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GAP SEQUENCES OF MCMULLEN SETS

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ABSTRACT. We study the gap sequence of totally disconnected McMullen sets. Our result shows that if every horizontal line in the McMullen set is nonempty, then the gap sequence is unrelated to the box dimension. This implies that in such situations, the separation properties of McMullen sets are quite different from that of self-similar sets.

1. INTRODUCTION

Fix integers m, n and r such that $n > m \geq 2$ and $1 \leq r \leq mn$. Let $R = \{d_0, \dots, d_{r-1}\}$ be a subset of $\{0, \dots, n-1\} \times \{0, \dots, m-1\}$. For each $d_k = (d_k^{(1)}, d_k^{(2)}) \in R$, we define a self-affine transformation on $\mathbb{R}$ by

$$S_k(x) = T(x + d_k), \qquad k = 0, 1, \dots, r-1, \tag{1}$$

where $T = \operatorname{diag}(n^{-1}, m^{-1})$; then the family $\{S_k\}_{k=0}^{r-1}$ forms a self-affine iterated function system. According to Hutchinson [6, 12], there exists an attractor E, called a *McMullen set* [2, 19], such that $E = \bigcup_{k=0}^{r-1} S_k(E)$; the set E may also be written as

$$E = \left\{ \left(\sum_{k=1}^{\infty} \frac{d_{i_k}^{(1)}}{n^k}, \sum_{k=1}^{\infty} \frac{d_{i_k}^{(2)}}{m^k} \right) : i_k \in \{0, 1, \dots, r-1\} \right\}.$$

As stated in [2, 19], the box-counting dimension of the McMullen set E is

$$\dim_{\mathrm{B}} E = \frac{\log r - \log s}{\log n} + \frac{\log s}{\log m}, \tag{2}$$

where $s = \operatorname{card}\{j \colon (i,j) \in R,\ 0 \leq i \leq n-1,\ 0 \leq j \leq m-1\}$. The formula indicates that the box-counting dimension depends not only on r but also on s. That is to say, moving selected rectangles from one line to another may cause a change of dimension, which is quite different from the self-similar construction.

Self-affine sets are one of the most important objects in fractal geometry and related fields. In the self-affine construction, contraction ratios assume different values in different directions. This property makes self-affine sets more flexible than self-similar sets in many applications, which also causes huge difficulties in studying self-affine fractals.

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It is one of the most challenging problems to determine the Hausdorff and box dimensions of self-affine sets. There remain many questions (see [22]) open in this direction. Roughly speaking, there are two different approaches in the dimension theory of self-affine sets. The first one is to study the special cases. Bedford [2] and McMullen [19] first introduced McMullen sets and obtained the Hausdorff and box dimensions. Later on, more general settings for special cases were studies in [1, 15, 24]. The other approach, initiated by Falconer, is to study the general case. In two seminal papers [7, 8], Falconer determined the Hausdorff and box dimensions of almost all self-affine sets (in the sense of Lebesgue measure).

Since the gap sequence often characterizes the geometric properties of a set, it has been widely used to explore properties of fractals. In particular, it always gives an upper bound for box dimension, and we refer the reader to [3,9,16,17,23] for various results on dimension. Currently, most work on gap sequences focuses on self-similar and self-conformal fractals, and there has been little work on general self-affine sets, even on McMullen sets, a special case of self-affine sets. As the simplest self-affine models, McMullen sets often serve as a testing ground for questions, conjectures or counterexamples; we refer the reader to [1, 2, 10, 13–15, 18–21] for various studies and generalizations on McMullen sets.

In this paper, we study the geometry of McMullen sets by making use of gap sequences. Gap sequences often play an important role in studying the box-counting dimension of self-similar sets or cutting-out fractals; see [3, 5, 11, 23, 26]. In these cases, it has been shown that the gap sequence determines the box-counting dimension. In this paper, we will show that this is not always the case for McMullen sets. This gives another indication of the complexity of self-affine fractals.

Before stating our result, we recall the definition of gap sequences. Gap sequences of cutting-out sets in the line were first studied by Besicovitch [3]. Rao, Ruan and Yang [23] generalized this concept to sets in higher dimensional spaces.

Let A be a compact subset of $\mathbb{R}^2$. For distinct $x, y \in A$, we say x and y are *δ-equivalent* if there exists a sequence of points $a_0 = x, a_1, \dots, a_k = y$ of A such that $|a_{i+1} - a_i| \le \delta$ for $i = 0, 1, \dots, k-1$. Let $\mathfrak{N}(\delta)$ be the cardinality of the set of δ-equivalent classes of A. Clearly the mapping $\mathfrak{N}\colon \mathbb{R} \to \mathbb{N}$ is nonincreasing. We write $\mathfrak{N}(\delta^-) = \lim_{h\to 0^+} \mathfrak{N}(\delta - h)$. We say a sequence $\{\alpha_k\}_{k\ge 1}$ is the *gap sequence* of A if the elements of $\{\alpha_k\}_{k\ge1}$ are made of the jump points of the function δ with multiplicity $\mathfrak{N}(\delta^-) - \mathfrak{N}(\delta)$; that is, the value of δ is in the gap sequence $\{\alpha_k\}_k$ if and only if $\mathfrak{N}(\delta) < \mathfrak{N}(\delta^-)$ and the multiplicity

$$\operatorname{card}\{k : \alpha_k = \delta\} = \mathfrak{N}(\delta^-) - \mathfrak{N}(\delta).$$

We also write $\mathfrak{N}(\delta, A)$ to emphasize the dependence on the set A.

Given an index set I, we say that $\{a_i\}_{i\in I}$ and $\{b_i\}_{i\in I}$ are comparable if there exists a constant $C > 1$ such that $C^{-1}a_i \le b_i \le Ca_i$ for all $i \in I$. We denote this by $a_i \asymp b_i$. The main result in the paper is

Theorem 1. *Let E be a totally disconnected McMullen set. Let $\{\alpha_k\}_{k\ge1}$ be the gap sequence of E. Then*

$$\alpha_k \asymp \begin{cases} k^{-1/\dim_{\mathrm{B}} E}, & \text{if } s < m; \\ k^{-\log n/\log r}, & \text{if } s = m. \end{cases}$$

We remark that the asymptotic formula for the gap sequence $\{\alpha_k\}_{k\ge1}$ is unrelated to the box dimension in the situation $s = m$ (recall (2)). Geometrically, this implies that if every horizontal line in a McMullen set E is nonempty, then the separation properties of E are quite different from that of self-similar sets.

As a direct corollary of Theorem 1, we have

Corollary 1. *Let E be a McMullen set satisfying the SSC, i.e., $S_i(E) \cap S_j(E) = \emptyset$ for $i \neq j$. Let $\{\alpha_k\}_{k\geq 1}$ be the gap sequence of E. Then*

$$\alpha_k \asymp \begin{cases} k^{-1/\dim_{\mathrm{B}} E}, & \text{if } s < m; \\ k^{-\log n/\log r}, & \text{if } s = m. \end{cases}$$

It is obvious that the SSC implies total disconnectedness. On the other hand, there exist McMullen sets which are totally disconnected but don't satisfy the SSC.

Example 1. Let $S_0(x,y) = ((x+3)/4, y/2)$, $S_1(x,y) = (x/4, (y+1)/2)$ and $S_2(x,y) = ((x+2)/4, (y+1)/2)$. Let $E = \bigcup_{i=0}^{2} S_i(E)$ be the McMullen set. One can verify that E is totally disconnected but doesn't satisfy the SSC.

This paper is organized as follows. In Section 2, we study a geometrical property, the so-called "finite type", which plays an important role in our study of gap sequence. Section 3 is devoted to the proof of Theorem 1.

2. Finite type

For $k = 0, 1, 2, \ldots$, let Ω^k be the set of all k-term sequences of integers $0, 1, \ldots, r-1$, that is, $\Omega^k = \{(\sigma_1 \ldots \sigma_k) \colon 0 \leq \sigma_j \leq r-1\}$. We regard Ω^0 as just containing the empty sequence, that is, $\Omega^0 = \{\emptyset\}$. We abbreviate members of Ω^k by $\sigma = (\sigma_1 \ldots \sigma_k)$ and write $|\sigma| = k$ for the number of terms in σ. We write $\Omega = \bigcup_{k=0}^{\infty} \Omega^k$ for the set of all such finite sequences, and Ω^∞ for the corresponding set of infinite sequences, so $\Omega^\infty = \{(\sigma_1\sigma_2 \ldots \sigma_k \ldots) \colon 0 \leq \sigma_k \leq r-1\}$. For $\sigma = \sigma_1 \ldots \sigma_k \in \Omega^k$, $\tau = \tau_1 \ldots \tau_l \in \Omega^l$, write $\sigma * \tau = \sigma_1 \ldots \sigma_k \tau_1 \ldots \tau_l \in \Omega^{k+l}$. We write $\sigma|k = (\sigma_1 \ldots \sigma_k)$ for the k-term prefix of $\sigma = (\sigma_1\sigma_2 \ldots) \in \Omega^\infty$. We write $\sigma \preceq \tau$ if σ is a curtailment of τ. We call the set $[\sigma] = \{\tau \in \Omega^\infty \colon \sigma \preceq \tau\}$ the *cylinder* of σ, where $\sigma \in \Omega$. If $\sigma = \emptyset$, its cylinder is $[\sigma] = \Omega^\infty$. Let S_k be as in (1). For $\sigma = (\sigma_1 \ldots \sigma_k) \in \Omega^k$, write $S_\sigma = S_{\sigma_1} \circ \cdots \circ S_{\sigma_k}$.

We denote the unit square $[0,1]^2$ by Q. For each integer $k \geq 1$, write

$$\Psi_k = \bigcup_{\sigma \in \Omega_k} S_\sigma(Q),$$

$$\Xi_k = \bigcup_{(i,j)\in\{-1,0,1\}^2} \Big(\bigcup_{\sigma \in \Omega_k} S_\sigma(Q) + (i,j) \Big).$$

We say the McMullen set is of *finite type* if there is an integer M such that for every integer $k \geq 1$, each connected component of Ψ_k contains at most M rectangles of width n^{-k} and height m^{-k}.

The finite type property appeared in the study of Lipschitz equivalence of self-similar sets (see [25]). We will prove that the totally disconnected McMullen sets are of finite type (Theorem 2) by making use of ideas in [25].

To study finite type, we denote the *Hausdorff metric* between two subsets A and B of $\mathbb{R}^2$ by

$$d_H(A,B) = \inf\{\delta \ : \ A \subset B_\delta \text{ and } B \subset A_\delta\},$$

where A_δ and B_δ are the δ-neighbourhood of A and B, that is, $A_\delta = \{x \in \mathbb{R}^2 \colon |x - a| \leq \delta \text{ for some } a \in A\}$.

We begin with two topological lemmas coming from [25].

Lemma 1. *Let* $\{D_k\}_{k=1}^{\infty}$ *be a sequence of connected compact subsets in* $\mathbb{R}^2$*. If the union* $\bigcup_{k=1}^{\infty} D_k$ *is bounded, then there exist a subsequence* $\{D_{k_i}\}_{i=1}^{\infty}$ *and a connected compact set* D *in* $\mathbb{R}^2$ *such that* $D_{k_i} \overset{d_H}{\to} D$*, as* $i \to \infty$*.*

Lemma 2. *Let* $\{D_k\}_{k=1}^{n}$ *be a finite family of totally disconnected and compact subsets of a Hausdorff topology space. Then the set* $\bigcup_{k=1}^{n} D_k$ *is also totally disconnected.*

We denote the boundary of a set $A \subset \mathbb{R}^2$ by ∂A. Because of these two lemmas, we have the following property for a totally disconnected McMullen set.

Lemma 3. *Suppose the McMullen set* E *is totally disconnected. Then there exists an integer* k *such that for all connected components* χ *in* Ξ_k*, we have that* $\chi \cap \partial[0,1]^2 = \emptyset$ *or* $\chi \cap \partial[-1,2]^2 = \emptyset$*.*

Proof. We assume on the contrary that for all integers $k > 0$, there exists a connected component χ_k in Ξ_k such that $\chi_k \cap \partial[0,1]^2 \neq \emptyset$ and $\chi_k \cap \partial[-1,2]^2 \neq \emptyset$.

We take $x_k \in \chi_k \cap \partial[0,1]^2$ and $y_k \in \chi_k \cap \partial[-1,2]^2$. Clearly, χ_k connects the two points x_k and y_k. By Lemma 1, there is a subsequence $\{\chi_{k_i}\}_i$ such that $x_{k_i} \to x \in \partial[0,1]^2$ and $y_{k_i} \to y \in \partial[-1,2]^2$ and $\chi_{k_i} \overset{d_H}{\to} \chi$, where d_H is Hausdorff metric. By Lemma 1, χ is connected. Since $\chi \subset \bigcup_{(i,j)\in\{-1,0,1\}^2}(E+(i,j))$ and χ contains two distinct points x and y, it implies that $\bigcup_{(i,j)\in\{-1,0,1\}^2}(E+(i,j))$ is not totally disconnected, which contradicts Lemma 2. □

The next theorem shows that totally disconnected McMullen sets are of finite type.

Theorem 2. *Suppose that the McMullen set* E *is totally disconnected. Then* E *is of finite type.*

Proof. Since E is totally disconnected, by Lemma 3, there exists an integer k such that for all connected components χ in Ξ_k, we have that $\chi \cap \partial[0,1]^2 = \emptyset$ or $\chi \cap \partial[-1,2]^2 = \emptyset$. That is, for each connected component $\chi \in \Xi_k$ such that $\chi \cap \partial[0,1]^2 \neq \emptyset$, we have that $\chi \subset (-1,2)^2$. It implies that, for all integers $l \geq k$, every connected component of Ψ_l contains at most $9r^k$ rectangles of width n^{-l} and height m^{-l}. Hence E is of finite type. □

Example 2. Let $n = 4$, $m = 3$ and $r = 4$. Set $A = \{(0,0),(0,2),(1,1),(3,1)\}$; see Figure 1. As stated in Lemma 3, we can take $k = 2$ so that no connected components in Ξ_2 connect $\partial[0,1]^2$ and $\partial[-1,2]^2$; see Figure 2.

3. Gap sequence

By the definition of δ-equivalence, the following facts are straightforward.

Lemma 4. *Let* $\{\beta_k\}_{k\geq 1}$ *be the gap sequence of a set* F*. Then* $\delta = \beta_k$ *for some* k *if and only if there exist* $x, y \in F$ *such that* x *and* y *are* δ*-equivalent, but not* $(\delta - \epsilon)$*-equivalent for all* $\epsilon > 0$*.*

Lemma 5. *Assume that* $\beta_{k_1} > \beta_{k_1+1} = \cdots = \beta_k = \cdots = \beta_{k_2} > \beta_{k_2+1}$*. Then* $\mathfrak{N}(\beta_k) = k_1 + 1$*, and* $\mathfrak{N}(\beta_k^-) = k_2 + 1$*.*

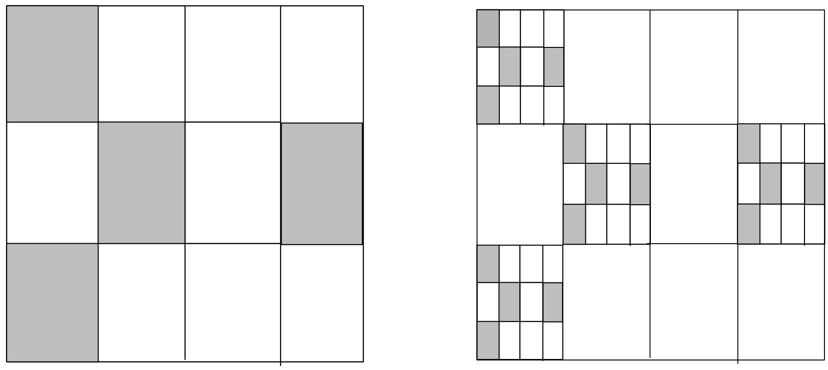

FIGURE 1. McMullen set for $n = 4$, $m = 3$ and $r = 4$.

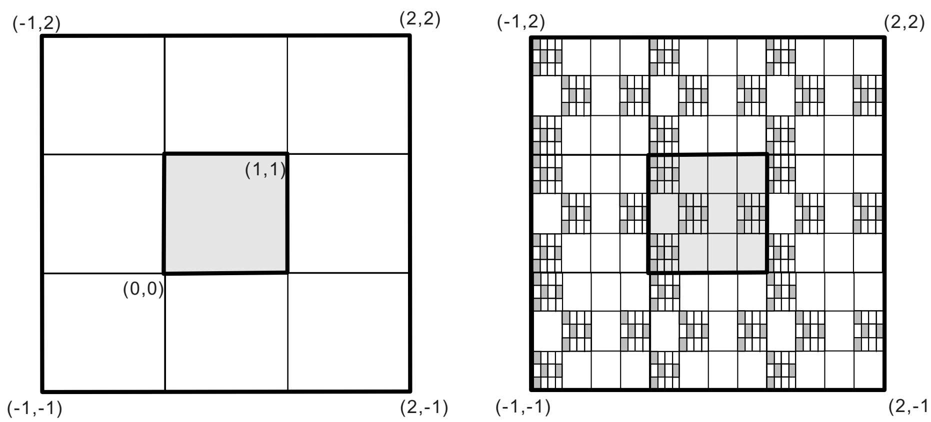

FIGURE 2. Connected components in Ξ_1 and Ξ_2.

Proposition 1. *Let $\{\beta_k\}_{k\geq 1}$ be the gap sequence of a set F. Then*

$$\beta_k^\gamma \asymp \frac{1}{k} \iff \mathfrak{N}(\delta, F) \asymp \delta^{-\gamma}.$$

Here $\gamma > 0$.

Proof. This proposition comes from [4]. However, the proof in [4] makes use of a redundant condition that $\inf_{k\geq 1} \beta_{k+1}/\beta_k > 0$. So we give a proof without this condition here.

$\Longrightarrow$: Assume that $\mathfrak{N}(\delta, F) = k$; then

$$\beta_{k-1} > \delta \geq \beta_k. \tag{3}$$

For this, it suffices to show that $\beta_{k-1} > \beta_k$, since if $\beta_{k-1} = \beta_k$, then $\mathfrak{N}(\beta_k, F) = \mathfrak{N}(\beta_{k-1}, F) \leq k - 1$ and $\mathfrak{N}(\beta_k^-, F) > k$, which contradicts that $\mathfrak{N}(\delta, F) = k$. Now it follows from $\beta_k^\gamma \asymp \frac{1}{k}$ and (3) that $\delta^{-\gamma} \asymp k = \mathfrak{N}(\delta, F)$.

$\Longleftarrow$: For all $k \geq 1$, we have $\mathfrak{N}(\beta_k, F) \leq k < \mathfrak{N}(\beta_k^-, F)$. Together with the condition $\mathfrak{N}(\delta, F) \asymp \delta^{-\gamma}$, we have $k \asymp \beta_k^{-\gamma}$. $\square$

Proof of Theorem 1. First, we define approximate squares, which are one of the fundamental ideas in studying self-affine fractals; see [1,15,19]. Let $x \in E$, and let $\{\sigma_i\}_{i=1}^{\infty}$ be the Ω-sequences of x. For each given k, let l be the unique integer such that $m^{-l-1} < n^{-k} \leq m^{-l}$. Note that $l > k$ since $n > m$. We write

$$(x_1, x_2) = S_{\sigma_1 \ldots \sigma_k}(0,0) \quad \text{and} \quad (x_3, x_4) = S_{\sigma_1 \ldots \sigma_l}(0,0).$$

Obviously, the sets $S_{\sigma_1 \ldots \sigma_k}(Q)$ and $S_{\sigma_1 \ldots \sigma_l}(Q)$ satisfy

$$S_{\sigma_1 \ldots \sigma_l}(Q) \subset [x_1, x_1 + n^{-k}] \times [x_4, x_4 + m^{-l}] \subset S_{\sigma_1 \ldots \sigma_k}(Q),$$

where the rectangle $[x_1, x_1 + n^{-k}] \times [x_4, x_4 + m^{-l}]$ is often named *the approximate square* of x, written as $\triangle_x(k)$, and clearly $x \in \triangle_x(k)$. We write $\Psi_{k,l} = \bigcup_{x \in E} \triangle_x(k)$.

Let $\{\alpha_k\}_{k \geq 1}$ and $\{\beta_k\}_{k \geq 1}$ be the gap sequences of E with respect to the Euclidean norm and the maximum norm, respectively. Rao, Ruan and Yang [23] proved that if two compact metric spaces with infinite gap sequences are bilipschitz equivalent, their gap sequences are comparable. Consequently, we have that

$$\alpha_k \asymp \beta_k.$$

From now on, we consider $\mathbb{R}^2$ with the maximum norm.

By Proposition 1, it is sufficient to show that $\mathfrak{N}(n^{-k}, E) \asymp n^{\gamma k}$ for all $k \geq 1$, where

$$\gamma = \begin{cases} \dim_{\mathrm{B}} E, & \text{if } s < m; \\ \log_n r, & \text{if } s = m. \end{cases}$$

We divide the proof into two cases.

Case 1: $s < m$. We need to show that

$$\mathfrak{N}(n^{-k}, E) \asymp n^{k \dim_{\mathrm{B}} E}.$$

For each given integer $k \geq 1$, by Theorem 2 and the definition of finite type, every connected component of Ψ_k contains at most M rectangles.

We split each rectangle of Ψ_k into approximate squares of width n^{-k} and height m^{-l}, where $m^{-l-1} < n^{-k} \leq m^{-l}$ (see Figure 3). Now fix a rectangle of Ψ_k and let A be the union of all the approximate squares of the fixed rectangle. According to the definition of approximate squares and the construction of the McMullen set, every connected component of A contains at most s approximate squares since $s < m$.

Recall that the union of all the approximate squares is $\Psi_{k,l}$, which is a subset of Ψ_k. Since all rectangles of Ψ_k are split into approximate squares in the same manner, every connected component of $\Psi_{k,l}$ contains at most sM approximate squares.

The number of all connected components of $\Psi_{k,l}$ is at least $(r^k s^{l-k})/(sM)$, since the number of all approximate squares in $\Psi_{k,l}$ is $r^k s^{l-k}$. Moreover, for any two distinct connected components χ, χ' of $\Psi_{k,l}$, we have $d(\chi, \chi') \geq n^{-k}$. It gives the lower bound for $\mathfrak{N}(n^{-k}, E)$, that is,

$$\mathfrak{N}(n^{-k}, E) \geq \frac{r^k s^{l-k}}{sM},$$

since $E \subset \Psi_{k,l}$ and E has nonempty intersection with every approximate square.

Now we turn to the upper bound. For each given integer $k \geq 1$, we split rectangles again into approximate squares but of width n^{-k} and height m^{-l-1}, where l is the unique integer such that $m^{-l-1} < n^{-k} \leq m^{-l}$ (see Figure 3).

The number of all such approximate squares is $r^k s^{l+1-k}$. Moreover, any two points in a same approximate square are n^{-k}-equivalent. Hence we have the upper bound $\mathfrak{N}(n^{-k}, E) \leq r^k s^{l+1-k}$.

Combining the lower bound and the upper bound, we have proved $\mathfrak{N}(n^{-k}, E) \asymp r^k s^{l-k}$, where l is the unique integer satisfying $m^{-l-1} < n^{-k} \leq m^{-l}$. Recall that $\dim_B E = (\log r - \log s)/\log n + \log s/\log m$. We have $\mathfrak{N}(n^{-k}, E) \asymp n^{k \dim_B B}$, which completes the proof of case 1.

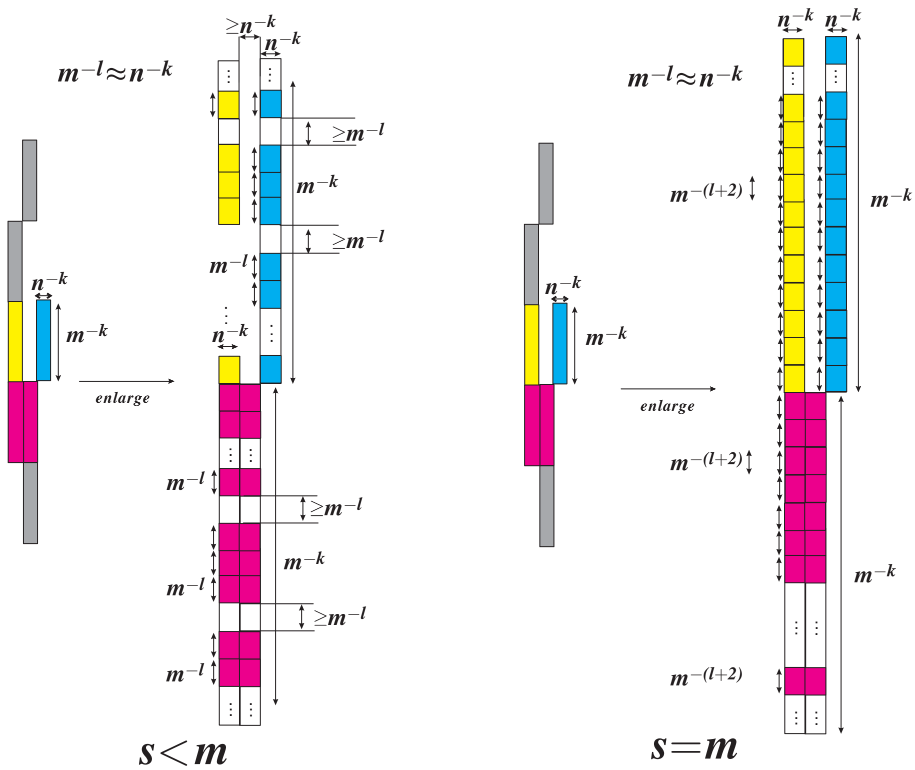

FIGURE 3. Connected components made of approximate squares.

Case 2: $s = m$. We need to show that

$$\mathfrak{N}(n^{-k}, E) \asymp r^k.$$

For each k, there are r^k rectangles of width n^{-k} and height m^{-k} in Ψ_k. By Theorem 2 and the definition of finite type, there are at least $\frac{r^k}{M}$ connected components in Ψ_k. Moreover, for any two distinct connected components χ, χ' of Ψ_k, we have $d(\chi, \chi') \geq n^{-k}$. It gives the lower bound for $\mathfrak{N}(n^{-k}, E)$, that is,

$$\mathfrak{N}(n^{-k}, E) \geq \frac{r^k}{M},$$

since $E \subset \Psi_k$ and E has nonempty intersection with every connected component of Ψ_k.

We turn to the upper bound. Let R be a rectangle of Ψ_k. We claim that any two points in $E \cap R$ are n^{-k}-equivalent. For this, we split R into approximate squares of width n^{-k} and height m^{-l-2}, where $m^{-l-1} < n^{-k} \leq m^{-l}$. There are m^{l+2-k} rectangles $R_1, \ldots, R_{m^{l+2-k}}$ of width n^{-k} and height m^{-l-2} in R. Since $s = m$, all these m^{l+2-k} rectangles are approximate squares in $\Psi_{k,l+2}$, i.e., $E \cap R_i \neq \emptyset$ for $i = 1, \ldots, m^{l+2-k}$. Pick $x_i \in E \cap R_i$. It is clear that $d(x_i, x_{i+1}) \leq \max(n^{-k}, 2m^{l+2-k}) = n^{-k}$ for $i = 1, \ldots, m^{l+2-k} - 1$. Thus, x_i and x_j are n^{-k}-equivalent for $i, j \in \{1, \ldots, m^{l+2-k}\}$. Now for any $x, y \in E \cap R$, suppose that $x \in R_i$ and $y \in R_j$; then x, x_i are n^{-k}-equivalent and y, x_j are n^{-k}-equivalent. Therefore, x, y are n^{-k}-equivalent.

Since there are r^k rectangles in Ψ_k, we have $\mathfrak{N}(n^{-k}, E) \leq r^k$. Combining the lower bound and the upper bound, we have $\mathfrak{N}(n^{-k}, E) \asymp r^k$, which completes the proof of case 2. $\square$

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