4), \quad \lim_{r\to 1} v(r) = -\frac{\pi}{8}.$$

Proof. From (1.5) we obtain

$$\lim_{r\to 0} w(r) = \log(4). \tag{2.5}$$

Thus,

$$\lim_{r\to 0} v(r) = \lim_{r\to 0} \frac{\mathcal{E}(\sqrt{1-r}) - rw(r) + \frac{1}{2}r\log(r)}{2(r-1)} = \frac{1-0+0}{2\cdot(0-1)} = -\frac{1}{2}. \tag{2.6}$$

Using (2.5) and (2.6) leads to

$$\lim_{r\to 0} A(r) = -\frac{1}{2}\cdot\log(4) + 2\cdot\frac{1}{2}\cdot\log(4) = \log(2),$$

$$\lim_{r\to 0} B(r) = -\frac{1}{2}+\frac{1}{2} = 0, \quad \lim_{r\to 0} C(r) = (0-1)\cdot\big(2\cdot\log(4)-\log(4)\big) = -\log(4).$$

Applying (1.6) gives

$$\lim_{r\to 1} v(r) = \lim_{r\to 1}\Big(-\frac{1}{2}\cdot\frac{\mathcal{E}(\sqrt{1-r})-r\mathcal{K}(\sqrt{1-r})}{1-r}\Big) = -\frac{1}{2}\cdot\frac{\pi}{4} = -\frac{\pi}{8}.$$

□

3. Inequalities

Our first theorem offers an improvement of inequality (1.7).

Theorem 3.1. *For all $r\in(0,1)$ we have*

$$\frac{1}{1+\lambda r} < \frac{\mathcal{K}(r)}{\mathcal{K}(\sqrt{r})} < \frac{1}{1+\mu r} \tag{3.1}$$

with the best possible constant factors $\lambda = 1/4$ and $\mu = 0$.

Proof. Let $r\in(0,1)$. We define

$$f(r) = \frac{2}{\pi}\Big(1+\frac{1}{4}r\Big)\mathcal{K}(r) \quad\text{and}\quad g(r) = \frac{2}{\pi}\mathcal{K}(\sqrt{r}).$$

Applying (1.1) and (1.2) yields the series representations

$$f(r) = \sum_{n=0}^{\infty}\Big(a_n + \frac{1}{4}a_n r\Big)r^{2n}$$

and

$$g(r) = \sum_{n=0}^{\infty}\big(a_{2n}+a_{2n+1}r\big)r^{2n}.$$

Let

$$P_n(r) = \frac{a_n + \frac{1}{4}a_n r}{a_{2n}+a_{2n+1}r}.$$

Then, $P_0(r) = 1$. We obtain

$$P_n(r) = \frac{1+\frac{1}{4}r}{1+(\frac{4n+1}{4n+2})^2 r}\,\frac{a_n}{a_{2n}} > \frac{1+\frac{1}{4}}{1+(\frac{4n+1}{4n+2})^2}\,\frac{a_n}{a_{2n}}.$$

Let $x>0$ and

$$Q(x) = \frac{\Gamma(x+1/2)\Gamma(2x+1)}{\Gamma(x+1)\Gamma(2x+1/2)}.$$

Using Lemma 2.1 and Jensen's inequality gives

$$\frac{Q'(x)}{Q(x)} = \psi(x+1/2)+2\psi(2x+1)-\psi(x+1)-2\psi(2x+1/2)$$

$$= 2\psi(x+1/2) - \big[\psi(x+1/4)+\psi(x+3/4)\big] \ge 2\psi(x+1/2)-2\psi(x+1/2) = 0.$$

It follows that Q is increasing on $(0,\infty)$. Thus, for $n\ge 1$,

$$\frac{a_n}{a_{2n}} = Q^2(n) \ge Q^2(1) = \frac{16}{9}.$$

This yields

$$P_n(r) > \frac{16}{9}\frac{1+\frac{1}{4}}{1+(\frac{4n+1}{4n+2})^2} = 1 + \frac{32n^2+104n+35}{9(32n^2+24n+5)} > 1.$$

Hence, we have $f(r) > g(r)$ which is equivalent to the left-hand side of (3.1) with $\lambda = 1/4$.

Since $\mathcal{K}$ is strictly increasing on $(0,1)$, we obtain $\mathcal{K}(r)/\mathcal{K}(\sqrt{r}) < 1$. This leads to the right-hand side of (3.1) with $\mu = 0$.

It remains to show that the given constants are best possible. Double-inequality (3.1) can be written as

$$\mu < S(r) = \Big(\frac{\mathcal{K}(\sqrt{r})}{\mathcal{K}(r)} - 1\Big)\frac{1}{r} < \lambda. \tag{3.2}$$

Since

$$S(r) = \frac{a_1 + (a_2 - a_1)r + \dots}{a_0 + a_1 r^2 + \dots},$$

we obtain

$$\lim_{r\to 0} S(r) = \frac{a_1}{a_0} = \frac{1}{4}. \tag{3.3}$$

Using (2.2) gives

$$\lim_{r\to 1} S(r) = 0. \tag{3.4}$$

From (3.2), (3.3) and (3.4) we conclude that the constants $\lambda = 1/4$ and $\mu = 0$ are sharp.

Note added in proof. After the paper was accepted for publication we discovered that G.D. Anderson, M.K. Vamanamurthy and M. Vuorinen [5] proved the inequalities

$$\frac{1}{\sqrt[4]{1+r}} < \frac{\mathcal{K}(r)}{\mathcal{K}(\sqrt{r})} < \frac{\min\{\sqrt[4]{2}, 1/\sqrt{r'}\}}{\sqrt[4]{1+r}}, \qquad (0 < r < 1). \tag{3.5}$$

The lower bound in (3.5) improves the left-hand side of (3.1) with $\lambda = 1/4$, whereas the upper bounds given in (3.5) and (3.1) cannot be compared in general. We also refer to S.-L. Qiu and M.K. Vamanamurthy [13] who showed that the function $r \mapsto \sqrt[4]{1+r}\mathcal{K}(r)/\mathcal{K}(\sqrt{r})$ is strictly increasing from $[0,1)$ onto $[1, \sqrt[4]{2})$. □

An application of (3.1) leads to the following result.

Corollary 3.2. *For all $r \in (0,1)$ we have*

$$0 < \frac{\mathcal{K}(\sqrt{r}) - \mathcal{K}(r)}{\mathcal{K}\Big(\frac{2\sqrt{r}}{1+r}\Big) - \mathcal{K}(\sqrt{r})} < \frac{1}{3}. \tag{3.6}$$

Both bounds are sharp.

Proof. Let $r \in (0,1)$. We denote the ratio in (3.6) by $T(r)$. Since $\mathcal{K}$ is strictly increasing on $(0,1)$, we conclude from $0 < r < \sqrt{r} < 2\sqrt{r}/(1+r) < 1$ that

$$0 < \mathcal{K}(\sqrt{r}) - \mathcal{K}(r) \quad \text{and} \quad 0 < \mathcal{K}\Big(\frac{2\sqrt{r}}{1+r}\Big) - \mathcal{K}(\sqrt{r}).$$

Thus, $T(r) > 0$. Using (1.4) and the left-hand side of (3.1) with $\lambda = 1/4$ gives

$$\begin{aligned}\Big(\mathcal{K}\Big(\frac{2\sqrt{r}}{1+r}\Big) - \mathcal{K}(\sqrt{r})\Big)\big(1 - 3T(r)\big) &= (4+r)\mathcal{K}(r) - 4\mathcal{K}(\sqrt{r}) \\ &> (4+r)\mathcal{K}(r) - 4\Big(1 + \frac{1}{4}r\Big)\mathcal{K}(r) = 0.\end{aligned}$$

This leads to $T(r) < 1/3$.

Applying (2.2) gives

$$\lim_{r\to 1} T(r) = \lim_{r\to 1} \frac{1 - \frac{\mathcal{K}(r)}{\mathcal{K}(\sqrt{r})}}{(1+r)\frac{\mathcal{K}(r)}{\mathcal{K}(\sqrt{r})} - 1} = \frac{1-1}{2-1} = 0. \tag{3.7}$$

We have

$$T(r) = \frac{S(r)}{1 - S(r)},$$

where S is defined in (3.2). Using (3.3) leads to

$$\lim_{r\to 0} T(r) = \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{1}{3}. \tag{3.8}$$

From (3.7) and (3.8) we conclude that the constant bounds given in (3.6) are best possible. □

Now, we present a refinement of (1.8).

Theorem 3.3. *Let $\alpha, \beta \in \mathbf{R}$. The inequalities*

$$\frac{2}{1+r^{\alpha}} < \frac{\mathcal{K}'(r)}{\mathcal{K}'(\sqrt{r})} < \frac{2}{1+r^{\beta}} \tag{3.9}$$

are valid for all $r \in (0,1)$ if and only if $\alpha \le 0$ and $\beta \ge 1/2$.

Proof. First, we prove that if $\alpha = 0$ and $\beta = 1/2$, then (3.9) holds for all $r \in (0,1)$. Since $\mathcal{K}'$ is strictly decreasing on $(0,1)$, we obtain

$$\frac{2}{1+r^0} = 1 < \frac{\mathcal{K}'(r)}{\mathcal{K}'(\sqrt{r})}. \tag{3.10}$$

Let

$$\delta(r) = \frac{1-\sqrt{r}}{1+\sqrt{r}}.$$

We replace in (1.4) r by $\delta^2(r)$ and obtain

$$\mathcal{K}\Big(\frac{1-r}{1+r}\Big) = \big(1+\delta^2(r)\big)\mathcal{K}\big(\delta^2(r)\big). \tag{3.11}$$

Using (1.3) and (3.11) gives

$$\frac{\mathcal{K}'(r)}{\mathcal{K}'(\sqrt{r})} = \frac{1+\sqrt{r}}{1+r}\,\frac{\mathcal{K}\big(\frac{1-r}{1+r}\big)}{\mathcal{K}(\delta(r))} = \frac{1+\sqrt{r}}{1+r}\big(1+\delta^2(r)\big)\frac{\mathcal{K}\big(\delta^2(r)\big)}{\mathcal{K}(\delta(r))}. \tag{3.12}$$

Since $\mathcal{K}$ is strictly increasing on $(0,1)$, we get

$$\frac{\mathcal{K}'(r)}{\mathcal{K}'(\sqrt{r})} < \frac{1+\sqrt{r}}{1+r}\big(1+\delta^2(r)\big) = \frac{2}{1+\sqrt{r}}. \tag{3.13}$$

If $r \in (0,1)$, then the function $t \mapsto 2/(1+r^t)$ is increasing on $\mathbb{R}$. It follows from (3.10) and (3.13) that if $\alpha \le 0$ and $\beta \ge 1/2$, then (3.9) is valid for all $r \in (0,1)$.

Next, we assume that (3.9) holds for all $r \in (0,1)$. Then, we obtain

$$\alpha < F(r) < \beta \tag{3.14}$$

with

$$F(r) = \frac{\log(G(r))}{\log(r)} \quad \text{and} \quad G(r) = 2\frac{\mathcal{K}'(\sqrt{r})}{\mathcal{K}'(r)} - 1.$$

Using (2.3) leads to

$$\lim_{r\to 0} G(r) = 0.$$

Thus,

$$\lim_{r\to 0} F(r) = \lim_{r\to 0} \frac{rG_1(r)}{G(r)} \tag{3.15}$$

with $G_1(r) = (d/dr)G(r)$. We have

$$\frac{rG_1(r)}{G(r)} = 2\,\frac{\frac{A(r)}{\log(r)} + B(r)}{C(r)},$$

where A, B and C are defined in (2.4). Applying Lemma 2.4 gives

$$\lim_{r\to 0} \frac{rG_1(r)}{G(r)} = 2 \cdot \frac{0+0}{-\log(4)} = 0. \tag{3.16}$$

From (3.15) and (3.16) we conclude that

$$\lim_{r\to 0} F(r) = 0. \tag{3.17}$$

Since $G(1) = 1$, we obtain

$$\lim_{r\to 1} F(r) = \lim_{r\to 1} \frac{rG_1(r)}{G(r)} = \lim_{r\to 1} G_1(r). \tag{3.18}$$

We have

$$G_1(r) = \frac{2u(r^2)v(r) - 4u(r)v(r^2)}{ru^2(r^2)},$$

with $u(r) = \mathcal{K}'(\sqrt{r})$. Using $u(1) = \pi/2$ and Lemma 2.4 yields

$$\lim_{r\to 1} G_1(r) = \frac{1}{2}. \tag{3.19}$$

From (3.18) and (3.19) we get

$$\lim_{r\to 1} F(r) = \frac{1}{2}. \tag{3.20}$$

Applying (3.14) and the limit relations (3.17) and (3.20) leads to $\alpha \le 0$ and $\beta \ge 1/2$. This completes the proof of Theorem 3.3. □

Remark 3.4. From (3.12) and the left-hand side of (3.1) with $\lambda = 1/4$ we obtain for $r \in (0,1)$:

$$\frac{\mathcal{K}'(r)}{\mathcal{K}'(\sqrt{r})} > \frac{1+\sqrt{r}}{1+r}\,\frac{1+\delta^2(r)}{1+\frac{1}{4}\delta^2(r)} = \frac{2(1+\sqrt{r})}{(1+\sqrt{r})^2 + \frac{1}{4}(1-\sqrt{r})^2} > 1.$$

This is a refinement of the first inequality in (3.9) with $\alpha = 0$.

Remark 3.5. The following companion of (3.9) holds for $r \in (0,1)$:

$$0 < \frac{2}{1+\sqrt{r}} - \frac{\mathcal{K}'(r)}{\mathcal{K}'(\sqrt{r})} \le 0.2254\ldots.$$

The constant bounds are sharp. Does there exist a closed expression for the given numerical upper bound?

In the literature we can find many refinements of the classical arithmetic mean - geometric mean inequality; see, for example, [7, II.3]. The second inequality in (3.9) plays a key role in the proof of the following new refinement.

Corollary 3.6. *For all $r \in (0,1)$ we have*

$$0 < \frac{\mathcal{K}\left(\frac{1-r}{1+r}\right) - \mathcal{K}\left(\frac{1-\sqrt{r}}{1+\sqrt{r}}\right)}{\mathcal{K}(\sqrt{1-r})} < \frac{1+r}{2} - \sqrt{r}. \tag{3.21}$$

Proof. Let $r \in (0,1)$. Since

$$0 < \frac{1-\sqrt{r}}{1+\sqrt{r}} < \frac{1-r}{1+r} < 1,$$

we obtain

$$\mathcal{K}\Big(\frac{1-\sqrt{r}}{1+\sqrt{r}}\Big) < \mathcal{K}\Big(\frac{1-r}{1+r}\Big).$$

This leads to the left-hand side of (3.21). Applying the second inequality in (3.9) with $\beta = 1/2$ gives

$$\frac{\mathcal{K}'(r)}{\mathcal{K}'(\sqrt{r})} < \frac{2}{1+\sqrt{r}} < \frac{2+r-\sqrt{r}}{1+r}.$$

It follows that

$$(1+r)\mathcal{K}'(r) < (2+r-\sqrt{r})\mathcal{K}'(\sqrt{r}) = (1+\sqrt{r})\mathcal{K}'(\sqrt{r}) + (1+r-2\sqrt{r})\mathcal{K}'(\sqrt{r}).$$

Using (1.3) we find

$$2\mathcal{K}\Big(\frac{1-r}{1+r}\Big) < 2\mathcal{K}\Big(\frac{1-\sqrt{r}}{1+\sqrt{r}}\Big) + (1+r-2\sqrt{r})\mathcal{K}(\sqrt{1-r}).$$

This is equivalent to the right-hand side of (3.21). □

4. Concluding remarks

The generalized elliptic integrals are defined for $a \in (0,1/2]$ and $r \in (0,1)$ by

$$\mathcal{K}_a(r) = \frac{\pi}{2}F(a,1-a;1;r^2) \quad \text{and} \quad \mathcal{E}_a(r) = \frac{\pi}{2}F(a-1,1-a;1;r^2).$$

The special case $a = 1/2$ gives

$$\mathcal{K}_{1/2} = \mathcal{K} \quad \text{and} \quad \mathcal{E}_{1/2} = \mathcal{E}.$$

For more information on these functions we refer the reader to [3].

The referee pointed out that Theorems 3.1 and 3.3 seem to hold not only for the complete elliptic integral $\mathcal{K}$ but also for $\mathcal{K}_a$. Indeed, using similar methods as in the proof of Theorem 3.1 we obtain the following result.

Theorem 4.1. *Let $a \in (0, 1/2]$. For all $r \in (0, 1)$ we have*

$$\frac{1}{1+\lambda_a r} < \frac{\mathcal{K}_a(r)}{\mathcal{K}_a(\sqrt{r})} < \frac{1}{1+\mu_a r}$$

with the best possible factors $\lambda_a = a(1-a)$ and $\mu_a = 0$.

It remains an open problem to find a corresponding generalization of Theorem 3.3. M.E.H. Ismail even asked whether our inequalities can be extended to the zero-balanced hypergeometric function $F(a, b; a+b; r^2)$.

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Morsbacher Str. 10, D-51545 Waldbröl, Germany
E-mail address: h.alzer@gmx.de

Department of Mathematics and Computer Science, Southwestern University, Georgetown, Texas
E-mail address: richards@southwestern.edu